Behavior of Entire Solutions of a Nonlinear Elliptic Equation with An Inhomogeneous Singular Term

ARIJ BOUZELMATE, HIKMAT EL BAGHOURI
LaR2A Laboratory, Department of Mathematics,
Faculty of Sciences, Abdelmalek Essaadi University
Tetouan, MOROCCO

Abstract: - In this paper, we are interested in the singular positive solutions of the inhomogeneous radial equation

$$\left(|u'|^{p-2}u'\right)'(r) + \frac{N-1}{r}|u'|^{p-2}u'(r) + u^q(r) + f(r) = 0,\ r > 0,$$

where $N \geq 1, p > 2,\ q > 1$ and f is a continuous radial and strictly positive function on $(0, +\infty)$.
More precisely, we study the solutions u that cannot be extended by continuity at zero, that is, $\lim_{r\to 0} u(r) = +\infty$. We give existence and nonexistence results and we describe the behavior of entire solutions near infinity. The study needs some assumptions on $p,\ q,\ N$ and explicit conditions on the inhomogeneous term f.

Key-Words: - Inhomogeneous elliptic equation; entire solutions; strictly positive solutions; energy function; asymptotic behavior near infinity.

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1 Introduction

The purpose of this paper is to study the following equation

$$(|u'|^{p-2}u')' + \frac{N-1}{r}|u'|^{p-2}u' + u^q + f(r) = 0,\ r > 0, \tag{1}$$

where $N \geq 1, p > 2,\ q > 1$ and f is a continuous radial and strictly positive function on $(0, +\infty)$.
We are interested in the singular positive solutions of (1) that satisfy $\lim_{r\to 0} u(r) = +\infty$. The study is a continuation of the work carried out by [6], where the authors proved the existence of a maximal solution u defined on $]0, r_{max}[$ such that $u \in C^0(]0, r_{max}[) \cap C^1(]0, r_{max}[)$ and $\left(|u'|^{p-2}u'\right)' \in C^1(]0, r_{max}[)$ where $0 < r_{max} \leq +\infty$ and satisfying

$$(\mathbf{P}) \begin{cases} (|u'|^{p-2}u')' + \frac{N-1}{r}|u'|^{p-2}u' + u^q + f = 0,\ r > 0, \\ \lim_{r\to 0} u(r) = +\infty, \quad \lim_{r\to 0} r^{(N-1)/(p-1)}u'(r) = 0, \end{cases}$$

where $N \geq 1, p > 2,\ q > 1$ and f is a strictly positive, continuous radial function on $(0, +\infty)$. They presented also the behavior of singular solutions near the origin. In this work, we give the existence of entire solutions of problem (P), present their behavior near infinity, and prove the nonexistence results.

Equation (1) can be considered as a natural generalization of pure Laplacian case $p = 2$ studied in the papers, [2], [3]. It is presented as follows

$$u''(r) + \frac{N-1}{r}u'(r) + u^q(r) + f(r) = 0,\ \ r > 0, \tag{2}$$

where $p > 1, N \geq 3$ and f is a strictly positive, continuous radial function on $(0, +\infty)$. Equation (2) appeared in probability theory in the study of stochastic processes. It plays a central role in establishing some limit theorems for super-Brownian motion, [15]. Therefore, it has been extensively studied in much literature. [3], studied the existence and nonexistence of solutions of equation (2). He proved that if $q \leq \dfrac{N}{N-2}$ or if $q > \dfrac{N}{N-2}$ and $f \geq Cr^{\frac{2q}{q-1}}$ for some constant $C > 0$, equation (2) does not have any solution. But when $q > \dfrac{N}{N-2}$ and f is dominated by a function of the form $C/(1+r)^{\frac{N-2}{q}}$ for $C > 0$ sufficiently small, equation (2) has a solution. [2], studied the existence of global positive solutions of equation (2), and in the paper, [1], presented the asymptotic behavior near the origin and infinity of positive radial solutions. We also refer the readers to see, [14], [8], [17] for more details about the equation (2) and the references therein.

When $f \equiv 0$, equation (2) becomes the classic Emden-Fowler equation. In [9], [10], [11], Emden-Fowler gave the existence results and a classification of global radial solutions on $\mathbb{R}^N$ and $\mathbb{R}^N\backslash\{0\}$. In the case $N > 2$, two critical values $\dfrac{N}{N-2}$ and $\dfrac{N+2}{N-2}$ appear. [12], presented local and global results in the non-radial case when $q < \dfrac{N+2}{N-2}$. [7], have just extended them to the critical case $q = \dfrac{N+2}{N-2}$.

The motivation to study the equation (1) grew from earlier work of [16], in case $f \equiv 0$ and $p > 2$ for the equation

$$\left(|u'|^{p-2}u'\right)' + \frac{N-1}{r}|u'|^{p-2}u' + u^q(r) = 0,\ r > 0. \tag{3}$$

They have shown the existence of two critical values $\dfrac{N(p-1)}{N-p}$ and $\dfrac{N(p-1)+p}{N-p}$. [13], studied the existence of global solutions and asymptotic behavior near the origin of radial solutions when $q < \dfrac{N(p-1)}{N-p}$. The non-radial case was proved by [5].

In this paper, we shall further extend the analysis of the equation (3) by adding an inhomogeneous singular term f which has an impact on the existence and the asymptotic behavior of solutions of equation (1). We show under some assumptions that if the term f behaves like $r^{-pq/(q+1-p)}$ near infinity, then the solution u of problem (P) behaves like $r^{-p/(q+1-p)}$ near infinity. In particular, we have the following results in the case $N > p$ and $q > \dfrac{N(p-1)}{N-p}$,

(i) If

$$f(r) = \frac{q+1-p}{p-1} \times \left(\frac{p-1}{q}\left(N - \frac{pq}{q+1-p}\right)\left(\frac{p}{q+1-p}\right)^{p-1}\right)^{q/(q+1-p)} \times r^{-pq/(q+1-p)},$$

then problem (P) possesses a positive explicit solution given by

$$u(r) = \left(\frac{p-1}{q}\left(N - \frac{pq}{q+1-p}\right)\left(\frac{p}{q+1-p}\right)^{p-1}\right)^{1/(q+1-p)} \times r^{-p/(q+1-p)}. \tag{4}$$

(ii) If $f(r) \underset{+\infty}{\sim} lr^{-pq/(q+1-p)}$ for some l such that

$$0 < l \le \frac{q+1-p}{p-1} \times \left(\frac{p-1}{q}\left(N - \frac{pq}{q+1-p}\right)\left(\frac{p}{q+1-p}\right)^{p-1}\right)^{q/(q+1-p)},$$

then there exists a solution u of problem (P) such that $u(r) \underset{+\infty}{\sim} br^{-p/(q+1-p)}$ for some $b > 0$.

(iii) If $f(r) \underset{+\infty}{\sim} lr^{-pq/(q+1-p)}$ for some l such that

$$l > \frac{q+1-p}{p-1} \times \left(\frac{p-1}{q}\left(N - \frac{pq}{q+1-p}\right)\left(\frac{p}{q+1-p}\right)^{p-1}\right)^{q/(q+1-p)},$$

then problem (P) does not possess any entire solution.

The rest of the paper is divided in three sections. In section 2, we give the existence of entire solutions of problem (P). In section 3, we present the asymptotic behavior of solutions near infinity in the cases $q \neq \dfrac{N(p-1)+p}{N-p}$ and $q = \dfrac{N(p-1)+p}{N-p}$. The last section is devoted to the nonexistence results.

2 Existence of Entire Solutions

In this section, we study the existence of entire solutions of problem (P) while recalling that the existence of maximal solutions of problem (P) defined on $(0, r_{max})$, $0 < r_{max} \le +\infty$ has been established if $N > p$ and $q > \dfrac{N(p-1)}{N-p}$ in the paper, [6].

Theorem 2.1. *Assume that $N > p$ and $q > \dfrac{N(p-1)}{N-p}$. Then problem (P) possesses an entire solution u.*

Proof. Let u be a maximal solution of problem (P) defined on $(0, r_{max})$, where $0 < r_{max} \le +\infty$. We will proceed in four steps to prove that u is entire.
Step 1. $u' < 0$ on the set $\{r \in (0, r_{max}) : u(r) > 0\}$.
First, we prove that u is strictly decreasing near 0. We argue by contradiction. Suppose that there exists a small r such that $u'(r) = 0$, then by equation (1), we have $\left(|u|^{p-2}u'\right)'(r) = -u^q(r) - f(r) < 0$ (because $f > 0$ and $u > 0$ near 0). Therefore $u' \neq 0$ near 0. Moreover, since $\lim\limits_{r\to 0} u(r) = +\infty$, then u must be decreasing near 0. If there exists a first zero $r_0 \in (0, r_{max})$ of u' such that $u'(r_0) = 0$ and $u(r_0) > 0$, then $\left(|u|^{p-2}u'\right)'(r_0) \ge 0$, but by equation (1) we have $\left(|u'|^{p-2}u'\right)'(r_0) = -u^q(r_0) - f(r_0) < 0$. We deduce that $u' < 0$ on the set $\{r \in (0, r_{max}) : u(r) > 0\}$.
Step 2. $u > 0$ and $u' < 0$ on $(0, r_{max})$.
Let $r_1 \in (0, r_{max})$ the first zero of u. Since $u \ge 0$ on $(0, r_{max})$, then necessarily $u'(r_1) = 0$. Using equation (1), we have

$$\left(r^{N-1}|u'|^{p-2}u'\right)'(r) = -r^{N-1}u^q(r) - r^{N-1}f(r). \tag{5}$$

We integrate (5) on (r, r_1) for $r \in (0, r_1)$ and since $(u^q + f) > 0$ on $(0, r_1)$ and $u'(r_1) = 0$, we obtain $u'(r) > 0$ on $(0, r_1)$. This contradicts the first step. Consequently $u > 0$ on $(0, r_{max})$ and by the first step $u' < 0$ on $(0, r_{max})$.

Step 3. $u(r) = O\left(r^{-\frac{p}{q+1-p}}\right)$ for any $r \in (0, r_{max})$.
Since f is positive, then by equation (5), we have for any $r \in (0, r_{max})$

$$\left(r^{N-1}|u'|^{p-2}u'\right)'(r) \leq -r^{N-1}u^q(r). \tag{6}$$

Integrating this last inequality on $\left(\frac{r}{2}, r\right)$ for $r \in (0, r_{max})$ and using the fact that $u'(r) < 0$ on $(0, r_{max})$, we obtain for any $r \in (0, r_{max})$

$$|u'|^{p-2}u'(r) < -\left(\frac{2^N - 1}{N\,2^N}\right) r u^q(r).$$

Since $u > 0$ and $u' < 0$ on $(0, r_{max})$, then for any $r \in (0, r_{max})$ we have

$$u'(r)u^{-q/(p-1)}(r) < -\left(\frac{2^N - 1}{N\,2^N}\right)^{1/(p-1)} r^{1/(p-1)}.$$

Since $q > \dfrac{N(p-1)}{N-p} > p-1$, then for any $r \in (0, r_{max})$ we have

$$\left(u^{(p-1-q)/(p-1)}\right)'(r) > \frac{q+1-p}{p-1}\left(\frac{2^N - 1}{N\,2^N}\right)^{1/(p-1)} \times r^{1/(p-1)}.$$

Integrating this last inequality on $(0, r)$ for $r \in (0, r_{max})$ and using the fact that $\lim_{r\to 0} u(r) = +\infty$ and $q > p-1$, we obtain

$$0 < u(r) \leq M r^{-p/(q+1-p)} \quad \text{for any } \ r \in (0, r_{max}), \tag{7}$$

where

$$M = \left(\frac{N\,2^N}{2^N - 1}\left(\frac{p}{q+1-p}\right)^{p-1}\right)^{1/(q+1-p)}. \tag{8}$$

Step 4. $r_{max} = +\infty$.
Suppose by contradiction that $r_{max} < +\infty$. Then $\lim_{r\to r_{max}} u(r) = +\infty$. But by letting $r \to r_{max}$ in (7), we get a contradiction. Consequently the solution u is entire. □

3 Asymptotic Behavior Near Infinity

In this section, we investigate the asymptotic behavior near the infinity of solutions of problem (P). Under some additional assumptions on f, we prove that any solution of (P) must behave like $r^{-p/(q+1-p)}$ at infinity. The study requires some ideas from [4].

First, let us define for any $c \neq 0$ the following function

$$(r^c u(r))' = r^{c-1}E_c(r), \tag{9}$$

where

$$E_c(r) = cu(r) + ru'(r). \tag{10}$$

By equation (1), for any $r > 0$ such that $u'(r) \neq 0$ we have

$$\begin{aligned}(p-1)\left|u'\right|^{p-2}(r)E_c'(r) =& (p-1)\left(c - \frac{N-p}{p-1}\right) \times \\ & \left|u'\right|^{p-2}u'(r) - ru^q(r) \\ & - rf(r).\end{aligned} \tag{11}$$

Therefore, if $E_c(r_0) = 0$ for some $r_0 > 0$, equation (1) implies that

$$\begin{aligned}(p-1)\,r_0^{p-1}\left|u'\right|^{p-2}(r_0)E_c'(r_0) = & -(p-1)\left(c - \frac{N-p}{p-1}\right) \times \\ & |c|^{p-2}c\,u^{p-1}(r_0) - r_0^p u^q(r_0) \\ & - r_0^p f(r_0)\end{aligned} \tag{12}$$

We start with the following preliminary results.

Proposition 3.1. *Let u be a solution of problem (P). Then*

$$u(r) > 0 \quad \text{and} \quad u'(r) < 0, \quad \text{for any } r > 0, \tag{13}$$

Moreover, if $q > p-1$, then

$$0 < u(r) \leq M\,r^{-p/(q+1-p)}, \tag{14}$$

where M is given by (8).

Proof. We follow the same reasoning of the first three steps of the proof of Theorem 2.1 by taking $r_{max} = +\infty$. □

Proposition 3.2. *Assume that $N > p$. Let u be a solution of problem (P). Then $E_{(N-p)/(p-1)}(r) > 0$ for large r.*

Proof. Taking $c = \dfrac{N-p}{p-1}$ in (11), we see that $E'_{(N-p)/(p-1)}(r) < 0$ for any $r > 0$. Therefore, $E_{(N-p)/(p-1)}(r) \neq 0$ for large r. Suppose by contradiction that $E_{(N-p)/(p-1)}(r) < 0$ for large r. Then, $\lim_{r\to+\infty} E_{(N-p)/(p-1)}(r) \in [-\infty, 0[$. We distinguish two cases.

Case 1. $\lim_{r\to+\infty} E_{(N-p)/(p-1)}(r) = -\infty$.

Using Proposition 3.1, we have $u(r) > 0$ and $u'(r) < 0$ on $(0, +\infty)$, which implies that $\lim_{r\to+\infty} u(r)$ exists and is finite. Now, by relation (10) we have necessarily $\lim_{r\to+\infty} ru'(r) = -\infty$, which is impossible since u is positive.

Case 2. $-\infty < \lim_{r\to+\infty} E_{(N-p)/(p-1)}(r) < 0$.

Then $ru'(r)$ converges necessarily to 0 when $r \to +\infty$. This implies that $\lim_{r\to+\infty} u(r) < 0$ (because $N > p$). Which is impossible.

We conclude that $E_{(N-p)/(p-1)}(r) > 0$ for large r. □

Proposition 3.3. *Assume that $N > p$ and $q > p - 1$. Let u be a solution of problem (P). Then the function $r^{p/(q+1-p)+1}u'(r)$ is bounded for large r.*

Proof. Using Propositions 3.1 and 3.2, we have u is strictly decreasing and $E_{(N-p)/(p-1)} > 0$, for large r. This implies that

$$0 < r|u'(r)| < \frac{N-p}{p-1}u(r) \quad \text{for large } r. \tag{15}$$

Recall by Proposition 3.1, that $r^{p/(q+1-p)}u(r)$ is bounded for any $r > 0$. Therefore, by (15), we easily get that $r^{p/(q+1-p)+1}u'(r)$ is bounded for large r. □

Now, to prove the next Theorems, we introduce the following change of variable which will be very useful. So let us define the function

$$\upsilon(t) = r^{\frac{p}{q+1-p}}u(r) \quad \text{where } t = \ln r. \tag{16}$$

So υ satisfies the following equation

$$y'(t) + \left(N - \frac{pq}{q+1-p}\right) y(t) + \upsilon^q(t) + j(t) = 0, \tag{17}$$

where

$$j(t) = e^{\frac{pq}{q+1-p}t}f(e^t), \tag{18}$$

$$y(t) = |k|^{p-2}k(t), \tag{19}$$

$$k(t) = \upsilon'(t) - \frac{p}{q+1-p}\upsilon(t). \tag{20}$$

It's easy to see that

$$k(t) = r^{\frac{p}{q+1-p}+1}u'(r). \tag{21}$$

Proposition 3.4. *Assume that $N > p$ and $q > \dfrac{N(p-1)}{N-p}$. Let u be a solution of problem (P). Suppose that $r^{pq/(q+1-p)}f(r)$ and $r^{p/(q+1-p)}u(r)$ converge when $r \to +\infty$. Then $r^{p/(q+1-p)+1}u'(r)$ converges also when $r \to +\infty$ and*

$$\lim_{r\to+\infty} r^{p/(q+1-p)+1}u'(r) = \frac{-p}{q+1-p}\lim_{r\to+\infty} r^{p/(q+1-p)}u(r). \tag{22}$$

Proof. According to Proposition 3.1 and logarithmic change (16), we have υ is bounded for large t, which gives by Proposition 3.3 that k is bounded for large t. Therefore by expression (19), we have y is bounded for large t. We show that y is monotone for large t. Suppose by contradiction that there exist two sequences $\{\gamma_i\}$ and $\{\rho_i\}$ going to $+\infty$ as $i \to +\infty$ such that $\{\gamma_i\}$ and $\{\rho_i\}$ are respectively local minimum and local maximum of y, satisfying $\gamma_i < \rho_i < \gamma_{i+1}$ and

$$\begin{aligned} -\infty < \liminf_{t\to+\infty} y(t) &= \lim_{i\to+\infty} y(\gamma_i) \\ < \limsup_{t\to+\infty} y(t) &= \lim_{i\to+\infty} y(\rho_i) \le 0. \end{aligned} \tag{23}$$

Using equation (17) and the fact that $y'(\gamma_i) = y'(\rho_i) = 0$, we obtain

$$\begin{aligned} &(N - \frac{pq}{q+1-p})y(\gamma_i) + \upsilon^q(\gamma_i) + j(\gamma_i) = \\ &(N - \frac{pq}{q+1-p})y(\rho_i) + \upsilon^q(\rho_i) + j(\rho_i) = 0. \end{aligned}$$

Since $\upsilon(t)$ and $j(t)$ converge when t tends to $+\infty$ and $N > \dfrac{pq}{q+1-p}$, then $\lim_{i\to+\infty} y(\gamma_i) = \lim_{i\to+\infty} y(\rho_i)$, which contradicts the estimate (23). Therefore $k(t)$ converges when $t \to +\infty$. So, by (20), $\upsilon'(t)$ converges necessarily to 0 when $t \to +\infty$. Consequently, $r^{p/(q+1-p)+1}u'(r)$ converges when $r \to +\infty$ and (22) is satisfied. □

Proposition 3.5. *Assume that $N > p$ and $q > \dfrac{N(p-1)}{N-p}$. Let u be a solution of problem (P). Suppose that $\lim_{r\to+\infty} r^{pq/(q+1-p)}f(r) = l > 0$.*

If $\lim_{r\to+\infty} r^{p/(q+1-p)}u(r) = b$. Then b is one of the two roots λ_1 and λ_2 of the equation

$$s^q - \left(N - \frac{pq}{q+1-p}\right)\left(\frac{p}{q+1-p}\right)^{p-1} s^{p-1} + l = 0, \tag{24}$$

such that $0 < \lambda_1 \le \lambda_2$.

In particular, if

$$l = \frac{q+1-p}{p-1} \times \left(\frac{p-1}{q}\left(N - \frac{pq}{q+1-p}\right)\left(\frac{p}{q+1-p}\right)^{p-1}\right)^{q/(q+1-p)}, \tag{25}$$

then

$$b = \left(\frac{p-1}{q} \left(N - \frac{pq}{q+1-p} \right) \left(\frac{p}{q+1-p} \right)^{p-1} \right)^{1/(q+1-p)}. \tag{26}$$

Proof. Using logarithmic change (16), we have $v(t)$ converges to $b \geq 0$ when $t \to +\infty$. Hence, combining with Proposition 3.4, $k(t)$ converges also and $\lim_{t \to +\infty} k(t) = \frac{-p}{q+1-p} b$. Therefore, by (19), $\lim_{t \to +\infty} y(t) = -\left(\frac{p}{q+1-p} \right)^{p-1} b^{p-1}$. Since $\lim_{t \to +\infty} j(t) = l$, then by equation (17), $y'(t)$ converges necessarily to 0. Therefore, by letting $t \to +\infty$ in the same equation, we obtain

$$b^q - \Lambda^{q+1-p} b^{p-1} + l = l - \psi(b) = 0, \tag{27}$$

where

$$\Lambda = \left(\left(N - \frac{pq}{q+1-p} \right) \left(\frac{p}{q+1-p} \right)^{p-1} \right)^{1/(q+1-p)}. \tag{28}$$

and

$$\psi(s) = \Lambda^{q+1-p} s^{p-1} - s^q, \quad s \geq 0. \tag{29}$$

A simple study gives

$$\max_{s \geq 0} \psi(s) = \psi \left(\left(\frac{p-1}{q} \right)^{1/(q+1-p)} \Lambda \right) = L, \tag{30}$$

where

$$L = \frac{q+1-p}{p-1} \left(\frac{p-1}{q} \right)^{q/(q+1-p)} \Lambda^q = \frac{q+1-p}{p-1} \times \left(\frac{p-1}{q} \left(N - \frac{pq}{q+1-p} \right) \left(\frac{p}{q+1-p} \right)^{p-1} \right)^{q/(q+1-p)}. \tag{31}$$

Since $l > 0$, we can easily see that the equation $l - \psi(s) = 0$ has two roots λ_1 and λ_2 such that $0 < \lambda_1 \leq \lambda_2$. Then, by (27), $b = \lambda_1 > 0$ or $b = \lambda_2 > 0$. Moreover if $l = L$, this last equation has only one explicit root, that is, $\lambda_1 = \lambda_2 = \left(\frac{p-1}{q} \right)^{1/(q+1-p)} \Lambda$.

Hence by (27) and (30), we have explicitly

$$b = \left(\frac{p-1}{q} \right)^{1/(q+1-p)} \Lambda = \left(\frac{p-1}{q} \left(N - \frac{pq}{q+1-p} \right) \left(\frac{p}{q+1-p} \right)^{p-1} \right)^{1/(q+1-p)}. \tag{32}$$

□

Proposition 3.6. *Assume that $N > p$ and $q > \frac{N(p-1)}{N-p}$. Let u be a solution of problem (P). Suppose that $\lim_{r \to +\infty} r^{pq/(q+1-p)} f(r) = l > 0$. Then*

$$\liminf_{r \to +\infty} r^{p/(q+1-p)} u(r) > 0 \tag{33}$$

and

$$\limsup_{r \to +\infty} r^{p/(q+1-p)+1} u'(r) < 0. \tag{34}$$

Proof. The proof will be done in two steps.
Step 1. $\liminf_{r \to +\infty} r^{p/(q+1-p)} u(r) > 0$.
Assume by contradiction that $\liminf_{r \to +\infty} r^{p/(q+1-p)} u(r) = 0$. This means that $\liminf_{t \to +\infty} v(t) = 0$. Since $v(t)$ is bounded for large t, we distinguish two cases.

- $v(t)$ is monotone for large t.

Then $\lim_{t \to +\infty} v(t) = 0$. Since $u'(r) < 0$ for any $r > 0$ and $E_{(N-p)/(p-1)}(r) > 0$ for large r (by Proposition 3.2), then we have for large t,

$$0 < |k(t)| < \frac{N-p}{p-1} v(t). \tag{35}$$

It follows that $\lim_{t \to +\infty} k(t) = 0$ and by relation (19), $\lim_{t \to +\infty} y(t) = 0$. Therefore, by equation (17), $\lim_{t \to +\infty} y'(t) = -l < 0$. But this is a contradiction with $\lim_{t \to +\infty} y(t) = 0$.

- $v(t)$ is oscillating for large t.

Then there exists a sequence $\{\mu_i\}$ going to $+\infty$ as $i \to +\infty$ such that v has a local minimum in μ_i. Therefore using our hypotheses, we have $\lim_{i \to +\infty} v(\mu_i) = 0$, $v'(\mu_i) = 0$ and $v''(\mu_i) \geq 0$ (v'' exists because $u' < 0$). Which gives $\lim_{i \to +\infty} k(\mu_i) = 0$ and $k'(\mu_i) \geq 0$ and so $\lim_{i \to +\infty} y(\mu_i) = 0$ and $y'(\mu_i) \geq 0$. Therefore, by equation (17), we have $\lim_{i \to +\infty} y'(\mu_i) = -l < 0$. This is a contradiction.
It follows from both cases that

$\liminf_{r\to+\infty} r^{p/(q+1-p)}u(r) > 0$.

Step 2. $\limsup_{r\to+\infty} r^{p/(q+1-p)+1}u'(r) < 0$.

Since $u' < 0$, we assume by contradiction that $\limsup_{r\to+\infty} r^{p/(q+1-p)+1}u'(r) = 0$. This means that $\limsup_{t\to+\infty} k(t) = 0$. In the same way as the first step, since $k(t)$ is bounded for large t (by Proposition 3.3), we distinguish two cases.

- $k(t)$ is monotone for large t.

Then $\lim_{t\to+\infty} k(t) = 0$. That is, $\lim_{r\to+\infty} r^{p/(q+1-p)+1}u'(r) = 0$. This yields by L'Hôpital's rule that $\lim_{r\to+\infty} r^{p/(q+1-p)}u(r) = 0$. But this contradicts the fact that $\liminf_{r\to+\infty} r^{p/(q+1-p)}u(r) > 0$ by the first step.

- $k(t)$ oscillates for large t.

Then there exists a sequence $\{\rho_i\}$ going to $+\infty$ as $i \to +\infty$ such that k has a local maximum in ρ_i. Therefore, $\lim_{i\to+\infty} k(\gamma_i) = 0$ and $k'(\gamma_i) = 0$ and so $\lim_{i\to+\infty} y(\gamma_i) = 0$ and $y'(\gamma_i) = 0$. Therefore, by equation (17), we have $\lim_{i\to+\infty} \upsilon^q(\gamma_i) = -l < 0$. This is impossible since υ is positive.

We deduce that $\limsup_{r\to+\infty} r^{p/(q+1-p)+1}u'(r) < 0$. The proof is complete. $\square$

The following theorem gives a sufficient condition to obtain explicitly $\liminf_{r\to+\infty} r^{p/(q+1-p)}u(r)$.

Theorem 3.7. *Assume that $N > p$ and $q > \dfrac{N(p-1)}{N-p}$. Let u be a solution of problem (P). Suppose that*

$$\lim_{r\to+\infty} r^{pq/(q+1-p)}f(r) = \frac{q+1-p}{p-1}\times \left(\frac{p-1}{q}\left(N - \frac{pq}{q+1-p}\right)\left(\frac{p}{q+1-p}\right)^{p-1}\right)^{q/(q+1-p)}. \tag{36}$$

Then

$$\liminf_{r\to+\infty} r^{p/(q+1-p)}u(r) = \left(\frac{p-1}{q}\left(N - \frac{pq}{q+1-p}\right)\left(\frac{p}{q+1-p}\right)^{p-1}\right)^{1/(q+1-p)}.$$

Proof. Note that if υ converges, we obtain the result directly by using Proposition 3.5. Otherwise, since υ is bounded, it must oscillate. Then there exists a sequence $\{\mu_i\}$ going to $+\infty$ as $i \to +\infty$ such that υ has a local minimum in μ_i. Therefore $\upsilon'(\mu_i) = 0$ and $\upsilon''(\mu_i) \geq 0$. Hence, using (20), we have $k(\mu_i) = \dfrac{-p}{q+1-p}\upsilon(\mu_i)$ and $k'(\mu_i) = \upsilon''(\mu_i) \geq 0$. This yields

$$y(\mu_i) = -\left(\frac{p}{q+1-p}\right)^{p-1}\upsilon^{p-1}(\mu_i)$$

and

$$y'(\mu_i) = (p-1)\,|k(\mu_i)|^{p-2}\,k'(\mu_i) \geq 0.$$

Using equation (17) with $t = \gamma_i$, we obtain

$$\begin{aligned} 0 \leq y'(\mu_i) &= \Lambda^{q+1-p}\upsilon^{p-1}(\mu_i) - \upsilon^q(\mu_i) - j(\mu_i) \\ &= \psi(\upsilon(\mu_i)) - j(\mu_i) \\ &\leq L - j(\mu_i), \end{aligned}$$

where ψ and L are respectively given by (29) and (31). Since $\lim_{t\to+\infty} j(t) = L$, then $\lim_{i\to+\infty} \psi(\upsilon(\mu_i)) = \lim_{i\to+\infty} j(\mu_i) = L$. Hence, according to (30),

$$\lim_{i\to+\infty} \upsilon(\mu_i) = \liminf_{t\to+\infty} \upsilon(t) = \left(\frac{p-1}{q}\right)^{1/(q+1-p)}\Lambda,$$

where Λ is given by (28). This completes the proof. $\square$

Now, we study the convergence of $r^{p/(q+1-p)}u(r)$ at infinity to improve the previous result which gives only $\liminf_{r\to+\infty} r^{p/(q+1-p)}u(r)$. For this, we assume that f is differentiable and satisfies the following conditions:

(K_1) $\displaystyle\int_1^{+\infty} \left(r^{pq/(q+1-p)}f\right)_r^{+} dr < +\infty$,

(K_2) $\displaystyle\int_1^{+\infty} \left(r^{pq/(q+1-p)}f\right)_r^{-} dr < +\infty$.

The study depends on the comparison of q with $\dfrac{N(p-1)}{N-p}$ and $\dfrac{N(p-1)+p}{N-p}$. We start with the case $\dfrac{N(p-1)}{N-p} < q \neq \dfrac{N(p-1)+p}{N-p}$.

Theorem 3.8. *Assume that $N > p$ and $q > \dfrac{N(p-1)}{N-p}$. Let u be a solution of problem (P). Suppose that $\lim_{r\to+\infty} r^{pq/(q+1-p)}f(r) = l > 0$ and f satisfies*

(i) (K_1) if $q > \dfrac{N(p-1)+p}{N-p}$,

or

(ii) (K_2) if $\dfrac{N(p-1)}{N-p} < q < \dfrac{N(p-1)+p}{N-p}$.

Then

$$l \leq \frac{q+1-p}{p-1} \times \left(\frac{p-1}{q} \left(N - \frac{pq}{q+1-p} \right) \left(\frac{p}{q+1-p} \right)^{p-1} \right)^{q/(q+1-p)} \tag{37}$$

and $\lim_{r \to +\infty} r^{p/(q+1-p)} u(r) = b$ *where* b *is one of the two roots* λ_1 *or* λ_2 *of equation (24) such that* $0 < \lambda_1 \leq \lambda_2$.

To demonstrate Theorem 3.8, we need the classic result of [12], of which we recall the proof.

Lemma 3.9. *Let* g *be a positive differentiable function satisfying*

(i) $\int_{t_0}^{+\infty} g(t)\,dt < +\infty$ *for large* t_0.

(ii) $g'(t)$ *is bounded for large* t.

Then $\lim_{t \to +\infty} g(t) = 0$.

Proof. We argue by contradiction and we suppose that $\lim_{t \to +\infty} g(t) \neq 0$. Then there exist $\varepsilon > 0$ and a sequence $\{t_i\}$ going to $+\infty$ as $i \to +\infty$ satisfying $g(t_i) \geq 2\varepsilon$. Since $g'(t)$ is bounded for large t, then there exists a constant $C > 0$ such that $|g'(t)| \leq C$ for large t. By the mean value Theorem for g, we have $g(t) \geq \varepsilon$ for $|t - t_i| < \dfrac{\varepsilon}{C}$.
Choose a subsequence t'_i such that $t'_0 > t_0$ and $t'_i > t'_{i-1} + \dfrac{2\varepsilon}{C}\, t'_0$ for $i > 1$. Therefore

$$\sum_{i=1}^{n} \int_{t'_{i-1}}^{t'_i} g(t)\,dt > \sum_{i=1}^{n} \int_{t'_{i-1}}^{t'_{i-1}+\varepsilon/C} g(t)\,dt \geq \frac{\varepsilon^2}{C} n \to +\infty \text{ as } n \to +\infty.$$

This implies that

$$\int_{t_0}^{+\infty} g(t)\,dt = +\infty.$$

This contradiction completes the proof. □

Now, we can prove Theorem 3.8.

Proof. Define the following energy function associated with equation (17),

$$I(t) = \frac{p-1}{p} |k(t)|^p + \frac{p}{q+1-p} y(t)\upsilon(t) - \frac{A}{p} \left(\frac{p}{q+1-p} \right)^{p-1} \upsilon^p(t) + \frac{\upsilon^{q+1}(t)}{q+1} + l\upsilon(t), \tag{38}$$

where

$$A = \frac{q(N-p) - (N(p-1)+p)}{q+1-p}. \tag{39}$$

Note that the energy function I plays a central role in the study of the convergence of $r^{p/(q+1-p)} u(r)$. First, using Proposition 3.3 we have $k(t)$ is bounded for large t, which yields that $y(t)$ is bounded for large t. Hence $I(t)$ is bounded for large t.
The rest of the proof will be done in three steps.
Step 1. The function $I(t)$ converges when $t \to +\infty$. A simple computation gives

$$I'(t) = -AZ(t) - (j(t) - l)\upsilon'(t), \tag{40}$$

where

$$Z(t) = \left[|k(t)|^{p-1} - \left(\frac{p}{q+1-p} \right)^{p-1} \upsilon^{p-1}(t) \right] \times \left[|k(t)| - \frac{p}{q+1-p} \upsilon(t) \right]. \tag{41}$$

Integrating (40) on (T, t) for large T, we get

$$I(t) = C(T) - AR(t) - (j(t) - l)\upsilon(t) + \int_T^t j'(s)\upsilon(s)\,ds, \tag{42}$$

where

$$C(T) = I(T) + (j(T) - l)\upsilon(T) \tag{43}$$

and

$$R(t) = \int_T^t Z(s)\,ds. \tag{44}$$

Since the function $s \to s^{p-1}$ is monotone, then $Z(t) \geq 0$, which in turn implies that the function $R(t)$ is positive and increasing. By our hypotheses, we have $A \neq 0$, then relation (42) can be expressed as follows:

$$R(t) = -\frac{I(t)}{A} - \frac{1}{A}(j(t) - l)\upsilon(t) + \frac{1}{A} \int_T^t j'(s)\upsilon(s)\,ds + \frac{C(T)}{A}. \tag{45}$$

Using the fact that $v(t)$ and $I(t)$ are bounded for large t, $\lim_{t\to+\infty} j(t) = l$, $-(j'(s))^- \leq j'(s) \leq (j'(s))^+$ and $\int_T^{+\infty} (j'(s))^+ \, ds < +\infty$ from (K_1) if $A > 0$ or $\int_T^{+\infty} (j'(s))^- \, ds < +\infty$ from (K_2) if $A < 0$, we obtain $R(t)$ is bounded for large t. Consequently, $R(t)$ converges when $t \to +\infty$, which yields that $\int_T^{+\infty} Z(s)\, ds$ exists. By letting $t \to +\infty$ in (42), we deduce that $I(t)$ converges to a real number noted d when $t \to +\infty$.

Step 2. $\lim_{t\to+\infty} v'(t) = 0$.

By expressions (41) and (20) and the fact that $k(t) < 0$ for large t, we have just to prove that $\lim_{t\to+\infty} Z(t) = 0$. Since $\int_T^{+\infty} Z(s)\, ds < +\infty$, then by Lemma 3.9, it suffices to prove that $Z'(t)$ is bounded for large t. Using expression (41), $Z(t)$ can be written as follows:

$$Z(t) = |y(t)|^{p/(p-1)} + \frac{p}{q+1-p} v(t)y(t) + \left(\frac{p}{q+1-p}\right)^{p-1} v^{p-1}(t)v'(t). \qquad (46)$$

Therefore

$$Z'(t) = \frac{p}{p-1} k(t)y'(t) + \frac{p}{q+1-p} y(t)v'(t) + \frac{p}{q+1-p} v(t)y'(t) + (p-1)\left(\frac{p}{q+1-p}\right)^{p-1} \times v^{p-2}(t)v'^2(t) + \left(\frac{p}{q+1-p}\right)^{p-1} v^{p-1}(t)v''(t). \qquad (47)$$

Since $v(t)$, $k(t)$ and $j(t)$ are bounded for large t, then combining with (20) and (17), $v'(t)$ and $y'(t)$ are bounded also for large t. Hence, it remains to prove that $v''(t)$ is bounded for large t. According to (20), we have

$$v''(t) = k'(t) + \frac{p}{q+1-p} v'(t). \qquad (48)$$

Then, it suffices to prove that $k'(t)$ is bounded for large t.

Since $k(t) < 0$ for large t, we have by (19)

$$k'(t) = \frac{1}{p-1} |k(t)|^{2-p} y'(t). \qquad (49)$$

According to Proposition 3.6, we have $\limsup_{t\to+\infty} k(t) < 0$. Therefore, there exists a constant $M > 0$ such that $k(t) \leq -M$ for large t. Which implies that, $|k(t)|^{2-p}$ is bounded for large t. Therefore $k'(t)$ is bounded for large t and by relations (48) and (47), $Z'(t)$ is bounded for large t. Hence, by Lemma 3.9, we get $\lim_{t\to+\infty} Z(t) = 0$ and therefore $\lim_{t\to+\infty} v'(t) = 0$.

Step 3. $v(t)$ converges when $t \to +\infty$.

Suppose by contradiction that $v(t)$ is oscillating for large t. Then there exist two sequences $\{\mu_i\}$ and $\{\nu_i\}$ tend to $+\infty$ when $i \to +\infty$ such that $\{\mu_i\}$ and $\{\nu_i\}$ are respectively local minimum and local maximum of v, satisfying $\mu_i < \nu_i < \mu_{i+1}$ and

$$0 < \liminf_{t\to+\infty} v(t) = \lim_{i\to+\infty} v(\mu_i) = \alpha < \limsup_{t\to+\infty} v(t) = \lim_{i\to+\infty} v(\nu_i) = \beta < +\infty. \qquad (50)$$

Since $v'(\mu_i) = v'(\nu_i) = 0$, then by relations (38), (50), (19) and (20), we obtain

$$\lim_{i\to+\infty} I(\mu_i) = \varphi(\alpha) \quad \text{and} \quad \lim_{i\to+\infty} I(\nu_i) = \varphi(\beta), \qquad (51)$$

where

$$\varphi(s) = ls - \frac{\Lambda^{q+1-p}}{p} s^p + \frac{s^{q+1}}{q+1} = ls - \int_0^s \psi(r)\, dr, \qquad (52)$$

for any $s \geq 0$ and ψ is given by (29). Since $\lim_{t\to+\infty} I(t) = d$ by the first step, then

$$\varphi(\alpha) = \varphi(\beta) = d. \qquad (53)$$

Then, there exist $\delta \in (\alpha, \beta)$ and $x_i \in (\mu_i, \nu_i)$ such that $v(x_i) = \delta$, $\varphi'(\delta) = 0$ and $\varphi(\delta) \neq d$. On the other hand, using step 2, we have $\lim_{i\to+\infty} v'(x_i) = 0$, so by (20), we obtain $\lim_{i\to+\infty} k(x_i) = \frac{-p}{q+1-p}\delta$. Therefore $\lim_{i\to+\infty} I(x_i) = \varphi(\delta) = d$. Which gives a contradiction. Hence $v(t)$ converges when $t \to +\infty$. Let $\lim_{t\to+\infty} v(t) = b$. So, by Proposition 3.5, b is one of the two roots λ_1 and λ_2 of equation (24) such that $0 < \lambda_1 \leq \lambda_2$. Finally, by (30), $l = \psi(b) \leq L$ where L is given by (31). The proof is complete. □

Now we give the following result which deals with the critical case $q = \frac{N(p-1)+p}{N-p}$.

Theorem 3.10. *Assume that $N > p$ and $q = \frac{N(p-1)+p}{N-p}$. Let u be a solution of problem (P). Suppose that $\lim_{r\to+\infty} r^{pq/(q+1-p)} f(r) = l > 0$ and f satisfies (K_1) or (K_2). Then u satisfies one of the following cases:*

(i) $\lim_{r\to+\infty} r^{p/(q+1-p)}u(r) = \lambda_1$ *or* λ_2 *given in Theorem 3.8.*
(ii) $r^{p/(q+1-p)}u(r)$ *is oscillating and*

$$\begin{aligned}\lambda_1 \leq \alpha = \liminf_{r\to+\infty} r^{p/(q+1-p)}u(r) < \lambda_2\\ < \beta = \limsup_{r\to+\infty} r^{p/(q+1-p)}u(r) \leq \Gamma,\end{aligned} \tag{54}$$

where $\Gamma \neq \lambda_1$ *is the root of the equation*

$$\begin{aligned}&l\Gamma + \frac{\Gamma^{q+1}}{q+1} - \frac{\Lambda^{q+1-p}}{p}\Gamma^p\\ &= \frac{p-1}{p}\left(\frac{N-p}{p}\right)^p \lambda_1^p - \frac{N(p-1)+p}{Np}\lambda_1^{Np/(N-p)}.\end{aligned} \tag{55}$$

Moreover, α *and* β *satisfy the following estimates*

$$\begin{aligned}l =&\frac{1}{p}\left(\frac{N-p}{p}\right)^p \frac{\beta^p-\alpha^p}{\beta-\alpha}\\ &-\frac{N-p}{Np}\frac{\beta^{Np/(N-p)}-\alpha^{Np/(N-p)}}{\beta-\alpha}\end{aligned} \tag{56}$$

and

$$\begin{aligned}\frac{p^p}{N(N-p)^{p-1}} &< \frac{\beta^p-\alpha^p}{\beta^{Np/(N-p)}-\alpha^{Np/(N-p)}}\\ &\leq \frac{1}{p-1}\left(\frac{p}{N-p}\right)^p\frac{N(p-1)+p}{N}.\end{aligned} \tag{57}$$

In both cases, we have the estimate (37).

Proof. According to the notations in the proof of Theorem 3.8, we have $A = 0$, where A is given by (39). Using the fact that $v(t)$ is bounded, $\lim_{t\to+\infty} j(t) = l$, $j'(s) \leq (j'(s))^+$ for (K_1) and $j'(s) \geq -(j'(s))^-$ for (K_2), we deduce that the energy function I converges when $t \to +\infty$.
Since $v(t)$ is bounded, we have two possibilities:
- $v(t)$ converges when $t \to +\infty$, then similarly to Theorem 3.8, $\lim_{t\to+\infty} v(t) = \lambda_1$ or λ_2, where λ_1 and λ_2 are two roots of the equation (24) such that $0 < \lambda_1 \leq \lambda_2$. Moreover, the estimate (37) is satisfied.
- $v(t)$ oscillates, then there exist two sequences $\{\mu_i\}$ and $\{\nu_i\}$ tend to $+\infty$ when $i \to +\infty$ such that $\{\mu_i\}$ and $\{\nu_i\}$ are respectively local minimum and local maximum of v, satisfying $\mu_i < \nu_i < \mu_{i+1}$ and relation (50). Therefore $k'(\mu_i) = v''(\mu_i) \geq 0$ and $k'(\nu_i) = v''(\nu_i) \leq 0$ and so by equation (17), we have

$$0 \leq y'(\mu_i) = \psi\left(v(\mu_i)\right) - j(\mu_i) \tag{58}$$

and

$$0 \geq y'(\nu_i) = \psi\left(v(\nu_i)\right) - j(\nu_i), \tag{59}$$

where ψ is given by (29). On the other hand, according to the proof of Theorem 3.8, we have

$$\lim_{t\to+\infty} I(t) = \varphi(\alpha) = \varphi(\beta),$$

where φ is given by (52). Since $q = \dfrac{N(p-1)+p}{N-p}$ and $\alpha < \beta$, then

$$\begin{aligned}l =&\frac{\Lambda^{q+1-p}}{p}\frac{\beta^p-\alpha^p}{\beta-\alpha} - \frac{1}{q+1}\frac{\beta^{q+1}-\alpha^{q+1}}{\beta-\alpha}\\ =&\frac{1}{p}\left(\frac{N-p}{p}\right)^p\frac{\beta^p-\alpha^p}{\beta-\alpha}\\ &-\frac{N-p}{Np}\frac{\beta^{Np/(N-p)}-\alpha^{Np/(N-p)}}{\beta-\alpha}.\end{aligned}$$

This proves (56). Now, a simple study of the function φ gives $\varphi'(\lambda_1) = \varphi'(\lambda_2) = 0$, $\varphi'(s) > 0$ for $0 < s < \lambda_1$, $\varphi'(s) < 0$ for $\lambda_1 < s < \lambda_2$, $\varphi'(s) > 0$ for $s > \lambda_2$ and $\lim_{s\to+\infty}\varphi(s) = +\infty$. Therefore, there exists $\Gamma > \lambda_2$ such that $\varphi(\Gamma) = \varphi(\lambda_1)$. Since $\psi(\lambda_1) = l$ and $q = \dfrac{N(p-1)+p}{N-p}$, then

$$\varphi(\Gamma) = \frac{p-1}{p}\left(\frac{N-p}{p}\right)^p\lambda_1^p - \frac{N(p-1)+p}{Np}\lambda_1^{Np/(N-p)}, \tag{60}$$

which gives (55). To prove estimate (54), we let $i \to +\infty$ in (58) and (59), we obtain

$$\psi(\beta) \leq l \leq \psi(\alpha), \tag{61}$$

that is by (52)

$$\varphi'(\alpha) \leq 0 \leq \varphi'(\beta). \tag{62}$$

Combining this with the study of φ and the fact that $\varphi(\alpha) = \varphi(\beta)$, we deduce that $\lambda_1 \leq \alpha < \lambda_2 < \beta$. Moreover, we have $\beta \leq \Gamma$, otherwise $\varphi(\beta) > \varphi(\Gamma) = \varphi(\lambda_1) \geq \varphi(\alpha)$, which contradicts $\varphi(\alpha) = \varphi(\beta)$. Consequently, (54) is satisfied. Concerning (57), we use the fact that $l > 0$ and (56) to obtain the left inequality. To prove the right inequality of (57), we have $\beta > \alpha > 0$, then by (61), we have

$$\beta\psi(\beta) \leq l\beta = \varphi(\beta) + \frac{\Lambda^{q+1-p}}{p}\beta^p - \frac{\beta^{q+1}}{q+1}$$

and

$$\alpha\psi(\alpha) \geq l\alpha = \varphi(\alpha) + \frac{\Lambda^{q+1-p}}{p}\alpha^p - \frac{\alpha^{q+1}}{q+1},$$

which in turn implies that

$$\begin{aligned}&\beta\psi(\beta) - \frac{\Lambda^{q+1-p}}{p}\beta^p + \frac{\beta^{q+1}}{q+1} \leq \varphi(\beta) = \varphi(\alpha) \leq\\ &\alpha\psi(\alpha) - \frac{\Lambda^{q+1-p}}{p}\alpha^p + \frac{\alpha^{q+1}}{q+1}.\end{aligned}$$

With the expression of ψ, we get

$$\frac{p-1}{p}\Lambda^{q+1-p}\beta^p - \frac{q}{q+1}\beta^{q+1} \leq \frac{p-1}{p}\Lambda^{q+1-p}\alpha^p - \frac{q}{q+1}\alpha^{q+1}.$$

The right inequality of (57) can be easily obtained since $q = \dfrac{N(p-1)+p}{N-p}$. The proof is complete. $\square$

4 Nonexistence Results

This section concerns the nonexistence theorems of entire solutions of problem (P). The study depends on the parameters N, p, q and the comparison of $f(r)$ with $r^{-pq/(q+1-p)}$.

Theorem 4.1. *Assume that $N \leq p$ or $N > p$ and $p-1 \leq q \leq \dfrac{N(p-1)}{N-p}$ or $N > p$ and $q < p-1$. Then problem (P) does not possess any entire solution.*

Proof. Let u be a solution of problem (P). We distinguish five cases.

Case 1. $N \leq p$.

It's easy to see by Proposition 3.1 and equation (5) that the function $r \to r^{N-1}|u'|^{p-2}u'(r)$ is decreasing and strictly negative on $(0,+\infty)$. Then $\lim\limits_{r\to+\infty} r^{N-1}|u'|^{p-2}u'(r) \in [-\infty, 0[$. Therefore, there exists a constant $M_0 > 0$ such that

$$r^{N-1}|u'|^{p-2}u'(r) < -M_0, \tag{63}$$

for large r. So

$$|u'(r)| > M_0^{1/(p-1)}\, r^{(1-N)/(p-1)},$$

for large r. We get a contradiction by integrating this last inequality for large r and using the fact that $N \leq p$.

Case 2. $N > p$ and $q = p-1$.

Recall, by Proposition 3.2, that $E_{(N-p)/(p-1)}(r) > 0$ for large r. Then estimation (15) is satisfied. Combining this with equation (1) and the fact that $q = p-1$, we obtain

$$\left(|u'|^{p-2}u'\right)'(r) < u^{p-1}(r) \times \left[(N-1)\left(\frac{N-p}{p-1}\right)^{p-1} r^{-p} - 1\right],$$

for large r. In particular, we get $\left(|u'|^{p-2}u'\right)'(r) < 0$ for large r. Since $u'(r) < 0$, then $\lim\limits_{r\to+\infty} |u'|^{p-2}u'(r) \in [-\infty, 0[$. Therefore $\lim\limits_{r\to+\infty} u(r) = -\infty$, which is impossible.

Case 3. $N > p$ and $p-1 < q < \dfrac{N(p-1)}{N-p}$.

In this case we have necessarily $\dfrac{N-p}{p-1} < \dfrac{p}{q+1-p}$, which implies by (14) that $\lim\limits_{r\to+\infty} r^{\frac{N-p}{p-1}}u(r) = 0$. But this contradicts the fact that $r^{\frac{N-p}{p-1}}u(r)$ is positive and strictly increasing by relation (9) and Proposition 3.2.

Case 4. $N > p$ and $q = \dfrac{N(p-1)}{N-p}$.

Using relations (15) and (5), then for large r we obtain

$$\left(r^{N-1}|u'|^{p-2}u'\right)'(r) \leq -\left(\frac{N-p}{p-1}\right)^{-q} r^{N+q-1}|u'|^q. \tag{64}$$

Which can be written as

$$-\phi'(r) \leq -\left(\frac{N-p}{p-1}\right)^{-q} r^{-1}\phi^{q/(p-1)}(r) \quad \text{for large } r, \tag{65}$$

where

$$\phi(r) = r^{N-1}|u'|^{p-1}. \tag{66}$$

Integrating relation (65) on (R, r) for large R and using the fact that $q > p-1$, we obtain for large r,

$$\phi^{(p-1-q)/(p-1)}(r) \leq \phi^{(p-1-q)/(p-1)}(R) + \frac{p-1-q}{p-1}\left(\frac{N-p}{p-1}\right)^{-q}\ln\left(\frac{r}{R}\right). \tag{67}$$

By letting $r \to +\infty$ in the last inequality, we get a contradiction with the fact that ϕ is positive.

Case 5. $N > p$ and $q < p-1$.

Since f is positive, then by equation (5), we have

$$\left(r^{N-1}|u'|^{p-2}u'\right)'(r) \leq -r^{N-1}u^q(r) \quad \text{for any } r > 0. \tag{68}$$

Integrating this last inequality on $\left(\dfrac{r}{2}, r\right)$ for $r > 0$ and using the fact that $u'(r) < 0$ on $(0,+\infty)$, we obtain

$$|u'|^{p-2}u'(r) < -\left(\frac{2^N-1}{N\,2^N}\right) r u^q(r) \quad \text{for any } r > 0.$$

Since $u > 0$ and $u' < 0$ on $(0,+\infty)$, then for any $r > 0$

$$u'(r)u^{-q/(p-1)}(r) < -\left(\frac{2^N-1}{N\,2^N}\right)^{1/(p-1)} r^{1/(p-1)}.$$

Integrating this last inequality on (R, r) for large R and using the fact that $q < p-1$, we obtain

$$u^{(p-1-q)/(p-1)}(r) \leq u^{(p-1-q)/(p-1)}(R) - \frac{p-1-q}{p}\left(\frac{2^N-1}{N\,2^N}\right)^{1/(p-1)} \times \left(r^{p/(p-1)} - R^{p/(p-1)}\right). \quad (69)$$

By letting $r \to +\infty$ in the last inequality, we get a contradiction with the fact that u is positive.
we deduce that problem (P) does not possess any entire solution in all cases. $\square$

Theorem 4.2. *Assume that $N > p$ and $q > \dfrac{N(p-1)}{N-p}$. Suppose that*

$$\liminf_{r\to+\infty} r^{pq/(q+1-p)} f(r) > \frac{q+1-p}{p-1}\times \left(\frac{p-1}{q}\left(N - \frac{pq}{q+1-p}\right)\left(\frac{p}{q+1-p}\right)^{p-1}\right)^{q/(q+1-p)}.$$

Then problem (P) does not possess any entire solution.

Proof. Let u be a solution of problem (P). First we show that $r^{p/(q+1-p)}u(r)$ is strictly monotone for large r. This amounts to proving by (9) that $E_{p/(q+1-p)}(r) \neq 0$ for large r.
Suppose that there exists a large r such that $E_{p/(q+1-p)}(r) = 0$. Taking $c = \dfrac{p}{q+1-p}$ in (12) and multiplying by $r^{pq/(q+1-p)-1}$, we obtain

$$(p-1)\, r^{pq/(q+1-p)-1} \left|u'\right|^{p-2} E'_{p/(q+1-p)}(r) = \Lambda^{q+1-p}\, r^{p(p-1)/(q+1-p)} u^{p-1}(r) - r^{pq/(q+1-p)} u^q(r) - r^{pq/(q+1-p)} f(r), \quad (70)$$

where Λ is given in relation (28).
Using the change of variable (16), we see that the relation (70) is equivalent to

$$(p-1)\, r^{pq/(q+1-p)-1} \left|u'\right|^{p-2}(r) E'_{p/(q+1-p)}(r) = \psi(v(t)) - j(t), \quad (71)$$

where ψ is given respectively by (29). Since $\liminf_{r\to+\infty} r^{pq/(q+1-p)} f(r) > L$, then there exists $\varepsilon > 0$ such that

$$j(t) \geq L + \varepsilon \quad \text{for large } t. \quad (72)$$

Recall, by (30), that $\max_{s\geq 0} \psi(s) = L$, where L is given by (31).Then by (71), $E'_{p/(q+1-p)}(r) < 0$ and so $E_{p/(q+1-p)}(r) \neq 0$ for large r.
Now, we have $v(t)$ is strictly monotone for large t and bounded by Proposition 3.1, then $v(t)$ converges when $t \to +\infty$. Let $\lim_{t\to+\infty} v(t) = b \geq 0$.
We distinguish two cases according to the monotonicity of $v'(t)$ for large t.
Case 1. $v'(t)$ is monotone for large t.
Since $k(t)$ is bounded for large t by Proposition 3.3, then $v'(t)$ is bounded for large t. Therefore $v'(t)$ converges and necessarily $\lim_{t\to+\infty} v'(t) = 0$ (because v is bounded). Therefore $\lim_{t\to+\infty} k(t) = -\dfrac{p}{q+1-p} b$ and so $\lim_{t\to+\infty} y(t) = -\left(\dfrac{p}{q+1-p}\right)^{p-1} b^{p-1}$. According to equation (17) and estimate (72), we have

$$y'(t) \leq -\left(N - \frac{pq}{q+1-p}\right) y(t) - v^q(t) - L - \varepsilon,$$

for large t. But

$$\lim_{t\to+\infty}\left(-\left(N - \frac{pq}{q+1-p}\right) y(t) - v^q(t) - L - \varepsilon\right) = \psi(b) - L - \varepsilon \leq -\varepsilon,$$

Hence there exists a constant $M > 0$ such that $y'(t) \leq -M$ for large t. Integrating this last inequality on (T, t) for large T, we get $\lim_{t\to+\infty} y(t) = -\infty$. Which gives a contradiction.
Case 2. $v'(t)$ is not monotone for large t.
Since $v(t)$ is strictly monotone for large t, then we have two possibilities.

- $v'(t) > 0$ for large t. Then $\liminf_{t\to+\infty} v'(t) = 0$. Otherwise, there exits $C > 0$ such that $v'(t) \geq C$ for large t, then integrating this last inequality near $+\infty$, we obtain $\lim_{t\to+\infty} v(t) = +\infty$, which is impossible. Hence there exists ζ_i tends to $+\infty$ when $i \to +\infty$ such that ζ_i is a local minimum of v' satisfying $\lim_{i\to+\infty} v'(\zeta_i) = 0$. Consequently, we have $\lim_{i\to+\infty} y(\zeta_i) = -\left(\dfrac{p}{q+1-p}\right)^{p-1} b^{p-1}$. On the other hand, by deriving relation (20) and taking account that $v''(\zeta_i) = 0$, we get $\lim_{i\to+\infty} k'(\zeta_i) = 0$. Therefore $\lim_{i\to+\infty} y'(\zeta_i) = 0$. Taking $t = \zeta_i$ in equation (17) and tending $i \to +\infty$, we obtain $\lim_{i\to+\infty} j(\zeta_i) = \psi(b) \leq L = \max_{s\geq 0} \psi(s)$. But this contradicts the fact that $\liminf_{t\to+\infty} j(t) > L$.
- $v'(t) < 0$ for large t. In the same way, using the fact that v is bounded, we deduce that $\lim_{t\to+\infty} \sup v'(t) = 0$. Then there exists τ_i tends to $+\infty$ when $i \to$

$+\infty$ such that τ_i is a local minimum of υ' satisfying $\lim_{i\to+\infty} \upsilon'(\tau_i) = 0$ and $\upsilon''(\tau_i) = 0$. Therefore $\lim_{i\to+\infty} y'(\tau_i) = 0$ and $\lim_{i\to+\infty} j(\tau_i) = \psi(b) \leq L$. Which is impossible like the previous case.
We conclude that problem (P) does not possess any entire solution. □

5 Conclusion

In this paper, we have studied the existence, the nonexistence and the asymptotic behavior near infinity of global singular solutions of problem (P). The difficulty of this work lies in the influence of the inhomogeneous term f which is positive and is equivalent to the function $r^{-pq/(q+1-p)}$ near infinity. Under some conditions, we prove that the singular solution of problem (P) is equivalent to the function $r^{-p/(q+1-p)}$ near infinity. The cases where f is not positive or negligible in front to the function $r^{-pq/(q+1-p)}$ near infinity are not yet treated and will be the subject of a future study.

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