On the Solutions of a Class of Nonlinear Fractional Differential Equations with Boundary Conditions

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Abstract: In this article, we consider a class of fractional boundary value problems with Caputo fractional derivative of order $\alpha \in (2,3)$. The existence and uniqueness of solutions are discussed and the Adomian decomposition method is proposed to obtain an approximation of the solution. Finally, an example is given to demonstrate the validity of results.

Key-Words: Banach contraction principle, Adomian decomposition method, Fractional differential equations, Caputo derivative

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1 Introduction

Fractional calculus is a field of mathematical analysis that can be viewed as a generalization of integer differential calculus, involving derivatives and integrals of real or complex order [12].Although its origins date back to 1695, with a famous correspondence between Leibniz and L'Hôpital, the scientific interest in this area of mathematics has become evident only in the last decades. Nowadays, it is one of the most intensively developed areas of mathematical analysis due to its numerous applications in various sciences and engineering, such as mechanics, biology economics, control theory, image and signal processing, etc. [13]. One of the major difficulties that arise in dealing with fractional differential equations is the great difficulty in solving such problems analytically. In most cases, we do not know the exact solution of the problem. Some numerical and approximate methods for solving fractional differential equations have been proposed, including the residual power series method, the homotopy perturbation method, fractional Adams-Moulton methods, variational interaction methods, and the Adomian decomposition method [3], [4], [9], [7], [10], [5], [15]. The Adomian decomposition method (ADM) was introduced by Adomian in the 1980s [2], [3], [4]. The method provides an effective procedure to obtain explicit and numerical solutions for a variety of differential systems to solve physical problems [16].

Continuing the results presented in [14], we consider the fractional boundary value problem with (left) Caputo fractional derivative (FBVP)

$$\begin{cases} {}^C\mathcal{D}^{\alpha}_{a+}x(t) - f(t, x(t), x'(t), x''(t)) = 0, \\ x(a) - \beta x'(a) = 0, \quad x'(a) = x'(b), \quad x''(a) = 0, \end{cases}$$

where $t \in [a,b]$, $\beta \in \mathbb{R}$, $0 \le a < b$, $2 < \alpha < 3$ and $f : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous. First, a result is presented that guarantees that the problem under study has a solution and, resorting to Banach's contraction principle, sufficient conditions are established that guarantee that the solution sought is unique. The method of Adomian decomposition is explained in the Section 2, and applied to approximate the solution in an illustrative example.

2 Preliminaries

In this section we introduce some notations, definitions and results used in this work.

Definition 1. *The Riemann-Liouville fractional integral of order $\alpha \in \mathbb{R}^+$ of a function x is defined by*

$$I^{\alpha}_{a}x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s)\mathrm{d}s,$$

provided the right-hand side is pointwise defined on (a,∞), where Γ is Euler Gamma function (given by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\mathrm{d}t,\ \alpha > 0$).

Definition 2. *The Caputo fractional derivative of order $\alpha > 0$ of a continuous function x is given by*

$${}^C\mathcal{D}^{\alpha}_{a}x(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{x^{(n)}(s)}{(t-s)^{\alpha-n+1}}\mathrm{d}s,$$

provided that the right-hand side is pointwise defined on (a,∞), where $n \in \mathbb{N}$ is such that $n-1 < \alpha < n$. If $\alpha \in \mathbb{N}$, then ${}^C\mathcal{D}^{\alpha}_{a}x(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\alpha} x(t)$.

The following lemma establishes an important relationship between the Riemann-Liouville integral and the Caputo derivative and will be essential for the study of the proposed fractional boundary value problem.

Lemma 1. [12] *Let* $n-1<\alpha<n$, $n\in\mathbb{N}$. *If* $x\in C^{n-1}([a,b])$ *or* $x\in AC^{n-1}([a,b])$, *then the following relation holds:*

$$(I_a^{\alpha}\,{}^C\mathcal{D}_a^{\alpha}x)(t)=x(t)-\sum_{k=0}^{n-1}\frac{x^{(k)}(a)}{k!}(t-a)^k. \quad (1)$$

Moreover, for $\alpha>0$ *and* $x\in C([a,b])$,

$$({}^C\mathcal{D}_{a+}^{\alpha}I_a^{\alpha}x)(t)=x(t).$$

Fixed point theory is a useful tool for studying the existence and uniqueness of solutions to boundary value problems. In this sense, we recall the Banach's contraction principle.

Theorem 1. (Banach's contraction principle) *Let* (X,d) *be a complete metric space and let* $T:X\to X$ *be a contraction on* X. *Then,* T *has a unique fixed point* $x\in X$.

In this paper, we consider $C^2([a,b])$ with the usual norm

$$\|x\|_{C^2}=\max_{t\in[a,b]}\{\|x\|_\infty+\|x'\|_\infty+\|x''\|_\infty\},$$

where $\|x\|_\infty=\max_{t\in[a,b]}|x(t)|$. It is known that $C^2([a,b])$, endowed with such norm, is a Banach space.

2.1 Adomian decomposition method

Consider a nonlinear differential equation, which can be decomposed in the following form

$$Lx+Rx+Nx=g, \quad (2)$$

where L is the highest order differential operator which is easily or trivially invertible, R is the remaining linear part of order less than L, N represents the nonlinear part and g is given known function.

Because L is invertible, applying L^{-1} to both members of equation (2), we get

$$x=\varphi-L^{-1}Rx-L^{-1}Nx+L^{-1}g, \quad (3)$$

where φ is the integration constant and satisfies $L\varphi=0$. The idea of this method is to represent the unknown function x by the infinite series

$$x(t)=\sum_{n=0}^{\infty}x_n(t).$$

In this regard, the nonlinear term Nx is represented by the infinite series of Adomian polynomials

$$Nx=\sum_{n=0}^{\infty}A_n(x_0,x_1,\cdots,x_n),$$

where A_n's are the Adomian polynomials [1], [2], [4], depending on $x_0,x_1,\cdots,x_n$, that can be formulated by

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{n=0}^{\infty}\lambda^n x_n\right)\right]_{\lambda=0},\ n=0,1,2\cdots.$$

Note that if the non linearity has the form $Nx=h(x)$, with h as a smooth function of x, then the polynomials A_n are generated for each nonlinearity in the way that A_0 depends only on x_0, A_1 depends on x_0 and x_1, A_2 depends on x_0,x_1,x_2, and so forth [2]. In particular, the first five Adomian polynomials are the following

$$\begin{aligned}
A_0 &= h(x_0),\\
A_1 &= x_1h'(x_0),\\
A_2 &= x_2h'(x_0)+\frac{1}{2!}x_1^2h''(x_0),\\
A_3 &= x_3h'(x_0)+x_1x_2^2h''(x_0)+\frac{1}{3!}x_1^3h^{(3)}(x_0),\\
A_4 &= x_4h'(x_0)+\left(\frac{1}{2!}x_2^2+x_1x_3\right)h''(x_0)\\
&\quad +\frac{1}{2!}x_1^2x_2h^{(3)}(x_0)+\frac{1}{4!}x_1^4h^{(4)}(x_0).
\end{aligned}$$

Therefore, the general solution becomes

$$\begin{aligned}
x &= -L^{-1}R\sum_{n=0}^{\infty}x_n-L^{-1}\sum_{n=0}^{\infty}A_n(x_0,x_1,\cdots,x_n)\\
&\quad +L^{-1}g+\varphi.
\end{aligned}$$

where

$$\begin{aligned}
x_0 &= \varphi+L^{-1}g,\\
x_1 &= -L^{-1}(Rx_0)-L^{-1}(A_0),\\
x_2 &= -L^{-1}(Rx_1)-L^{-1}(A_1),\\
&\vdots\\
x_{n+1} &= -L^{-1}Rx_n-L^{-1}A_n,\ n\geq 0.
\end{aligned}$$

Thus, using the known term x_0, all components $x_1,\cdots,x_n,\cdots$ can be determined. The n-th term approximation solution for the Adomian decomposition method is

$$\phi_n=\sum_{k=0}^{n-1}x_k(t),\ n\geq 1$$

and the solution $x(t)=\lim_{n\to+\infty}\phi_n(t)$. The convergence of this method has been proved in [5], [6], [8], [11].

3 Existence and uniqueness of solutions

Consider the equation

$${}^C\mathcal{D}_{a+}^{\alpha}x(t)-f(t,x(t),x'(t),x''(t))=0,\ t\in[a,b], \quad (4)$$

subject to the boundary conditions

$$x(a) - \beta x'(a) = 0, \;\; x'(a) = x'(b), \;\; x''(a) = 0.$$

In what follows, we use the notation

$$\vartheta = x'(a).$$

Applying I_a^α to both members of equation (4), using Lemma 1, it yields that

$$x(t) = x(a)+x'(a)(t-a)+x''(a)(t-a)^2+I_a^\alpha(f_x)(t).$$

From $x''(a) = 0$ and $x(a) = \beta x'(a) = \beta\vartheta$, it follows that

$$x(t) = \beta\vartheta + \vartheta(t-a) + I_a^\alpha(f_x)(t), \qquad (5)$$

where $f_x(t) = f(t, x(t), x'(t), x''(t))$.

3.1 Existence of solutions

The next theorem was proved in [14, Theorem 3] and establishes sufficient conditions for the existence of solutions to the fractional boundary value problem (1). This result was obtained by applying Mawhin's coincidence theory.

Theorem 2. *Let $f : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ be continuous, and suppose the following conditions are verified:*

(H1) There exist nonnegative constants p_1, p_2, p_3 and q satisfying $\eta \cdot p^* < 1$, with

$$\eta = \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(b-a)^{\alpha-2}}{\Gamma(\alpha-1)} \qquad (6)$$

and $p^* = \max_{t\in[a,b]}\{p_1, p_2, p_3\}$ such that for all $(u,v,w) \in \mathbb{R}^3$

$$\begin{aligned} |f(t,u,v,w)| &\leq p_1|u(t)| + p_2|v(t)| \\ &+ p_3|w(t)| + q, \;\; t \in [a,b]. \end{aligned}$$

(H2) There exists a constant $R > 0$ such that for $x \in \mathrm{dom}L$, if $|x'(t)| > R$ for all $t \in [a,b]$, then

$$\int_a^b (b-s)^{\alpha-2} f(s, x(s), x'(s), x''(s))\mathrm{d}s \neq 0.$$

(H3) There exists a positive constant R^* such that for $c_1 \in \mathbb{R}$, if $|c_1| > R^*$ for $t \in [a,b]$, either

$$c_1 f(t, c_1(t-a+\beta), c_1, 0) > 0, \; t \in [a,b],$$

or

$$c_1 f(t, c_1(t-a+\beta), c_1, 0) < 0, \; t \in [a,b].$$

Then the fractional boundary value problem FBVP has at least one solution in $C^2([a,b])$.

3.2 Uniqueness of solution

The following theorem establishes sufficient conditions for the uniqueness of solutions.

Theorem 3. *Suppose that (H1)–(H3) are verified and assume that the following condition is satisfied:*

(H4) There exists nonnegative constants d_1, d_2 and d_3 such that

$$\begin{aligned} |f(t,u,v,w) &- f(t,\overline{u},\overline{v},\overline{w})| \leq d_1|u-\overline{u}| \\ &+ d_2|v-\overline{v}| + d_3|w-\overline{w}|, \end{aligned}$$

for every $t \in [a,b]$, $(u,v,w) \in \mathbb{R}^3$, $(\overline{u},\overline{v},\overline{w}) \in \mathbb{R}^3$.

If

$$\eta \cdot d^* < 1 \qquad (7)$$

with η as defined in (6) and $d^* = \max\{d_1, d_2, d_3\}$, then the fractional boundary value problem FBVP has a unique solution in $C^2([a,b])$.

Proof. Let us prove that we have a unique solution in $C^2([a,b])$. For this purpose, and since the solution of the problem can be rewritten in terms of the integral equation (5), we consider the operator $T : C^2([a,b]) \to C^2([a,b])$ defined by

$$\begin{aligned} (Tx)(t) &= \beta\vartheta + \vartheta(t-a) + I_a^\alpha(f_x)(t) \\ &= \beta\vartheta + \vartheta(t-a) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f_x(s)ds. \end{aligned}$$

Let $B_R = \{x \in C^2(\mathbb{R}) : \|X\|_{C^2} \leq R\}$ and choose

$$R \geq \frac{(|\vartheta| + b - a + 1)|\vartheta| + \eta q}{1 - \eta p^*},$$

with $p^* = \max\{p_1, p_2, p_3\}$. Note that $1 - \eta p^* > 1$ according to (H1). We have that

$$\begin{aligned} &|Tx(t)| \\ &\leq |\vartheta|(|\beta| + t - a) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}|f_x(s)|ds \\ &\leq |\vartheta|(|\beta| + t - a) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(p_1|x(s)| \\ &\quad + p_2|x'(s)| + p_3|x''(s)| + q)ds \\ &\leq |\vartheta|(|\beta| + b - a) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\big(p^*(|x(s)| \\ &\quad + |x'(s)| + |x''(s)|) + q\big)ds \\ &\leq |\vartheta|(|\beta| + b - a) + \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}q + \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}p^*\|x\|_{C^2}. \end{aligned}$$

Moreover, we have

$$(Tx)'(t) = \vartheta + \frac{1}{\Gamma(\alpha-1)}\int_a^t (t-s)^{\alpha-2} f_x(s)ds.$$

Thus, we obtain

$$\begin{aligned}
&|(Tx)'(t)| \\
\leq\ & |\vartheta| + \frac{1}{\Gamma(\alpha-1)}\int_a^t (t-s)^{\alpha-2}|f_x(s)|ds \\
\leq\ & |\vartheta| + \frac{1}{\Gamma(\alpha-1)}\int_a^t (t-s)^{\alpha-2}(p_1|x(s)| \\
& +p_2|x'(s)| + p_3|x''(s)| + q)ds \\
\leq\ & |\vartheta| + \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}q + \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}p^*\|x\|_{C^2}.
\end{aligned}$$

Moreover, we get that

$$(Tx)''(t) = \frac{1}{\Gamma(\alpha-2)}\int_a^t (t-s)^{\alpha-3}f_x(s)ds$$

and

$$\begin{aligned}
&|(Tx)''(t)| \\
\leq\ & \frac{1}{\Gamma(\alpha-2)}\int_a^t (t-s)^{\alpha-3}|f_x(s)|ds \\
\leq\ & \frac{1}{\Gamma(\alpha-2)}\int_a^t (t-s)^{\alpha-3}(p_1|x(s)| \\
& +p_2|x'(s)| + p_3|x''(s)| + q)ds \\
\leq\ & \frac{(b-a)^{\alpha-2}}{\Gamma(\alpha-1)}q + \frac{(b-a)^{\alpha-2}}{\Gamma(\alpha-1)}p^*\|x\|_{C^2}.
\end{aligned}$$

Finally, we can conclude

$$\begin{aligned}
\|Tx\|_{C^2} &= \max \|Tx\|_\infty + \|Tx'\|_\infty + \|Tx''\|_\infty \\
&\leq (|\beta| + b - a + 1)|\vartheta| + \eta q + \eta p^*\|x\|_{C^2} \\
&\leq (|\beta| + b - a + 1)|\vartheta| + \eta q + \eta p^* R \\
&\leq R,
\end{aligned}$$

which shows that $T(B_R) \subset B_R$.

Let us now take $x, y \in C^2([a,b])$. Note that Ty is defined by

$$(Ty)(t) = \beta\vartheta + \vartheta(t-a) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f_y(s)ds.$$

Applying (H3), it follows that, for any $t \in [a,b]$,

$$\begin{aligned}
&\|Tx - Ty\| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}|f_x(s) - f_y(s)|ds \\
& + \frac{1}{\Gamma(\alpha-1)}\int_a^t (t-s)^{\alpha-2}|f_x(s) - f_y(s)|ds \\
& + \frac{1}{\Gamma(\alpha-2)}\int_a^t (t-s)^{\alpha-3}|f_x(s) - f_y(s)|ds \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\left(d_1|x(s) - y(s)|\right. \\
& \left. +d_2|x'(s) - y'(s)| + d_3|x''(s) - y''(s)|\right)|ds \\
& + \frac{1}{\Gamma(\alpha-1)}\int_a^t (t-s)^{\alpha-2}|\left(d_1|x(s) - y(s)|\right| \\
& \left. +d_2|x'(s) - y'(s) + d_3|x''(s) - y''(s)|\right)|ds \\
& + \frac{1}{\Gamma(\alpha-2)}\int_a^t (t-s)^{\alpha-3}|\left(d_1|x(s) - y(s)|\right. \\
& \left. +d_2|x'(s) - y'(s)| + d_3|x''(s) - y''(s)|\right)|ds \\
\leq\ & \frac{d^*\|x-y\|_{C^2}}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}ds \\
& + \frac{d^*\|x-y\|_{C^2}}{\Gamma(\alpha-1)}\int_a^t (t-s)^{\alpha-2}ds \\
& + \frac{d^*\|x-y\|_{C^2}}{\Gamma(\alpha-2)}\int_a^t (t-s)^{\alpha-3}|ds \\
=\ & \eta d^*\|x-y\|_{C^2}.
\end{aligned}$$

Since $\eta \cdot d^* < 1$, by Banach's contraction principle, T has a unique fixed point which is the unique solution of the problem (1), and the proof is complete. □

4 Illustrative example

Consider the following fractional boundary value

$$\begin{cases} {}^C\mathcal{D}^{\frac{5}{2}}_{1+}x(t) = \frac{1}{10}t + \frac{1}{15}\sin(x(t)) + \frac{2}{5}x'(t) \\ x(1) = x'(1) = x'(2),\ x''(1) = 0 \end{cases}, \quad (8)$$

$t \in [1,2]$, where $\alpha = \dfrac{5}{2}$, $\beta = 1$, $a = 1$, $b = 0$ and

$$f(t, x(t), x'(t), x''(t)) = \frac{1}{10}t + \frac{1}{15}\sin(x(t)) + \frac{2}{5}x'(t)$$

is a continuous function. It follows that

$$|f(t, x(t), x'(t), x''(t))| \leq \frac{1}{5} + \frac{1}{15}|x(t)| + \frac{2}{5}|x'(t)|$$

with $p_1 = \dfrac{1}{15}$, $p_2 = \dfrac{2}{5}$, $p_3 = 0$ and $q = \dfrac{1}{5}$. It follows that $p^* = \max\{p_1, p_2, p_2\} = \dfrac{2}{5}$. Moreover, $\eta = \dfrac{58}{15\sqrt{\pi}}$ and consequently, $p\eta = \frac{116}{75\sqrt{\pi}} < 1$. Thus, condition (H1) is verified.

Let $R = 1$, and for any $x \in \mathrm{dom}L$, assume $|x'(t)| > R$ holds for $t \in [1,2]$. From the continuity of x', either $x'(t) > R$ or $x'(t) < -R$ for $t \in [1,2]$.

If $x'(t) > 1$, one has

$$\begin{aligned}
& \int_1^2 (2-s)^{\frac{3}{2}}\left[\frac{1}{10}t + \frac{1}{15}\sin(x(t)) + \frac{2}{5}x'(t)\right]\mathrm{d}s \\
>\ & \left(\frac{1}{10} - \frac{1}{15} + \frac{2}{5}\right)\int_1^2 (2-s)^{\frac{3}{2}}\mathrm{d}s = \frac{13}{75} > 0.
\end{aligned}$$

If $x'(t) < -1$, one has

$$\begin{aligned} & \int_1^2 (2-s)^{\frac{3}{2}} \left[\frac{1}{10}t + \frac{1}{15}\sin(x(t)) + \frac{2}{5}x'(t)\right] \mathrm{d}s \\ < & \left(\frac{1}{5} + \frac{1}{15} - \frac{2}{5}\right) \int_1^2 (2-s)^{\frac{3}{2}} \mathrm{d}s = -\frac{4}{75} < 0. \end{aligned}$$

Thus, for $|x'(t)| > 1$,

$$\int_1^2 (2-s)^{\frac{3}{2}} f(s, u(s), u'(s), u''(s))\mathrm{d}s \neq 0$$

and condition (H2) is verified.

Finally, we observe that

$$\begin{aligned} & f\left(t, c_1\left(t-\frac{1}{2}\right), c_1, 0\right) \\ = & \frac{1}{10}t + \frac{2}{5}c_1 + \frac{1}{15}\sin\left(c_1\left(t-\frac{1}{2}\right)^2\right). \end{aligned}$$

Take $R^* = \dfrac{1}{2}$ and assume $|c_1| > \dfrac{1}{2}$. Thus, if $c_1 > \dfrac{1}{2}$,

$$\begin{aligned} c_1 f\left(t, c_1\left(t-\frac{1}{2}\right), c_1, 0\right) &> \frac{1}{2}\left(\frac{1}{10} - \frac{1}{15} + \frac{1}{5}\right) \\ &= \frac{7}{60} > 0 \end{aligned}$$

and if $c_1 < -1$, one has

$$\begin{aligned} c_1 f\left(t, c_1\left(t-\frac{1}{2}\right), c_1, 0\right) &< -\frac{1}{2}\left(\frac{1}{5} + \frac{1}{15} - \frac{1}{5}\right) \\ &= -\frac{1}{30} < 0. \end{aligned}$$

Therefore, condition (H3) is verified.

It follows from Theorem 2 that fractional boundary value problem (8) has at least one solution.

According to the Theorem 3, we have that $d_1 = \dfrac{1}{15}$, $d_2 = \dfrac{2}{5}$ and $d_3 = 0$. Thus, $d^* = \frac{2}{5}$ and $\eta \cdot d^* < 1$ and the solution of the problem (8) is unique.

4.1 Numerical part

The integral equation (8) we can be identified with $Lx + Rx + Nx = g$ where

$$(Rx)(t) = -\frac{2}{5}x''(t)$$

$$(Nx)(t) = -\frac{1}{15}\sin(x(t))$$

$$g(t) = \frac{1}{10}t.$$

As before, let $x'(1) = \vartheta$. Applying $L^{-1} = I_a^\alpha$ to both members of equation in the problem (8) and used ADM method presented in the Section 2, we get

$$\begin{aligned} x_0(t) &= \vartheta t + \frac{1}{10} I_{1+}^{\frac{5}{2}} t \\ &= \vartheta t + \frac{4}{3\sqrt{\pi}}\left(\frac{2(t-1)^{5/2}}{7} + \frac{4(t-1)^{5/2}t}{35}\right), \\ x_1(t) &= -I_{1+}^{\frac{5}{2}} R x_0 - I_{1+}^{\frac{5}{2}} A_0, \\ x_{n+1}(t) &= -I_{1+}^{\frac{5}{2}} R x_n - I_{1+}^{\frac{5}{2}} A_n, \ n = 1, 2, \cdots. \end{aligned}$$

The first few Adomian polynomials A_n that represent the nonlinear term $-\dfrac{1}{15}\sin(x(t))$ are defined as

$$\begin{aligned} A_0 &= -\frac{1}{15}\sin(x_0), \\ A_1 &= -\frac{1}{15}x_1\cos(x_0), \\ A_2 &= -\frac{1}{15}x_2\cos(x_0) + \frac{1}{30}x_1^2\sin(x_0), \\ A_3 &= -\frac{1}{15}x_3\cos(x_0) + \frac{1}{15}x_1 x_2^2\sin(x_0), \\ &\quad +\frac{1}{90}x_1^3\cos(x_0). \end{aligned}$$

Expressing the n-term approximation solution of fractional boundary value problem (1) as $\phi_n(t) = \sum_{k=0}^n x_k(t)$, the exact solution can be obtained by

$$x(t) = \lim_{n\to\infty} \phi_n(t).$$

5 Conclusions

In this paper, we continue the study of a class of boundary value problems, using fractional derivative of Caputo of order $\alpha \in (2,3)$. Following the results obtained in [14], it is presented a result of existence of solutions [14] and conditions are obtained to guarantee its uniqueness by applying the Banach contraction theorem. From a numerical point of view, it is considered the Adomian decomposition method, which provides a numerical approximation to the solution. Finally, an example of application of the previously presented theory is given.

We emphasize that it is presented a possible approach to the study of solutions of a class of nonlinear fractional differential equations under certain boundary conditions. The same problem can be studied from a different point of view, obtaining other sufficient conditions for the existence and uniqueness of solutions as well as other numerical techniques can be used. On the other hand, the approaches proposed here can also be applied to similar problems.

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Conflict of Interest

The author declares no conflict of interest.

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