Duality for Multiobjective Programming Problems with Equilibrium Constraints on Hadamard Manifolds under Generalized Geodesic Convexity

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Abstract: - This article is devoted to the study of a class of multiobjective mathematical programming problems with equilibrium constraints on Hadamard manifolds (in short, (MPPEC)). We consider (MPPEC) as our primal problem and formulate two different kinds of dual models, namely, Wolfe and Mond-Weir type dual models related to (MPPEC). Further, we deduce the weak, strong as well as strict converse duality relations that relate (MPPEC) and the corresponding dual problems employing geodesic pseudoconvexity and geodesic quasiconvexity restrictions. Several suitable numerical examples are incorporated to demonstrate the significance of the deduced results. The results derived in this article generalize and extend several previously existing results in the literature.

Key-Words: - Multiobjective optimization, Guignard constraint qualification, Generalized geodesic convexity, Duality, Hadamard manifolds

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1 Introduction

In recent times, the study of optimization problems in the setting of manifolds has emerged as a very significant area of research. It is possible to model various practical problems that arise in numerous areas related to engineering in a much more effective manner on the setting of a manifold, rather than that of Euclidean space, see, [1], [8]. In fact, extending and generalizing the methods of optimization from Euclidean spaces to manifolds have various important advantages from theoretical as well as practical standpoints. For instance, numerous non-convex mathematical programming problems can be converted into convex mathematical programming problems by employing the Riemannian geometry perspective (see, for instance, [17], [18]). Furthermore, it can be observed that the relative interior of several important constraints in certain mathematical programming problems can be viewed as Hadamard manifolds, for instance, the positive orthant $\mathbb{R}^n_{++}$ (see, for instance, [15]) equipped with the Dikin metric $Y^{-2} = diag\left(\frac{1}{y_1^2}, \ldots, \frac{1}{y_n^2}\right)$, and the hypercube $(0,1)^n$ (see, for instance, [16]) equipped with the metric Y^{-2} $(I-Y)^{-2}$=diag $(y_1^{-2}(1-y_1)^{-2}, \ldots, y_n^{-2}(1-y_n)^{-2})$ are Hadamard manifolds. As a result, several constrained optimization problems can be suitably transformed into much simpler unconstrained problems by appropriate use of the Riemannian geometry. Due to this fact, a wider range of optimization problems can be investigated by formulating the problems in the framework of Riemannian and Hadamard manifolds. The notions of geodesic convex sets and geodesic convex function in manifold setting are developed to generalize the definitions of convex sets and convex functions (see, [21], [26]). Further, the notions of geodesic pseudoconvex and geodesic quasiconvex functions were introduced in [26], by generalizing geodesic convex functions in the setting of Riemannian manifolds. In the last few years, several authors have extended many interesting ideas of mathematical programming from Euclidean spaces to the setting of Riemannian as well as Hadamard manifolds, see, [3], [5], [13], [29], [30], [31], [32], and the references cited therein.

In the theory of mathematical programming, an optimization problem that is accompanied by some complementarity constraints, or certain variational inequality constraints is termed a mathematical programming problem with equilibrium constraints (in brief, (MPEC)). One of the first attempts in investigating such optimization problems is due to [7], where the existence of efficient solutions for (MPECs) is explored. Due to its immense scope of applicability in numerous fields of science, technology, and engineering (see, for instance, [19], [20]), (MPECs) have been studied by numerous authors in recent years. For further details and an updated survey of (MPEC) and its applications, we refer the readers to [12], [14], [22], [23], [24], [27], [28], and the references cited therein.

Several regularity and optimality conditions for (MPECs) were deduced in [4]. The existence of efficient solutions for (MPECs) was investigated in [7]. In [6], the Wolfe type duality model for (MPECs) was explored and several interesting duality results were derived. Optimality conditions and duality for semi-infinite (MPECs) were studied in [12]. Several duality results for multiobjective (MPECs) were derived in [23]. Further, optimality criteria and duality for multiobjective (MPECs) were studied in [24]. Recently, optimality criteria for multiobjective (MPEC) on Hadamard manifolds were derived in [25].

Motivated by the results derived in [6], [14], [23], [24], in this article we consider a certain class of multiobjective mathematical programming problem with equilibrium constraints on the framework of Hadamard manifolds (in short, ((MPPEC)) as our primal problem. We formulate two different kinds of dual problems related to (MPPEC), namely, Wolfe and Mond-Weir type dual problems. Further, we deduce weak, strong as well as strict converse duality relations that relate (MPPEC) and the corresponding dual problems under geodesic quasiconvexity and pseudoconvexity restrictions. To the best of our knowledge, this is for the first time that duality results for (MPPEC) have been investigated in the context of Hadamard manifolds.

The main contributions and novelty of the work in this article are twofold. Firstly, the results in this paper generalize the corresponding duality results deduced in [24], in the setting of a more general space, that is, Hadamard manifolds. The results obtained in this article extend the corresponding results of [23], from Euclidean space to the context of Hadamard manifolds. Secondly, the duality results obtained in this article also extend the corresponding duality results derived in [6], for a more general category of optimization problems, that is, (MPPEC) and generalize them on the framework of a wider space, which is Hadamard manifolds.

This article is organized in the following manner. Some basic mathematical preliminaries and concepts are recalled in Section 2. Moreover, we discuss the generalized Guignard constraint qualification and Karush-Kuhn-Tucker type necessary optimality criteria for (MPPEC). In Section 3, we formulate the Wolfe type dual problem related to (MPPEC) and deduce the weak, strong as well as strict converse duality relations that relate (MPPEC) and the dual problem employing geodesic pseudoconvexity restrictions. In Section 4, the Mond-Weir dual problem related to (MPPEC) is formulated, and weak, strong as well as strict converse duality relations that relate (MPPEC) and the dual problem are derived using geodesic pseudoconvexity and quasiconvexity assumptions. We conclude our discussions in Section 5 along with some future research directions.

2 Problem Formulation

In this section, we recollect some notation, preliminary definitions, and concepts that will be used in the rest of the paper.

The $n-$dimensional Euclidean plane is indicated by using the standard symbol $\mathbb{R}^n$. The set containing every natural number is signified by $\mathbb{N}$. The symbol $\emptyset$ is used to denote any empty set. The symbol $\langle\cdot,\cdot\rangle$ is employed to indicate the standard Euclidean inner product on $\mathbb{R}^n$. Let $c, d \in \mathbb{R}^n$ be arbitrary pair of vectors in $\mathbb{R}^n$. The following notation for inequalities will be employed in this article:

$$c \prec d \Leftrightarrow c_j < d_j, \forall j = 1,2,\dots,n.$$
$$c \preceq d \Leftrightarrow c_j \leq d_j, \forall j = 1,2,\dots,n, \text{ and } c_r < d_r, \text{ for at least one } r \in \{1,2,\dots,n\}.$$

We now recollect some basic definitions and concepts from Riemannian manifolds as well as Hadamard manifolds which will be required in the sequel.

We shall be using the notation $\mathcal{M}$ to signify a smooth manifold having dimension n, where n is any natural number. Let $\hat{y} \in \mathcal{M}$ be arbitrary. The set that contains every tangent vector at the element $\hat{y} \in \mathcal{M}$ is known as the tangent space at $\hat{y} \in \mathcal{M}$, and is denoted by $T_{\hat{y}}\mathcal{M}$. For any element $\hat{y} \in \mathcal{M}$, $T_{\hat{y}}\mathcal{M}$ is a real vector space, having a dimension n. In case we are restricted to real manifolds, $T_{\hat{y}}\mathcal{M}$ is isomorphic to the $n-$dimensional Euclidean space $\mathbb{R}^n$.

A Riemannian metric, denoted by $\mathcal{G}$ on the set $\mathcal{M}$ is a 2-tensor field that is symmetric as well as positive-definite. For every pair of elements $w_1, w_2 \in T_{\hat{y}}\mathcal{M}$, the inner product of w_1 and w_2 is given by:

$$\langle w_1, w_2 \rangle_{\hat{y}} = \mathcal{G}_{\hat{y}}(w_1, w_2),$$

where the symbol $\mathcal{G}_{\hat{y}}$ denotes the Riemannian metric at the element $\hat{y} \in \mathcal{M}$. The norm corresponding to the inner product $\langle w_1, w_2 \rangle_{\hat{y}}$ is denoted by $|\cdot|_{\hat{y}}$ (or simply $|\cdot|$, when there is no ambiguity regarding the subscript).

Let $a, b \in \mathbb{R}$, $a < b$ and $\nu: [a,b] \to \mathcal{M}$ be any piecewise differentiable curve that joins the elements $\hat{y}$ and $\hat{z}$ in $\mathcal{M}$, that is, we have:

$$\nu(a) = \hat{y}, \qquad \nu(b) = \hat{z}.$$

The length of the curve ν is denoted by $l\,(\nu)$ and is defined in the following manner:

$$l(\nu) := \int_a^b |\,\nu'(t)| dt.$$

For any differentiable curve ν, a vector field Y is referred to be parallel along the curve ν, provided that the following condition is satisfied:

$$\nabla_{\nu'} Y = 0.$$

If $\nabla_{\nu'}\nu' = 0$, then ν is termed as a geodesic. If $||\nu|| = 1$, then the curve ν is said to be normalised. For any $\hat{y} \in \mathcal{M}$, the exponential function $\exp_{\hat{y}}: T_{\hat{y}}\mathcal{M} \to \mathcal{M}$ is given by $\exp_{\hat{y}}(\hat{w}) = \nu(1)$, where ν is a geodesic that satisfies $\nu(0) = \hat{y}$ and $\nu'(0) = \hat{w}$. A Riemannian manifold $\mathcal{M}$ is referred to as geodesic complete, provided that the exponential function $\exp_{\hat{y}}(\hat{w})$ is defined for every arbitrary $\hat{w} \in T_{\hat{y}}\mathcal{M}$ and $\hat{y} \in \mathcal{M}$.

A Riemannian manifold is referred to as a Hadamard manifold (or, Cartan-Hadamard manifold) provided that $\mathcal{M}$ is simply connected, geodesic complete, as well as has a nonpositive sectional curvature throughout. Henceforth, in our discussions, the notation $\mathcal{M}$ will always signify a Hadamard manifold of dimension n, unless it is specified otherwise.

Let $\hat{y} \in \mathcal{M}$ be some arbitrary element lying in the Hadamard manifold $\mathcal{M}$. Then, the exponential function on the tangent space $\exp_{\hat{y}}: T_{\hat{y}}\mathcal{M} \to \mathcal{M}$is a globally diffeomorphic function. Moreover, the inverse of the exponential function $\exp_{\hat{y}}^{-1}: \mathcal{M} \to T_{\hat{y}}\mathcal{M}$ satisfies $\exp_{\hat{y}}^{-1}(\hat{y}) = 0$. Furthermore, for every pair of arbitrary elements $\widehat{y_1}, \widehat{y_2} \in \mathcal{M}$, there will always exist some unique normalized minimal geodesic $\nu_{\widehat{y_1},\widehat{y_2}}: [0,1] \to \mathcal{M}$, such that the geodesic ν satisfies the following:

$$\gamma_{\widehat{y_1},\widehat{y_2}}(\tau) = \exp_{\widehat{y_1}}\left(\tau \exp_{\widehat{y_1}}^{-1}(\widehat{y_2})\right), \quad \forall \tau \in [0,1].$$

Thus, every Hadamard manifold $\mathcal{M}$ of dimension n is diffeomorphic to the n-dimensional Euclidean space $\mathbb{R}^n$. The gradient of any smooth function $\Theta: \mathcal{M} \to \mathbb{R}$ is symbolized by grad Θ and is a vector field on $\mathcal{M}$ that is defined as:

$$d\Theta(X) = \langle \text{grad}\,\Theta, X \rangle = X(\Theta),$$

where X is also some vector field on the manifold $\mathcal{M}$.

The following definition is from [26].

Definition 2.1. Any non-empty subset $\mathcal{D}$ of a Hadamard manifold $\mathcal{M}$ is termed as a geodesic convex set in $\mathcal{M}$, if for every $y, \tilde{y} \in \mathcal{D}$ and for every geodesic $\gamma_{y,\tilde{y}}: [0,1] \to \mathcal{M}$ joining the points y and $\tilde{y}$, we have

$$\gamma_{y,\tilde{y}}(t) \in \mathcal{D}, \forall t \in [0,1],$$

where, $\gamma_{y,\tilde{y}}(t) = \exp_y\left(t \exp_y^{-1} \tilde{y}\right)$.

The following definitions are from [2].

Definition 2.2. Let $\mathcal{D} \subseteq \mathcal{M}$ be a geodesic convex set. Let $\Psi: \mathcal{D} \to \mathbb{R}$ be any real valued function on the set $\mathcal{D}$.

(i) The function Ψ is termed as a geodesic (respectively, strictly geodesic) pseudoconvex function at $\tilde{z} \in \mathcal{D}$, provided that for any arbitrary element $z \in \mathcal{D}$ (respectively, $z \in \mathcal{D}, z \neq \tilde{z}$), we have

$$\Psi(z) - \Psi(\tilde{z}) < (\leq) 0 \Rightarrow \langle grad\, \Psi(\tilde{z}), \exp_{\tilde{z}}^{-1}(z)\rangle_{\tilde{z}} < 0.$$

(ii) The function Ψ is termed as a geodesic quasiconvex function at $\tilde{z} \in \mathcal{D}$, provided that for any arbitrary element $z \in \mathcal{D}$, we have

$$\Psi(z) - \Psi(\tilde{z}) \leq 0 \Rightarrow \left\langle \mathrm{grad}\, \Psi(\tilde{z}), \exp_{\tilde{z}}^{-1}(z)\right\rangle_{\tilde{z}} \leq 0.$$

Remark 1. (i) If $\mathcal{M} = \mathbb{R}^n$, then $\mathrm{grad}\, \Psi(\tilde{z}) = \nabla\Psi(\tilde{z})$, where $\nabla\Psi(\tilde{z})$ denotes the gradient of the function Ψ at $\tilde{z}$ in $\mathbb{R}^n$, and $\exp_{\tilde{z}}^{-1}(z) = z - \tilde{z}$. In this case, the definitions of geodesic pseudoconvex and quasiconvex functions correspond to the usual standard definitions of differentiable pseudoconvex and quasiconvex functions (see, for instance, [10], pp. 146) for Euclidean spaces.

(ii) If the function $\Psi: \mathcal{D} \to \mathbb{R}$ is a geodesic convex function, then the function Ψ is automatically geodesic pseudoconvex as well as geodesic quasiconvex (see, Definition 10.1 in [26], and Definition 13.2.1 in [21]).

For further detailed exposition on geodesic quasiconvexity and pseudoconvexity in the setting of Hadamard manifolds, we refer to [21], [26]. Unless specified otherwise, we shall employ the notation $\mathcal{M}$ to denote an $n-$dimensional Hadamard manifold.

In this article, the following multiobjective mathematical programming problem with equilibrium constraints on the Hadamard manifold is considered:

$$\begin{aligned} (MPPEC)\ \ & Minimize\ \Phi(z) := \left(\Phi_1(z), \ldots, \Phi_l(z)\right), \\ \text{subject to}\ \ & \Psi(z) \leq 0, \\ & \vartheta(z) = 0, \\ & P(z) \geq 0, \\ & Q(z) \geq 0, \\ & P(z)^T Q(z) = 0, \end{aligned}$$

where every component of the objective function $\Phi_j: \mathcal{M} \to \mathbb{R}$ $\left(j \in I^{\Phi} := \{1,2, \ldots, l\}\right)$, and constraints $\Psi_j: \mathcal{M} \to \mathbb{R}$ $\left(j \in I^{\Psi} := \{1,2, \ldots, p\}\right)$, $\vartheta_j: \mathcal{M} \to \mathbb{R}$ $\left(j \in I^{\vartheta} := \{1,2, \ldots, q\}\right)$, $P_j, Q_j: \mathcal{M} \to \mathbb{R}$, $(j \in J := \{1,2, \ldots, m\})$ are real-valued and smooth functions defined on some $n-$dimensional Hadamard manifold $\mathcal{M}$, where n is a natural number. We use the symbol $\mathcal{F}$ to indicate the set of all feasible solutions to (MPPEC).

Let $\tilde{y} \in \mathcal{F}$ be any arbitrary feasible element. We now define a few index sets as follows that will render the subsequent analysis convenient:

$$\begin{aligned} J_\Psi &:= J_\Psi(\tilde{y}) = \left\{j: \Psi_j(\tilde{y}) = 0\right\}, \\ \mathcal{I}_1 &:= \mathcal{I}_1(\tilde{y}) = \left\{j: P_j(\tilde{y}) = 0, Q_j(\tilde{y}) > 0\right\}, \\ \mathcal{I}_2 &:= \mathcal{I}_2(\tilde{y}) = \left\{j: P_j(\tilde{y}) = 0, Q_j(\tilde{y}) = 0\right\}, \\ \mathcal{I}_3 &:= \mathcal{I}_3(\tilde{y}) = \left\{j: P_j(\tilde{y}) > 0, Q_j(\tilde{y}) = 0\right\}. \end{aligned}$$

The following definition will be employed in the sequel.

Definition 2.3. ([11]). Any arbitrary feasible element $\tilde{y} \in \mathcal{F}$ is termed as a Pareto efficient (resp., weak Pareto efficient) solution of (MPPEC), provided that there does not exist any other feasible element $z \in \mathcal{F}$, that satisfies the following:

$$\Phi(z) \preceq (\text{resp.}, \prec) \Phi(\tilde{y}).$$

For any arbitrary feasible element $\tilde{y} \in \mathcal{F}$, we now define the sets $\mathcal{B}^k$ (for every $k \in I^{\Phi}$) and $\mathcal{B}$ that will be used in the discussion that follows.

$$\begin{aligned} \mathcal{B}^k &:= \{y \in \mathcal{F}: \Phi_j(y) \leq \Phi_j(\tilde{y}),\ \forall j \in I^{\Phi},\ j \neq k\}, \\ \mathcal{B} &:= \{y \in \mathcal{F}: \Phi_j(y) \leq \Phi_j(\tilde{y}),\ \forall j \in I^{\Phi}\}. \end{aligned}$$

Remark 2. (i) From the above definitions of the sets $\mathcal{B}^k$ and $\mathcal{B}$, it is clear that

$$\cap_{k=1}^{l} \mathcal{B}^k = \mathcal{B}.$$

(ii) In case $I^{\Phi} = \{1\}$, then (MPPEC) reduces to a single-objective (MPEC). In such cases, we have

$$\mathcal{B}^1 = \mathcal{F}.$$

The following definition of the Bouligand tangent cone on the Hadamard manifold is from [9].

Definition 2.4. ([9]). Let $\mathcal{D} \subseteq \mathcal{M}$ and $z \in \mathrm{cl}(\mathcal{D})$. The contingent cone (in other terms, Bouligand tangent cone) of $\mathcal{D}$ at z is symbolized by the notation $\mathcal{T}(\mathcal{D}, z)$, and is the set defined as follows:

$$\mathcal{T}(\mathcal{D},z):=\{w\in T_z\mathcal{M}:\exists t_n\downarrow 0,\exists w_n\in T_z\mathcal{M},w_n\to w,\forall n\in N,\exp_z(t_nw_n)\in\mathcal{D}\}.$$

The following definitions and theorem are from [25].

Definition 2.5. Let $\tilde{y}\in\mathcal{F}$ be any arbitrary feasible element. The modified linearizing cone to the set $\mathcal{B}$ at the feasible element $\tilde{y}$ is the set defined as follows:

$$\begin{gathered}T^{Lin}(\mathcal{B},\tilde{y})\coloneqq\{u\in T_{\tilde{y}}\mathcal{M}:\\ \langle grad\ \Phi_j(\tilde{y}),u\rangle\leq 0,\forall j\in I^{\Phi},\\ \langle grad\ \Psi_j(\tilde{y}),u\rangle\leq 0,\forall j\in J_{\Psi},\\ \langle grad\ \vartheta_j(\tilde{y}),u\rangle=0,\forall j\in I^{\vartheta},\\ \langle grad\ \mathcal{M}_j(\tilde{y}),u\rangle=0,j\in\mathcal{I}_1,\\ \langle grad\ \mathcal{N}_j(\tilde{y}),u\rangle=0,\forall j\in\mathcal{I}_3,\\ \langle grad\ \mathcal{M}_j(\tilde{y}),u\rangle\geq 0,\forall j\in\mathcal{I}_2,\\ \langle grad\ \mathcal{N}_j(\tilde{y}),u\rangle\geq 0,\forall j\in\mathcal{I}_2,\\ \langle grad\ \mathcal{M}_j(\tilde{y}),u\rangle\langle grad\ \mathcal{N}_j(\tilde{y}),u\rangle=0,\forall j\in\mathcal{I}_2\}.\end{gathered}$$

Definition 2.6. Let us assume that $\tilde{y}\in\mathcal{F}$ is any arbitrary feasible element of (MPPEC). The generalized Guignard constraint qualification (in short, (GGCQ)) is said to hold at $\tilde{y}$ provided that the following inclusion relation is satisfied:

$$T^{Lin}(\mathcal{B},\tilde{y})\subseteq\bigcap_{i=1}^{l}cl\ co\mathcal{T}(\mathcal{B}^i,\tilde{y}).$$

Theorem 2.7. Let $\tilde{y}\in\mathcal{F}$ be a Pareto efficient solution of (MPPEC). Moreover, let us suppose that (GGCQ) holds at $\tilde{y}$. Then there exist real numbers $\tau_j>0\ (j\in I^{\Phi})$, $\sigma_j^{\Psi}(j\in I^{\Psi})$, $\sigma_j^{\vartheta}(j\in I^{\vartheta})$, $\sigma_j^{\mathcal{M}}$ *and* $\sigma_j^{\mathcal{N}}(j\in J)$, which satisfies the following:

$$\sum_{j\in I^{\Phi}}\tau_j grad\ \Phi_j(\tilde{y})+\sum_{j=1}^{p}\sigma_j^{\Psi}grad\ \Psi_j(\tilde{y})+\sum_{j\in I^{\vartheta}}\sigma_j^{\vartheta}grad\ \vartheta_j(\tilde{y})-\sum_{j\in J}\left[\sigma_j^{\mathcal{M}}grad\mathcal{M}_j(\tilde{y})+\sigma_j^{\mathcal{N}}grad\mathcal{N}_j(\tilde{y})\right]=0,\tag{1}$$

$$\begin{gathered}\Psi_j(\tilde{y})\leq 0,\sigma_j^{\Psi}\geq 0,\sigma_j^{\Psi}\Psi_j(\tilde{y})=0,\forall j\in I^{\Psi},\\ \vartheta_j(\tilde{y})=0,\forall j\in I^{\vartheta},\\ \sigma_j^{\mathcal{M}}\text{ free, }\forall j\in\mathcal{I}_1,\sigma_j^{\mathcal{M}}\geq 0,\forall j\in\mathcal{I}_2,\\ \sigma_j^{\mathcal{M}}=0,\forall j\in\mathcal{I}_3,\\ \sigma_j^{\mathcal{N}}\text{ free, }\forall j\in\mathcal{I}_3,\sigma_j^{\mathcal{N}}\geq 0,\forall j\in\mathcal{I}_2,\\ \sigma_j^{\mathcal{N}}=0,\forall j\in\mathcal{I}_1.\end{gathered}\tag{2}$$

3 Wolfe Duality

In this section, the Wolfe type dual problem related to (MPPEC) is formulated. Subsequently, we prove the weak, strong as well as strict converse duality relations that relate (MPPEC) and the dual problem employing certain geodesic pseudoconvexity assumptions.

Let us now consider that $z\in\mathcal{M}$, $\tau\in\mathbb{R}^l,\tau_j>0\ (\forall j\in I^{\Phi})$, $\sigma=(\sigma^{\Psi},\sigma^{\vartheta},\sigma^{P},\sigma^{Q})\in\mathbb{R}^{p+q+2m}$ and $e=(1,1,\ldots,1)\in\mathbb{R}^l$. The Wolfe type dual model (in brief, (WDP)), related to (MPPEC) may be formulated as:

(WDP) Maximize $\mathcal{L}(z,\tau,\sigma)\coloneqq\Phi(z)+[\sum_{j\in J_{\Psi}}\sigma_j^{\Psi}\Psi_j(z)+\sum_{j\in I^{\vartheta}}\sigma_j^{\vartheta}\vartheta_j(z)-\sum_{j\in J}\sigma_j^{P}P_j(z)-\sum_{j\in J}\sigma_j^{Q}Q_j(z)]e$,

subject to

$$\sum_{j\in I^{\Phi}}\tau_j\ grad\ \Phi_j(z)+\sum_{j\in J_{\Psi}}\sigma_j^{\Psi}grad\ \Psi_j(z)+\sum_{j\in I^{\vartheta}}\sigma_j^{\vartheta}grad\ \vartheta_j(z)-\sum_{j\in J}\left[\sigma_j^{P}grad\ P_j(z)+\sigma_j^{Q}grad\ Q_j(z)\right]=0,\tag{3}$$

$$\begin{gathered}\sigma_j^{\Psi}\geq 0,\forall j\in J_{\Phi};\ \ \sigma_j^{P}=0,\forall j\in\mathcal{I}_3;\\ \sigma_j^{Q}=0,\forall j\in\mathcal{I}_1;\\ \forall j\in\mathcal{I}_2,\text{ either, }\sigma_j^{P}>0,\sigma_j^{Q}>0,\text{ or, }\sigma_j^{P}\sigma_j^{Q}=0.\end{gathered}\tag{4}$$

The set of all feasible elements of (WDP) is denoted by $\mathcal{F}_{\mathcal{W}}$. We define an auxiliary function $\mathcal{H}:\mathcal{M}\to\mathbb{R}$ as follows:

$$\mathcal{H}(z)\coloneqq\sum_{j\in I^{\Phi}}\tau_j\ \Phi_j+\sum_{j\in J_{\Psi}}\sigma_j^{\Psi}\ \Psi_j+\sum_{j\in I^{\vartheta}}\sigma_j^{\vartheta}\ \vartheta_j-\sum_{j\in J}\left[\sigma_j^{P}P_j+\sigma_j^{Q}Q_j(z)\right],$$

for every $z\in\mathcal{M}$.

In the following theorem, we establish weak duality relation that relates (MPPEC) and (WDP).

Theorem 3.1. Let $y \in \mathcal{F}$ and $(z, \tau, \sigma) \in \mathcal{F}_{\mathcal{W}}$. Let $\mathcal{H}$ be geodesic pseudoconvex at z. Then the inequality $\Phi(y) \preceq \mathcal{L}(z, \tau, \sigma)$ does not hold.

Proof. On contrary, we suppose that $\Phi(y) \preceq \mathcal{L}(z, \tau, \sigma)$. Then there exists some $k \in I^{\Phi}$, such that

$$\Phi_i(y) \leq \Phi_i(z) + \sum_{j \in J_{\Psi}} \sigma_j^{\Psi} \Psi_j(z) + \sum_{j \in I^{\vartheta}} \sigma_j^{\vartheta} \vartheta_j(z) - \sum_{j \in J} \left[\sigma_j^P P_j(z) + \sigma_j^Q Q_j(z)\right],$$

for all $i \in I^{\Phi}, i \neq k$ and the above inequality holds strictly for $i = k$. Since $y \in \mathcal{F}$, $(z, \tau, \sigma) \in \mathcal{F}_{\mathcal{W}}$, $\tau_j > 0 \left(\forall j \in I^{\Phi}\right)$, it follows from the above inequality that

$$\sum_{i \in I^{\Phi}} \tau_i \Phi_i(y) + \sum_{j \in J_{\Psi}} \sigma_j^{\Psi}(y) + \sum_{j \in I^{\vartheta}} \sigma_j^{\vartheta} \vartheta_j(y) - \sum_{j \in J} \left[\sigma_j^P P_j(y) + \sigma_j^Q Q_j(y)\right] < \sum_{i \in I^{\Phi}} \tau_i \Phi_i(z) + \sum_{j \in J_{\Psi}} \sigma_j^{\Psi}(z) + \sum_{j \in I^{\vartheta}} \sigma_j^{\vartheta} \vartheta_j(z) - \sum_{j \in J} \left[\sigma_j^P P_j(z) + \sigma_j^Q Q_j(z)\right].$$

From the definition of $\mathcal{H}$, it follows that $\mathcal{H}(y) < H(z)$.
By invoking the geodesic pseudoconvexity restriction on $\mathcal{H}$ at z, we get

$$\langle grad\ \mathcal{H}(z), \exp_z^{-1}(y) \rangle_z < 0.$$

which is a contradiction to (3). Thus, the proof is complete.

In the following theorem, we establish strong duality relation that relates (MPPEC) and (WDP).

Theorem 3.2. Let $\tilde{y} \in \mathcal{F}$ be any Pareto efficient solution of (MPPEC). Let us further suppose that (GGCQ) holds at $\tilde{y}$. Then there exist some $\tau \in R^l, \tau \succ 0, \sigma = \left(\sigma^{\Psi}, \sigma^{\vartheta}, \sigma^P, \sigma^Q\right) \in \mathbb{R}^{p+q+2m}$, such that

$$(\tilde{y}, \tau, \sigma) \in \mathcal{F}_{\mathcal{W}},$$

and $\Phi(\tilde{y}) = \mathcal{L}(\tilde{y}, \tau, \sigma)$. Further, if every assumption of the weak duality theorem (Theorem 3.1) holds, then $(\tilde{y}, \tau, \sigma)$ is a Pareto efficient solution of (WDP).

Proof. Since (GGCQ) is satisfied with the Pareto efficient solution $\tilde{y} \in \mathcal{F}$, it follows from Theorem 2.7 that there exist some $\tau \in \mathbb{R}^l, \tau \succ 0, \sigma = \left(\sigma^{\Psi}, \sigma^{\vartheta}, \sigma^P, \sigma^Q\right) \in \mathbb{R}^{p+q+2m}$ such that equations (1) and (2) of Theorem 2.7 are satisfied. From the feasibility conditions of (MPPEC), it follows that

$$\sum_{j \in J_{\Psi}} \sigma_j^{\Psi} \Psi_j(\tilde{y}) + \sum_{j \in I^{\vartheta}} \sigma_j^{\vartheta} \vartheta_j(\tilde{y}) - \sum_{j \in J} \left[\sigma_j^P P_j(\tilde{y}) + \sigma_j^Q Q_j(\tilde{y})\right] = 0.$$

This shows that $(\tilde{y}, \tau, \sigma) \in \mathcal{F}_{\mathcal{W}}$, and $\Phi(\tilde{y}) = \mathcal{L}(\tilde{y}, \tau, \sigma)$. On contrary, suppose that $(\tilde{y}, \tau, \sigma)$ is not a Pareto efficient solution of (WDP). Then there exists $(\overline{y}, \overline{\tau}, \overline{\sigma}) \in \mathcal{F}_{\mathcal{W}}$, such that $\mathcal{L}(\tilde{y}, \tau, \sigma) \preceq \mathcal{L}(\overline{y}, \overline{\tau}, \overline{\sigma})$, or, $\Phi(\tilde{y}) \preceq \mathcal{L}(\overline{y}, \overline{\tau}, \overline{\sigma})$, which contradicts the weak duality theorem (Theorem 3.1). Thus, the proof is complete.

In the following theorem, we establish the strict converse duality relation that relates (MPPEC) and (WDP).

Theorem 3.3. Let $\tilde{y} \in \mathcal{F}$ and $(\overline{x}, \overline{\tau}, \overline{\sigma}) \in \mathcal{F}_{\mathcal{W}}$ be arbitrary feasible elements of (MPPEC) and (WDP), respectively. Let us assume that the following inequality holds:

$$\sum_{j \in I^{\Phi}} \tau_j \Phi_j(\tilde{y}) \leq \sum_{j \in I^{\Phi}} \tau_j \Phi_j(\overline{x}).$$

If the assumption of weak duality theorem (Theorem 3.1) is satisfied, then $\tilde{y} = \overline{x}$.

Proof. On contrary, let $\tilde{y} \neq \overline{x}$. Given that

$$\sum_{j \in I^{\Phi}} \tau_j \Phi_j(\tilde{y}) \leq \sum_{j \in I^{\Phi}} \tau_j \Phi_j(\overline{x}).$$

From the feasibility conditions and definitions of index sets, we infer that

$$\sum_{i \in I^{\Phi}} \tau_i \Phi_i(\tilde{y}) + \sum_{j \in J_{\Psi}} \sigma_j^{\Psi}(y) + \sum_{j \in I^{\vartheta}} \sigma_j^{\vartheta} \vartheta_j(\tilde{y}) - \sum_{j \in J} \left[\sigma_j^P P_j(\tilde{y}) + \sigma_j^Q Q_j(\tilde{y})\right] \leq \sum_{i \in I^{\Phi}} \tau_i \Phi_i(\overline{x}) + \sum_{j \in J_{\Psi}} \sigma_j^{\Psi}(\overline{x}) + \sum_{j \in I^{\vartheta}} \sigma_j^{\vartheta} \vartheta_j(\overline{x}) - \sum_{j \in J} \left[\sigma_j^P P_j(\overline{x}) + \sigma_j^Q Q_j(\overline{x})\right].$$

By invoking the geodesic pseudoconvexity restriction on $\mathcal{H}$ at $\bar{x}$, we get

$$\left\langle grad\ \mathcal{H}(\bar{x}), \exp_{\bar{x}}^{-1}(\tilde{y})\right\rangle_{\bar{x}} < 0.$$

which is a contradiction to (3). Thus, the proof is complete.

Remark 3. (i) If $\mathcal{M} = \mathbb{R}^n$, then Theorem 3.1 and Theorem 3.2 reduce to Theorem 4 and Theorem 5 derived in [24].
(ii) The weak, strong as well as strict converse duality relations (Theorem 3.1, Theorem 3.2 and Theorem 3.3) extends Theorem 3, Theorem 4 and Theorem 5 of [23], on wider space, that is, Hadamard manifold, and generalize it for (MPPEC).

In the following numerical example, we demonstrate the results of Mond-Weir duality on the Poincaré half plane, which is a Hadamard manifold with negative sectional curvature.

Example 3.4. Consider the Poincaré half plane, which is the set defined as $\mathcal{M} := \{z = (z_1, z_2) \in R^2 : z_2 > 0\}$. $\mathcal{M}$ is then a Riemannian manifold (see, for instance, [26]). The tangent space at every element $z \in \mathcal{M}$ is given by $T_z\mathcal{M} = \mathbb{R}^2$. The Riemannian metric on the set $\mathcal{M}$ is given by $\langle w_1, w_2\rangle_z := \langle \mathcal{G}(z)w_1, w_2\rangle, \forall w_1, w_2 \in T_z\mathcal{M} = \mathbb{R}^2$, where

$$\mathcal{G}(z) = \begin{pmatrix} \frac{1}{z_1^2} & 0 \\ 0 & \frac{1}{z_2^2} \end{pmatrix}.$$

Furthermore, it can also be verified that $\mathcal{M}$ is also a Hadamard manifold having a sectional curvature of -1.

Consider the following problem (P) on the set $\mathcal{M}$, which is a (MPPEC):

$$\text{(P) Minimize} \quad \Phi(y) := \left(y_1^2, \frac{y_1^2 + y_2^2}{y_2}\right),$$

$$\text{subject to} \quad \Psi(y) := \frac{1}{y_2} - 2 \le 0,$$
$$P(y) := y_1^2 \ge 0,$$
$$Q(y) := \frac{1}{y_2} \ge 0,$$
$$P(y)^T Q(y) = 0,$$

where the functions $\Phi_1, \Phi_2 : \mathcal{M} \to \mathbb{R}, \Psi, P, Q : \mathcal{M} \to \mathbb{R}$ are considered to be smooth functions on $\mathcal{M}$. The set of feasible elements F for (P) is

$$F := \{x = (x_1, x_2) \in \mathcal{M} : x_1 = 0, x_2 \ge 1/2\}.$$

The Wolfe type dual problem related to (P) may be formulated in the following manner:

$$\text{(WD) Maximize} \quad \mathcal{L}(z, \tau, \sigma) := \Phi(z) + \sigma^{\Psi}\Psi(z) - \left[\sigma_j^P P(z) + \sigma^Q Q(z)\right]e,$$

subject to

$$\sum_{j=1}^{2} \tau_j grad\ \Phi_j(z) + \sigma^{\Psi} grad\Psi(z) - [\sigma^P gradP(z) + \sigma^Q gradQ(z)] = 0,$$

where, $\sigma^{\Psi} \in \mathbb{R} \ge 0, \sigma^P \in \mathbb{R}, \sigma^Q = 0 \in \mathbb{R}, \tau_1, \tau_2 \in \mathbb{R}, \tau_1, \tau_2 > 0, \sum_{j=1}^{2} \tau_j = 1$, $e = (1,1)$.
The set containing all feasible elements of (WD) is denoted by $\mathcal{F}_{\mathcal{W}}$. Consider the element $\tilde{y} = (0,0.5) \in F$. By simple calculations, $\tilde{y} = (0,0.5)$ is a Pareto efficient solution of the problem (P). Let us now choose $\tau_1 = \tau_2 = 1/2, \sigma^{\Psi} = 1/8$, $\sigma^P = 1, \sigma^Q = 0$. We observe that the constraint
is satisfied. Hence, $(\tilde{y}, \tau, \sigma) \in \mathcal{F}_{\mathcal{W}}$. Furthermore, we have $\Phi(\tilde{y}) = \mathcal{L}(\tilde{y}, \tau, \sigma)$. It can be verified that

$$\mathcal{H}(z) := \left(\sum_{j=1}^{2} \tau_j\, \Phi_j + \sigma^{\Psi}\Psi_j - \sigma^P P_j - \sigma^Q Q_j\right)(z)$$

is geodesic pseudoconvex at (0,0.5). Thus, it is illustrated that every assumption and implication of the weak duality theorem is verified.

4 Mond-Weir Duality

In this section, the Mond-Weir type dual problem related to (MPPEC) is formulated. Subsequently, we deduce the weak, strong, as well as strict converse duality relations that relate (MPPEC) and the dual problem employing certain generalized geodesic quasiconvexity and pseudoconvexity assumptions.

Let $z \in \mathcal{M}$. The Mond-Weir type dual model (in brief, (MWD)) related to (MPPEC) may be formulated as:

$$\text{(MWD) Maximize} \quad \Phi(z),$$

subject to

$$\sum_{j\in I^{\Phi}} \tau_j grad\, \Phi_j(z) + \sum_{j\in J_{\Psi}} \sigma_j^{\Psi} grad\, \Psi_j(z) + \sum_{j\in I^{\vartheta}} \sigma_j^{\vartheta} grad\, \vartheta_j(z) - \sum_{j\in J}\left[\sigma_j^{P} gradP_j(z) + \sigma_j^{Q} gradQ_j(z)\right] = 0, \quad (4)$$

$$\sum_{j\in J_{\Psi}} \sigma_j^{\Psi}\Psi_j(z) \geq 0, \sum_{j\in I^{\vartheta}} \sigma_j^{\vartheta}\vartheta_j(z) \geq 0,$$
$$\sum_{j\in J} \sigma_j^{P} P_j(z) \leq 0, \sum_{j\in J} \sigma_j^{Q}\, Q_j(z) \leq 0,$$
$$\sigma_{J_{\Psi}}^{\Psi} \geq 0, \sigma_{\mathcal{I}_3}^{P} = 0, \sigma_{\mathcal{I}_1}^{Q} = 0,$$

and $\forall j \in \mathcal{I}_2$, either $\sigma_j^P > 0, \sigma_j^Q > 0$ or, $\sigma_j^P\sigma_j^Q = 0$,

where $\sigma = (\sigma^{\Psi}, \sigma^{\vartheta}, \sigma^{P}, \sigma^{Q}) \in \mathbb{R}^{p+q+2m}, \tau(\succ 0) \in R^l$.
The set containing every feasible element of (MWD) is signified by the symbol $\mathcal{F}_{\mathcal{M}}$.

The following index sets will be helpful in deriving duality results in the rest of the paper.

$$\mathcal{I}_2^{+} := \left\{j \in \mathcal{I}_2 : \sigma_j^P > 0, \sigma_j^Q > 0\right\},$$
$$\mathcal{I}_2^{0+} := \left\{j \in \mathcal{I}_2 : \sigma_j^P = 0, \sigma_j^Q > 0\right\},$$
$$\mathcal{I}_2^{0-} := \left\{j \in \mathcal{I}_2 : \sigma_j^P = 0, \sigma_j^Q < 0\right\},$$
$$\mathcal{I}_2^{+0} := \left\{j \in \mathcal{I}_2 : \sigma_j^Q = 0, \sigma_j^P > 0\right\},$$
$$\mathcal{I}_2^{-0} := \left\{j \in \mathcal{I}_2 : \sigma_j^Q = 0, \sigma_j^P < 0\right\},$$
$$\mathcal{I}_1^{+} := \left\{j \in \mathcal{I}_1 : \sigma_j^P > 0\right\},$$
$$\mathcal{I}_1^{-} := \left\{j \in \mathcal{I}_1 : \sigma_j^P < 0\right\},$$
$$\mathcal{I}_3^{+} := \left\{j \in \mathcal{I}_3 : \sigma_j^Q > 0\right\},$$
$$\mathcal{I}_3^{-} := \left\{j \in \mathcal{I}_3 : \sigma_j^Q < 0\right\},$$
$$I^{\vartheta+} := \left\{j \in I^{\vartheta} : \sigma_j^{\vartheta} > 0\right\},$$
$$I^{\vartheta-} := \left\{j \in I^{\vartheta} : \sigma_j^{\vartheta} < 0\right\}.$$

Now, we derive weak duality relations that relate (MPPEC) and (MWD).

Theorem 4.1. Let $\tilde{y} \in \mathcal{F}$ and $(z, \tau, \sigma) \in \mathcal{F}_{\mathcal{M}}$. Let us suppose that the functions $\Psi_j(j \in J_{\Psi}), \vartheta_j(j \in I^{\vartheta+}), -\vartheta_j(j \in I^{\vartheta-}), P_j(j \in \mathcal{I}_1^- \cup \mathcal{I}_2^{-0}), -P_j(j \in \mathcal{I}_1^+ \cup \mathcal{I}_2^{+0} \cup \mathcal{I}_2^+), Q_j(j \in \mathcal{I}_3^- \cup \mathcal{I}_2^{0-}), -Q_j(j \in \mathcal{I}_3^+ \cup \mathcal{I}_2^{0+} \cup \mathcal{I}_2^+)$ are geodesic quasiconvex at z. Further, let $\mathcal{I}_1^- \cup \mathcal{I}_3^- \cup \mathcal{I}_2^{0-} \cup \mathcal{I}_2^{-0} = \emptyset$ and $\sum_{j\in I^{\Phi}} \tau_j \Phi_j(\cdot)$ be strictly geodesic pseudoconvex at z. Then the inequality $\Phi(\tilde{y}) \preceq \Phi(z)$ does not hold true.

Proof. On contrary, let us suppose that $\Phi(\tilde{y}) \preceq \Phi(z)$. Then, as $\tau \succ 0$, it follows that

$$\sum_{j\in I^{\Phi}} \tau_j \Phi_j(\tilde{y}) \leq \sum_{j\in I^{\Phi}} \tau_j \Phi_j(z).$$

From the geodesic strict pseudoconvexity of $\sum_{j\in I^{\Phi}} \tau_j \Phi_j(\cdot)$, it follows that

$$\left\langle \sum_{j\in I^{\Phi}} \tau_j grad\, \Phi_j(z), \exp_z^{-1}(\tilde{y})\right\rangle_z < 0. \quad (5)$$

For every $j \in J_{\Psi}(z)$, we have

$$\Psi_j(\tilde{y}) \leq 0 = \Psi_j(z).$$

Then, in light of the geodesic quasiconvexity assumption on Ψ_j, we obtain the following:

$$\left\langle grad\Psi_j(z), \exp_z^{-1}(\tilde{y})\right\rangle_z \leq 0, \quad \forall j \in J_{\Psi}.$$

For every $j \in I^{\vartheta}$, we have

$$\vartheta_j(\tilde{y}) \leq 0 = \vartheta_j(z).$$

Then, in view of the geodesic quasiconvexity assumption on ϑ_j, and definition of index sets, we obtain

$$\left\langle grad\vartheta_j(z), \exp_z^{-1}(\tilde{y})\right\rangle_z \leq 0, \qquad \forall j \in I^{\vartheta+},$$
$$\left\langle grad\vartheta_j(z), \exp_z^{-1}(\tilde{y})\right\rangle_z \leq 0, \qquad \forall j \in I^{\vartheta-}.$$

Again, $-P_j(\tilde{y}) \leq 0 = -P_j(z), \forall j \in \mathcal{I}_1^+ \cup \mathcal{I}_2^{+0}$ and $-Q_j(\tilde{y}) \leq 0 = -Q_j(z), \forall j \in \mathcal{I}_3^+ \cup \mathcal{I}_2^{0+}$. From the geodesic quasiconvexity assumption on P_j and Q_j and definitions of index sets, we obtain

$$\left\langle gradP_j(z), \exp_z^{-1}(\tilde{y})\right\rangle_z \geq 0, \quad \forall j \in \mathcal{I}_1^+ \cup \mathcal{I}_2^{+0},$$
$$\left\langle gradQ_j(z), \exp_z^{-1}(\tilde{y})\right\rangle_z \geq 0, \quad \forall j \in \mathcal{I}_3^+ \cup \mathcal{I}_2^{0+}.$$

Since by hypothesis $\mathcal{I}_1^- \cup \mathcal{I}_3^- \cup \mathcal{I}_2^{0-} \cup \mathcal{I}_2^{-0} = \emptyset$, it follows from above inequalities that

$$\left\langle \sum_{j\in J_{\Psi}} \sigma_j^{\Psi} grad\, \Psi_j(z), \exp_z^{-1}(\tilde{y})\right\rangle_z \leq 0,$$
$$\left\langle \sum_{j\in I^{\vartheta}} \sigma_j^{\vartheta}\, grad\vartheta_j(z), \exp_z^{-1}(\tilde{y})\right\rangle_z \leq 0,$$

$$\left\langle \sum_{j\in \mathcal{I}_1\cup\mathcal{I}_2} \sigma_j^{P} grad\, P_j(z), \exp_z^{-1}(\tilde{y})\right\rangle_z \geq 0,$$

$$\left\langle \sum_{j\in\mathcal{J}_2\cup\mathcal{J}_3} \sigma_j^Q grad\, Q_j(z), \exp_z^{-1}(\tilde{y}) \right\rangle_z \geq 0.$$

By combining each of the inequalities obtained above, we get the following expression:

$$\Big\langle \sum_{j\in J_\Psi} \sigma_j^\Psi grad\, \Psi_j(z) + \sum_{j\in I^\vartheta} \sigma_j^\vartheta grad\, \vartheta_j(z) - \sum_{j\in J}\big[\sigma_j^P gradP_j(z) + \sigma_j^Q gradQ_j(z)\big], \exp_z^{-1}(\tilde{y}) \Big\rangle_z \leq 0. \quad (6)$$

It follows from (4) and (6) that $\left\langle \sum_{j\in I^\Phi} \tau_j grad\, \Phi_j(z), \exp_z^{-1}(\tilde{y}) \right\rangle_z \geq 0$, which is a contradiction to (5). Thus, the proof is complete.

Now, we establish strong duality relation that relates (MPPEC) and (MWD).

Theorem 4.2. Let $\tilde{y} \in \mathcal{F}$ be a Pareto efficient solution of (MPPEC). Let us further suppose that (GGCQ) is satisfied at $\tilde{y}$. Then there exist some $\tau \in \mathbb{R}^l, \tau \succ 0, \sigma = (\sigma^\Psi, \sigma^\vartheta, \sigma^P, \sigma^Q) \in \mathbb{R}^{p+q+2m}$, such that $(\tilde{y}, \tau, \sigma) \in \mathcal{F}_\mathcal{M}$, and the corresponding objective function values are equal. Moreover, if every assumption of weak duality (Theorem 4.1) is satisfied, then $(\tilde{y}, \tau, \sigma)$ is a Pareto efficient solution of (MWD).

Proof. Since (GGCQ) is satisfied at the Pareto efficient solution $\tilde{y} \in \mathcal{F}$, it follows from Theorem 2.7 that there exist some $\tau \in \mathbb{R}^l \succ 0, \sigma = (\sigma^\Psi, \sigma^\vartheta, \sigma^P, \sigma^Q) \in \mathbb{R}^{p+q+2m}$, such that equations (1) and (2) of Theorem 2.7 are satisfied. From the feasibility conditions of (MPPEC) it follows that $(\tilde{y}, \tau, \sigma) \in \mathcal{F}_\mathcal{M}$, and the corresponding objective function values are equal.

On contrary, let us suppose that $(\tilde{y}, \tau, \sigma)$ is not a Pareto efficient solution of (MWD). Then there exists $(\overline{u}, \overline{\tau}, \overline{\sigma}) \in \mathcal{F}_\mathcal{M}$, such that

$$\Phi(\tilde{y}) \preceq \Phi(\overline{u}, \overline{\tau}, \overline{\sigma}),$$

which is a contradiction to the weak duality theorem (Theorem 4.1). Thus, the proof is complete.

Now, we deduce strict converse duality relation that relates (MPPEC) and (MWD).

Theorem 4.3. Let $\tilde{y} \in \mathcal{F}$ and $(\overline{x}, \overline{\tau}, \overline{\sigma}) \in \mathcal{F}_\mathcal{M}$ be arbitrary feasible elements of (MPPEC) and (MWD), respectively. Let us assume that the following inequality holds:

$$\sum_{j\in I^\Phi} \tau_j\, \Phi_j(\tilde{y}) \leq \sum_{j\in I^\Phi} \tau_j\, \Phi_j(\overline{x}).$$

If the assumption of weak duality theorem (Theorem 4.1) is satisfied, then $\tilde{y} = \overline{x}$.

Proof. On contrary, let $\tilde{y} \neq \overline{x}$. Given that

$$\sum_{j\in I^\Phi} \tau_j\, \Phi_j(\tilde{y}) \leq \sum_{j\in I^\Phi} \tau_j\, \Phi_j(\overline{x}).$$

By invoking the geodesic strict pseudoconvexity of $\sum_{j\in I^\Phi} \tau_j \Phi_j(\cdot)$ it follows that

$$\left\langle \sum_{j\in I^\Phi} \tau_j grad\, \Phi_j(\overline{x}), \exp_{\overline{x}}^{-1}(\tilde{y}) \right\rangle_{\overline{x}} < 0.$$

For every $j \in J_\Psi(\overline{x})$, we have $\Psi_j(\tilde{y}) \leq 0 = \Psi_j(\overline{x})$. Then, in light of the geodesic quasiconvexity assumption on Ψ_j, we obtain the following

$$\left\langle grad\Psi_j(\overline{x}), \exp_{\overline{x}}^{-1}(\tilde{y}) \right\rangle_{\overline{x}} \leq 0, \quad \forall j \in J_\Psi.$$

For every $j \in I^\vartheta$, we have $\vartheta_j(\tilde{y}) \leq 0 = \vartheta_j(\overline{x})$. Then, in view of the geodesic quasiconvexity assumption on ϑ_j, and definition of index sets, we obtain

$$\left\langle grad\, \vartheta_j(\overline{x}), \exp_{\overline{x}}^{-1}(\tilde{y}) \right\rangle_{\overline{x}} \leq 0, \quad \forall j \in I^{\vartheta+},$$
$$\left\langle grad\, \vartheta_j(\overline{x}), \exp_{\overline{x}}^{-1}(\tilde{y}) \right\rangle_{\overline{x}} \leq 0, \quad \forall j \in I^{\vartheta-}.$$

Again, $-P_j(\tilde{y}) \leq 0 = -P_j(\overline{x}), \forall j \in \mathcal{J}_1^+ \cup \mathcal{J}_2^{+0}$and $-Q_j(\tilde{y}) \leq 0 = -Q_j(\overline{x}), \forall j \in \mathcal{J}_3^+ \cup \mathcal{J}_2^{0+}$. From the geodesic quasiconvexity assumption on P_j and Q_j and definitions of index sets, we obtain

$$\left\langle gradP_j(\overline{x}), \exp_{\overline{x}}^{-1}(\tilde{y}) \right\rangle_{\overline{x}} \geq 0, \quad \forall j \in \mathcal{J}_1^+ \cup \mathcal{J}_2^{+0},$$
$$\left\langle gradQ_j(\overline{x}), \exp_{\overline{x}}^{-1}(\tilde{y}) \right\rangle_{\overline{x}} \geq 0, \quad \forall j \in \mathcal{J}_3^+ \cup \mathcal{J}_2^{0+}.$$

Since by hypothesis, $\mathcal{J}_1^- \cup \mathcal{J}_3^- \cup \mathcal{J}_2^{0-} \cup \mathcal{J}_2^{-0} =$it follows from above inequalities that

$$\left\langle \sum_{j\in J_\Psi} \sigma_j^\Psi grad\, \Psi_j(\overline{x}), \exp_{\overline{x}}^{-1}(\tilde{y}) \right\rangle_{\overline{x}} \leq 0,$$

$$\left\langle \sum_{j\in I^{\vartheta}} \sigma_j^{\vartheta} grad\, \vartheta_j(\bar{x}), \exp_{\bar{x}}^{-1}(\tilde{y}) \right\rangle_{\bar{x}} \leq 0,$$

$$\left\langle \sum_{j\in \mathcal{J}_1\cup\mathcal{J}_2} \sigma_j^{P} grad\, P_j(\bar{x}), \exp_{\bar{x}}^{-1}(\tilde{y}) \right\rangle_{\bar{x}} \geq 0,$$

$$\left\langle \sum_{j\in \mathcal{J}_2\cup\mathcal{J}_3} \sigma_j^{Q}\, gradQ_j(\bar{x}), \exp_{\bar{x}}^{-1}(\tilde{y}) \right\rangle_{\bar{x}} \geq 0.$$

By combining each of the inequalities obtained above, we obtain the following expression:

$$\Big\langle \sum_{j\in J_{\Psi}} \sigma_j^{\Psi}\, grad\Psi_j(\bar{x}) + \sum_{j\in I^{\vartheta}} \sigma_j^{\vartheta} grad\, \vartheta_j(\bar{x}) - \sum_{j\in J}\big[\sigma_j^{P} gradP_j(\bar{x}) + \sigma_j^{Q} gradQ_j(\bar{x})\big],\ \exp_{\bar{x}}^{-1}(\tilde{y}) \Big\rangle_{\bar{x}} \leq 0, \qquad (7)$$

It follows from (4) and (7) that $\langle \sum_{j\in I^{\Phi}} \tau_j grad\, \Phi_j(\bar{x}), \exp_{\bar{x}}^{-1}(\tilde{y})\rangle_{\bar{x}} \geq 0$, which is a contradiction. Thus, the proof is complete.

Remark 4. (i) If $\mathcal{M} = \mathbb{R}^n$, then Theorem 4.1 and Theorem 4.2 reduce to Theorem 6 and Theorem 7 derived in [24], for Euclidean spaces.

(ii) The weak, strong as well as strict converse duality relations (Theorem 4.1, Theorem 4.2 and Theorem 4.3) extend Theorem 6, Theorem 7 and Theorem 8, respectively, derived in [23], on the framework of wider space, namely, Hadamard manifold, and generalize it for (MPPEC).

In the following numerical example, we illustrate the results derived for Mond-Weir duality.

Example 4.4. Consider the problem (P) as formulated in Example 3.4.

The Mond-Weir dual problem related to (P), denoted by (MWD), may be formulated as follows

(MWD) Maximize $\Phi(z)$,

subject to

$$\sum_{j=1}^{2} \tau_j grad\, \Phi_j(z) + \sigma^{\Psi} grad\Psi(z) - [\sigma^{P} gradP(z) + \sigma^{Q} gradQ(z)] = 0,$$

$$\sigma^{\Psi}\Psi(z) \geq 0; \sigma^{P} P(z) \leq 0;$$
$$\sigma^{Q} Q(z) \leq 0;$$
$$\sigma^{\Psi} \geq 0, \sigma^{Q} = 0;$$

where $\sigma = (\sigma^{\Psi}, \sigma^{\vartheta}, \sigma^{P}, \sigma^{Q}) \in \mathbb{R}^4, \tau_1, \tau_2 > and \sum_{j=1}^{2} \tau_j = 1.$ The feasible set of (MWD) is denoted by $\mathcal{F}_{\mathcal{M}}$. Consider the element $\tilde{y} = (0,1/2) \in F$. By simple calculations, it can be verified that $\tilde{y}$ is a Pareto efficient solution of the problem (P). Then by choosing multipliers as $\tau_1 = \tau_2 = 1/2$, $\sigma^{\Psi} = 1/8, \sigma^{P} = 1, \sigma^{Q} = 0$, we observe that $(\tilde{y}, \tau, \sigma) \in \mathcal{F}_{\mathcal{M}}$. Moreover, the functions $\Psi(z), P(z), Q(z)$ are geodesic quasiconvex and $\sum_{j=1}^{2} \tau_j \Phi_j(\cdot)$ is strictly geodesic pseudoconvex at $\tilde{y}$. Thus, it can be verified that every assumption and implication of the weak duality theorem is valid.

5 Conclusion

In this article, we have investigated a certain category of multiobjective mathematical programming problems with equilibrium constraints on Hadamard manifolds (abbreviated as, (MPPEC)). We have formulated the Wolfe type dual model (WDP) and Mond-Weir type dual model (MWD) related to (MPPEC) and derived the weak, strong, as well as strict converse duality relations that relate (MPPEC) and the dual models under generalized geodesic convexity restrictions. Several non-trivial numerical examples have been provided to demonstrate the importance of the derived results.

The various results derived in this article extend and generalize several notable results present in the literature. For instance, the results that are established in this article generalize the corresponding results deduced in [24], on the framework of an even wider space, which is, Hadamard manifolds, as well as for an even more class of convex functions. Further, the results deduced in this article extend the duality results deduced in [23], to Hadamard manifolds. Moreover, the results established in the paper also extend the results derived in [6], to a more general class of programming problems, namely, (MPPEC), and generalize them in the context of a wider space, namely, Hadamard manifolds.

For future work, we would like to extend the duality results derived in this article for nonsmooth optimization problems with equilibrium constraints in the setting of Hadamard manifolds. Moreover, it would be an exciting challenge to study duality results for mathematical programming problems with vanishing constraints on Hadamard manifolds.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

-Balendu Bhooshan Upadhyay is responsible for the conceptualization of the research problem as well as the supervision of the work.
-Arnav Ghosh is responsible for formal analysis and writing the first draft of the paper.
-I. M. Stancu-Minasian revised and edited the first draft of the paper.

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Conflict of Interest

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