Kumarswamy Truncated Lomax Distribution with Applications

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Abstract: - This paper introduces a new flexible generalized family of distributions, named Kumarswamy Truncated Lomax Distribution. We study its statistical properties including quantile function, skewness, kurtosis, moments, generating functions, incomplete moments and order statistics. Maximum likelihood estimation of the model parameters is investigated. An application is carried out on real data set to illustrate the performance and flexibility of the proposed model.

Key-Words: - Truncated Lomax distribution; Kumaraswamy generated family of distributions; Lorenz curve; Moments; Estimation; Simulation.

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1 Introduction

Lifetime models are extremely valuable for explaining and forecasting real-world occurrences in medicine, research, economics, and a variety of other fields. These models are often useful in product lifetime evaluations such as reliability and survival analyses. Many researchers have been undertaken in order to develop appropriate lifespan models that best represent real-world occurrences. Throughout the last decade, research has mostly concentrated on establishing new families of distributions by adding more parameters to the standard and common families of continuous distributions. These additional parameters have produced far more flexible models that match real-world datasets better than standard models, [1].

Some of these approaches for generated family of distributions are as follows: exponentiated version of the M family in [2], Kumaraswamy Kumaraswamy $-G$ in [3], generalized Kumaraswamy $-G$ in [4], alpha power transformation family in [5], Weibull odd Burr III $-G$ in [6], odd Frechet $-G$ in [7], exponentiated Kumaraswamy $-G$ in [8], type-I half logistic Burr X $-G$ in [9], transmuted Burr X$-G$ in [10], exponentiated generalized $-G$ in [11], gamma Kumaraswamy $-G$ in [12], odd-generalized N-H $-G$ in [13], extended-gamma $-G$ in [14], Kumaraswamy type I half logistic $-G$ in [15], $T-X$ family in [16], gamma $-G$ in [17], Kumaraswamy Poisson $-G$ in [18], exponentiated power-generalized Weibull power series $-G$ in [19], beta generalized Marshall-Olkin Kumaraswamy $-G$ in [20], odd Burr X $-G$ in [21], the Weibull $-G$ in [22], type-II half logistic$-G$ in [23], sec$-G$ in [24], truncated Cauchy power Weibull $-G$ in [25], exponentiated truncated inverse Weibull $-G$ in [26], odd Perks $-G$ in [27], sine Topp-Leone $-G$ in [28], Kumaraswamy generalized Marshall-Olkin $-G$ in [29], a new power Topp-Leone$-G$ in [30], truncated inverted Kumaraswamy $-G$ in [31], transmuted odd Frechet $-G$ in [32], Kumaraswamy Marshal-Olkin $-G$ in [33], Kavya-Manoharan transformation family, [34], among others. Recently, [35], studied the Kumaraswamy (K) generated family of distributions and it has the following cumulative distribution function (CDF) and probability density function (PDF) as below:

$$F(y;\ \mu,\eta,\omega) = 1-[1-G(y;\omega)^{\mu}]^{\eta}, \quad y \in R,\ \mu,\eta>0, \tag{1}$$

and

$$f(y;\ \mu,\eta,\omega) = \mu\eta g(y;\omega)G(y;\omega)^{\mu-1}[1-G(y;\omega)^{\mu}]^{\eta-1}, \quad y \in R,\ \mu,\eta>0. \tag{2}$$

where ω and $H(.)$ are the vector of parameters and the CDF for the baseline distribution.

Many studies on the K-family of distributions have made significant contributions by providing new distributions. For instance, the K Weibull distribution was presented by [36]. In [37], the authors proposed K generalized gamma distribution. In [38], [39], the authors presented the K Weibull exponential and K quasi Lindley distributions, respectively. The K generalized power Lomax distribution was developed by [40].

A truncated distribution is a conditional distribution with a narrower scope than the parent distribution. The truncated distributions have had widespread application, primarily in reliability and life-testing studies. The recently truncated L (TL) distribution published by [41], with one shape parameter has attracted our interest. The CDF and PDF for the TL distribution are

$$G(y;\lambda)=\frac{1-(1+\ y)^{-\lambda}}{1-2^{-\lambda}},\quad 0<y<1,\quad \lambda>0, \tag{3}$$

and

$$g(y;\lambda)=\frac{\lambda(1+\ y)^{-\lambda-1}}{1-2^{-\lambda}}\quad,\quad 0<y<1,\quad \lambda>0. \tag{4}$$

The goal of creating this new generalization of the TL distribution is to improve the performance and the flexibility of TL distribution. The PDF of the KTL distribution is bathtub, uni-modal, decreasing, and right skewed, while the HRF is U-shaped, bathtub, increasing and J-shaped.

The remainder of the manuscript is arranged as follows: the construction of the KTL distribution is proposed in Section 2. Section 3 deduces some basic statistical features, like; quantile function, skewness, kurtosis, moments, generating functions, incomplete moments and order statistics. Maximum likelihood estimate is covered in Section 4. In section 5, we consider three real-world applications to assess the adaptability of the KTL model, followed by some final concluding remarks in Section 6.

2 The KTL Distribution

The KTL distribution is constructed by inserting the Equations (3) and (4) into the Equations (1) and (2). The random variable (RV) Y is said to have KTL model denoted by KTL $\sim \psi=(\mu,\eta,\lambda)$ if the CDF, PDF and survival function (SF) of Y are provided as below

$$F(y;\ \psi)=\ 1-\left[1-\left(\frac{1-(1+\ y)^{-\lambda}}{1-2^{-\lambda}}\right)^{\mu}\right]^{\eta}, \tag{5}$$

$$f(y;\ \psi)=\ \frac{\lambda\mu\eta(1+\ y)^{-\lambda-1}}{(1-2^{-\lambda})^{\mu}}\Big(1-(1+\ y)^{-\lambda}\Big)^{\mu-1} \left[1-\left(\frac{1-(1+\ y)^{-\lambda}}{1-2^{-\lambda}}\right)^{\mu}\right]^{\eta-1},\quad y>0,\psi>0, \tag{6}$$

and

$$R(y;\ \psi)=\ \left[1-\left(\frac{1-(1+\ y)^{-\lambda}}{1-2^{-\lambda}}\right)^{\mu}\right]^{\eta}.$$

The HRF, the reversed HRF and cumulative HRF for the KTL distribution are provided as below

$$h(y;\ \psi)=\ \frac{\lambda\mu\eta(1+\ y)^{-\lambda-1}\Big(1-(1+\ y)^{-\lambda}\Big)^{\mu-1}}{(1-2^{-\lambda})^{\mu}\left[1-\left(\frac{1-(1+\ y)^{-\lambda}}{1-2^{-\lambda}}\right)^{\mu}\right]},$$

$$\tau(y;\ \psi)=\ \frac{\lambda\mu\eta(1+\ y)^{-\lambda-1}\Big(1-(1+\ y)^{-\lambda}\Big)^{\mu-1}}{(1-2^{-\lambda})^{\mu}\left[1-\left[1-\left(\frac{1-(1+\ y)^{-\lambda}}{1-2^{-\lambda}}\right)^{\mu}\right]^{\eta}\right]} \times\left[1-\left(\frac{1-(1+\ y)^{-\lambda}}{1-2^{-\lambda}}\right)^{\mu}\right]^{\eta-1},$$

and

$$H(y;\ \psi)=-\eta\ln\left[1-\left(\frac{1-(1+\ y)^{-\lambda}}{1-2^{-\lambda}}\right)^{\mu}\right].$$

The KTL distribution is very flexible and includes three sub-models such as; When $\eta=1$, we get the exponentiated TL distribution; when $\mu=2$, we get the Topp-Leone TL distribution and when $\eta=\mu=1$, we get the TL distribution, [41].

Figure 1 and Figure 2 mention the PDF and HRF curves for the KTL distribution with different values of the parameter. The PDF is bathtub, uni-modal, decreasing, and right skewed, while the HRF is U-shaped, bathtub, increasing and J-shaped.

3 Important Mathematical Characterizations

This section investigates some important mathematical characterizations of the KTL distribution.

3.1 Quntile Function

The quantile function of KTL distribution is provided via

$$y_u=\left[1-\Big(1-2^{-\lambda}\Big)\left[1-(1-u)^{\frac{1}{\eta}}\right]^{\frac{1}{\mu}}\right]^{\frac{-1}{\lambda}}-1, \tag{7}$$

where $u\in(0,1)$. To get the first quantile ($Q1$), the second quantile ($Q2$) (median) and the third quantile ($Q3$) we insert u = 0.25, 0.5 and 0.75 respectively. The Q1, Q2 and Q3 are provided via

$$Q1=\left[1-\Big(1-2^{-\lambda}\Big)\left[1-(0.75)^{\frac{1}{\eta}}\right]^{\frac{1}{\mu}}\right]^{\frac{-1}{\lambda}}-1,$$

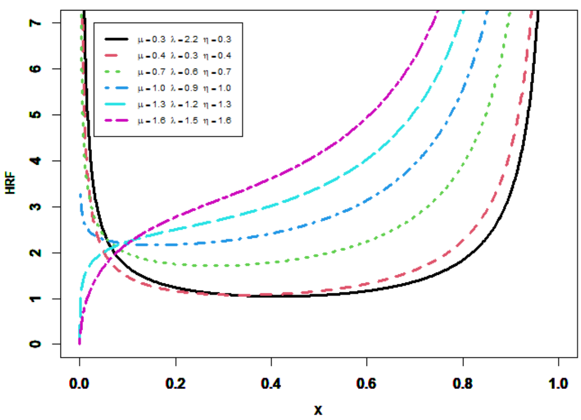

Figure 1: Plots of PDF for the KTL distribution.

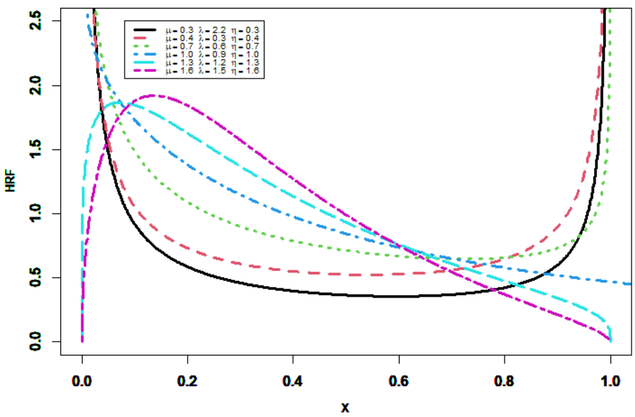

Figure 2: Plots of hrf for the KTL distribution.

$$Q2 = \left[1 - \left(1 - 2^{-\lambda}\right)\left[1 - (0.5)^{\frac{1}{\eta}}\right]^{\frac{1}{\mu}}\right]^{\frac{-1}{\lambda}} - 1,$$

and

$$Q3 = \left[1 - \left(1 - 2^{-\lambda}\right)\left[1 - (0.25)^{\frac{1}{\eta}}\right]^{\frac{1}{\mu}}\right]^{\frac{-1}{\lambda}} - 1.$$

The Bowley skewness (α_1) and the Moors kurtosis (α_2) are provided via

$$\alpha_1 = \frac{y_{0.75} + y_{0.25} - 2y_{0.5}}{y_{0.75} - y_{0.25}}.$$

and

$$\alpha_2 = \frac{y_{0.875} - y_{0.625} + y_{0.375} - y_{0.125}}{y_{0.75} - y_{0.25}}.$$

Table 1: Results of Some numerical values of $Q1$, $Q2$, $Q3$, α_1 and α_2 for the KTL model at $\lambda = 5$

η	μ	$Q1$	$Q2$	$Q3$	α_1	α_2
2.5	3	0.132	0.203	0.298	0.141	1.258
	3.1	0.137	0.209	0.305	0.139	1.257
	3.2	0.142	0.215	0.311	0.136	1.256
	3.3	0.146	0.22	0.317	0.134	1.255
	3.4	0.151	0.226	0.323	0.132	1.254
	3.5	0.155	0.231	0.329	0.13	1.253
	3.6	0.16	0.236	0.335	0.129	1.252
	3.7	0.164	0.241	0.341	0.127	1.251
	3.8	0.168	0.246	0.346	0.125	1.251
	3.9	0.172	0.251	0.351	0.124	1.25
4	3	0.107	0.161	0.229	0.122	1.249
	3.1	0.111	0.166	0.235	0.119	1.249
	3.2	0.115	0.171	0.241	0.117	1.248
	3.3	0.12	0.176	0.247	0.115	1.247
	3.4	0.124	0.181	0.252	0.113	1.247
	3.5	0.128	0.186	0.258	0.111	1.246
	3.6	0.132	0.19	0.263	0.109	1.246
	3.7	0.136	0.195	0.268	0.108	1.245
	3.8	0.14	0.199	0.273	0.106	1.245
	3.9	0.144	0.204	0.278	0.104	1.244
5	3	0.097	0.144	0.203	0.113	1.245
	3.1	0.101	0.149	0.209	0.111	1.244
	3.2	0.105	0.154	0.215	0.108	1.244
	3.3	0.109	0.159	0.22	0.106	1.243
	3.4	0.113	0.163	0.226	0.104	1.243
	3.5	0.117	0.168	0.231	0.102	1.242
	3.6	0.121	0.173	0.236	0.101	1.242
	3.7	0.125	0.177	0.241	0.099	1.242
	3.8	0.128	0.181	0.246	0.097	1.241
	3.9	0.132	0.186	0.251	0.096	1.241

From Table 1 we note that when η=2.5,4, 5, and μ increases, Q1, Q2 and Q3 increase, whereas α_1 and α_2 decrease.

3.2 Ordinary and Incomplete Moments

The n^{th} ordinary moments of the KTL distribution can be calculated from the next equation

$$\mu'_n = \int_0^\infty y^n f(y;\psi)\, dy, \tag{8}$$

by employing (6) in (8) we have

$$\mu'_n = \frac{\lambda\mu\eta}{(1-2^{-\lambda})^\mu}\int_0^\infty y^n(1+\ y)^{-\lambda-1} \times \left(1-(1+\ y)^{-\lambda}\right)^{\mu-1} \times \left[1-\left(\frac{1-(1+\ y)^{-\lambda}}{1-2^{-\lambda}}\right)^\mu\right]^{\eta-1} dy,$$

By utilizing the binomial theorem to the last term in the above equation

$$\left[1-\left(\frac{1-(1+\ y)^{-\lambda}}{1-2^{-\lambda}}\right)^\mu\right]^{\eta-1} = \sum_{i=0}^\infty (-1)^i \binom{\eta-1}{i} \times \left(\frac{1-(1+\ y)^{-\lambda}}{1-2^{-\lambda}}\right)^{\mu i}. \tag{9}$$

then,

$$\mu'_n = \sum_{i=0}^\infty (-1)^i \binom{\eta-1}{i} \frac{\lambda\mu\eta}{(1-2^{-\lambda})^{\mu(i+1)}} \int_0^\infty y^n(1+\ y)^{-\lambda-1}\left(1-(1+\ y)^{-\lambda}\right)^{\mu(i+1)-1} dy,$$

Again utilizing the binomial theorem (9) to the last term in the above equation, then we have

$$\mu'_n = \sum_{i,j=0}^\infty \Psi_{i,j} \int_0^\infty y^n(1+\ y)^{-\lambda(j+1)-1} dy, \tag{10}$$

where,

$$\Psi_{i,j} = \frac{(-1)^{i+j}\lambda\mu\eta}{(1-2^{-\lambda})^{\mu(i+1)}}. \times \binom{\eta-1}{i}\binom{\mu(i+1)-1}{j}.$$

By using the prime beta function the integration (10) leads to

$$\mu'_n = \sum_{i,j=0}^\infty \Psi_{i,j}\ B(n+1, \lambda(j+1)-n)\ \ , \lambda(j+1) > n. \tag{11}$$

To get the first four moments, we put n = 1, 2, 3 and 4 as below

$$\mu'_1 = \sum_{i,j=0}^\infty \Psi_{i,j}\ B(2, \lambda(j+1)-1)\ \ , \lambda(j+1) > 1,$$

$$\mu'_2 = \sum_{i,j=0}^\infty \Psi_{i,j}\ B(3, \lambda(j+1)-2)\ \ , \lambda(j+1) > 2,$$

$$\mu'_3 = \sum_{i,j=0}^\infty \Psi_{i,j}\ B(4, \lambda(j+1)-3)\ \ , \lambda(j+1) > 3,$$

and

$$\mu'_4 = \sum_{i,j=0}^\infty \Psi_{i,j}\ B(5, \lambda(j+1)-4)\ \ , \lambda(j+1) > 4.$$

The moment generating function of the KTL distribution is provided via

$$M_Y(t) = E\left(e^{tY}\right) = \sum_{n=0}^\infty \frac{t^n}{n!}\mu'_n$$

$$= \sum_{i,j,n=0}^\infty \frac{t^n}{n!}\Psi_{i,j}\ B(n+1, \lambda(j+1)-n)\ \ , \lambda(j+1) > n.$$

Table 2 shows the numerical values of the first four moments $\mu'_1, \quad \mu'_2, \quad \mu'_3, \quad \mu'_4$, also the numerical values of variance (σ^2), coefficient of skewness (CS), coefficient of kurtosis (CK) and coefficient of variation (CV) for the TKL model.

Table 2: Results of some moments, CS, CK, and CV for the KTL model at $\lambda = 5$

η	μ	μ'_1	μ'_2	μ'_3	μ'_4	σ^2	CS	CK	CV
2.5	3	0.233	0.284	3.308	109.787	0.23	28.354	2013	2.063
	3.1	0.214	0.184	1.741	53.521	0.138	32.129	2745	1.732
	3.2	0.202	0.126	0.933	26.384	0.085	35.223	3550	1.447
	3.3	0.192	0.092	0.51	13.148	0.055	36.995	4296	1.214
	3.4	0.185	0.071	0.285	6.625	0.037	36.829	4796	1.033
	3.5	0.179	0.058	0.163	3.375	0.026	34.425	4876	0.897
	3.6	0.175	0.05	0.096	1.739	0.019	30.067	4483	0.797
	3.7	0.17	0.044	0.059	0.906	0.015	24.586	3735	0.726
	3.8	0.166	0.04	0.038	0.478	0.013	18.993	2850	0.677
	3.9	0.163	0.037	0.026	0.256	0.011	14.062	2024	0.643
4	3	0.997	4.604	64.441	2159	3.609	7.679	147.862	1.905
	3.1	0.707	2.586	32.25	1004	2.086	9.116	211.41	2.043
	3.2	0.535	1.494	16.459	474.179	1.208	10.825	302.571	2.052
	3.3	0.432	0.892	8.563	227.106	0.705	12.781	428.679	1.945
	3.4	0.368	0.553	4.541	110.298	0.418	14.939	596.518	1.758
	3.5	0.327	0.358	2.455	54.306	0.252	17.213	809.594	1.536
	3.6	0.3	0.245	1.354	27.102	0.155	19.443	1063	1.315
	3.7	0.281	0.177	0.763	13.709	0.098	21.37	1337	1.116
	3.8	0.268	0.136	0.441	7.029	0.064	22.649	1592	0.948
	3.9	0.257	0.11	0.262	3.653	0.044	22.912	1771	0.814
5	3	3.511	20.843	300.086	10090	8.516	6.725	96.039	0.831
	3.1	2.166	11.215	145.352	4547	6.522	5.572	83.156	1.179
	3.2	1.405	6.182	71.966	2084	4.209	5.959	98.287	1.461
	3.3	0.965	3.494	36.393	971.414	2.563	6.843	129.083	1.659
	3.4	0.706	2.028	18.784	460.134	1.53	8.03	176.217	1.753
	3.5	0.55	1.213	9.891	221.389	0.911	9.461	243.054	1.735
	3.6	0.455	0.752	5.312	108.158	0.545	11.105	333.745	1.625
	3.7	0.394	0.486	2.909	53.639	0.331	12.914	451.78	1.458
	3.8	0.355	0.33	1.627	26.999	0.204	14.798	597.73	1.271
	3.9	0.329	0.237	0.93	13.791	0.129	16.601	765.724	1.089

Note that when η=2.5 and μ increases, σ^2, CS, CK and CV decrease. When η=4 and μ increases, σ^2 and CV decrease, whereas CS and CK increase. When η=5 and μ increases, σ^2 decreases and CV increases then decreases, whereas CS and CK increase.

3.3 Order Statistics

Assume that $Y_1, Y_2, ..., Y_n$ be independent and identically distributed (iid) RVs and their corresponding continuous $F(y)$. Suppose that $Y_{1:n} < Y_{2:n} < ... < Y_{n:n}$ the relevant ordered random sample from a population of size *n*. The pdf of the m^{th} ORS, is provided via

$$f_{m:n}(y) = Q(n,m) f(y) F(y)^{m-1} \left(1 - F(y)\right)^{n-m} \quad , (12)$$

where $Q(n,m) = \frac{n!}{(m-1)!(n-m)!}$. For the KTL distribution, the pdf of the m^{th} ORS is computed via

$$\begin{aligned} f_{m:n}(y) = {} & Q(m,n) \frac{\lambda\mu\eta(1+y)^{-\lambda-1}}{(1-2^{-\lambda})^{\mu}} \left(1-(1+y)^{-\lambda}\right)^{\mu-1} \\ & \times \left[1-\left(\frac{1-(1+y)^{-\lambda}}{1-2^{-\lambda}}\right)^{\mu}\right]^{\eta(n-m+1)-1} \\ & \times \left\{1-\left[1-\left(\frac{1-(1+y)^{-\lambda}}{1-2^{-\lambda}}\right)^{\mu}\right]^{\eta}\right\}^{m-1}. \end{aligned} \quad (13)$$

The first ORS of the KTL distribution can be computed by inserting For *m* =1 in Equation (13), as

$$\begin{aligned} f_{1:n}(y) = {} & \frac{n\lambda\mu\eta(1+y)^{-\lambda-1}}{(1-2^{-\lambda})^{\mu}} \left(1-(1+y)^{-\lambda}\right)^{\mu-1} \\ & \times \left[1-\left(\frac{1-(1+y)^{-\lambda}}{1-2^{-\lambda}}\right)^{\mu}\right]^{\eta(n-2)-1}. \end{aligned}$$

The first ORS of the KTL distribution can be computed by inserting For *m*=*n* in Equation (13), as

$$\begin{aligned} f_{n:n}(y) = {} & \frac{n\lambda\mu\eta(1+y)^{-\lambda-1}}{(1-2^{-\lambda})^{\mu}} \left(1-(1+y)^{-\lambda}\right)^{\mu-1} \\ & \times \left[1-\left(\frac{1-(1+y)^{-\lambda}}{1-2^{-\lambda}}\right)^{\mu}\right]^{\eta-1} \\ & \times \left\{1-\left[1-\left(\frac{1-(1+y)^{-\lambda}}{1-2^{-\lambda}}\right)^{\mu}\right]^{\eta}\right\}^{n-1}. \end{aligned}$$

4 Approach of ML Estimation

We used the ML estimates (MLEs) approach in this part to estimate the unknown parameters of the KTL distribution. Assume that $y_1, ..., y_n$ is the nth random sample (RS) from the KTL distribution (6). The KTL distribution's log-likelihood function is provided via

$$\begin{aligned} \log L = {} & n\log(\lambda) + n\log(\mu) + n\log(\eta) \\ & -n\mu\log\left(1-2^{-\lambda}\right) - (\lambda+1)\sum_{i=1}^{n}\log(1+y_i) \\ & + (\mu-1)\sum_{i=1}^{n}\log\left(1-(1+y_i)^{-\lambda}\right) \\ & + (\eta-1)\sum_{i=1}^{n}\log\left[1-\left(\frac{1-(1+y_i)^{-\lambda}}{1-2^{-\lambda}}\right)^{\mu}\right]. \end{aligned} \quad (14)$$

By differentiating Equation (14) with respect to μ, λ and η respectively as

$$\begin{aligned} \frac{\partial \log L}{\partial \mu} = {} & \frac{n}{\mu} - n\log\left(1-2^{-\lambda}\right) + \sum_{i=1}^{n}\log\left(1-(1+y_i)^{-\lambda}\right) \\ & -(\eta-1)\sum_{i=1}^{n}\frac{\ln\left(\frac{1-(1+y_i)^{-\lambda}}{1-2^{-\lambda}}\right)}{\left(\frac{1-(1+y_i)^{-\lambda}}{1-2^{-\lambda}}\right)^{-\mu}-1}, \end{aligned} \quad (15)$$

$$\begin{aligned} \frac{\partial \log L}{\partial \lambda} = {} & \frac{n}{\lambda} + \frac{n\mu 2^{-\lambda}\log(2)}{1-2^{-\lambda}} - \sum_{i=1}^{n}\log(1+y_i) \\ & -(\mu-1)\sum_{i=1}^{n}\frac{(1+y_i)^{-\lambda}\log(1+y_i)}{1-(1+y_i)^{-\lambda}} \\ & -\mu(\eta-1)\sum_{i=1}^{n}\frac{\left(\frac{1-(1+y_i)^{-\lambda}}{1-2^{-\lambda}}\right)^{\mu-1}}{1-\left(\frac{1-(1+y_i)^{-\lambda}}{1-2^{-\lambda}}\right)^{\mu}} \\ & \times\left\{\frac{(1+y_i)^{-\lambda}\left(1-2^{-\lambda}\right)\log(1+y_i)}{\left(1-2^{-\lambda}\right)^2} \right. \\ & \left. + \frac{2^{-\lambda}\left[1-(1+y_i)^{-\lambda}\right]\log(2)}{\left(1-2^{-\lambda}\right)^2}\right\}, \end{aligned} \quad (16)$$

and

$$\frac{\partial \log L}{\partial \eta} = \frac{n}{\eta} + \sum_{i=1}^{n}\log\left[1-\left(\frac{1-(1+y_i)^{-\lambda}}{1-2^{-\lambda}}\right)^{\mu}\right]. \quad (17)$$

The MLEs of the parameters μ, λ and η symbolize by $\hat{\mu}, \hat{\lambda}$ and $\hat{\eta}$ are calculated by equating the equations (15), (16) and (17) to 0 and simultaneously solving these nonlinear systems of equations.

5 Simulation

A brief simulation is run to evaluate the performance of the ML approach for estimating parameters. The

KTL is investigated for such reasons using Monte Carlo simulations. The computations in this section are performed utilizing Mathematica 9. The procedure is structured as follows:

1. We generate random samples from the KTL distribution by utilizing

$$y_u = \left[1 - \left(1 - 2^{-\lambda}\right)\left[1 - (1-u)^{\frac{1}{\eta}}\right]^{\frac{1}{\mu}}\right]^{\frac{-1}{\lambda}} - 1,$$

2. Monte Carlo simulations were performed 10000 times with n = 100, 200 and 300.

3. The chosen values for the parameters are listed in Table 3, Table 4, Table 5, Table 6 and Table 7.

4. Mean square error (MSE), lower limit (LL), upper limit (UL), and average length (AL) formulas of 90% and 95% are determined.

The simulation results for the KTL distribution are reported in Table 3, Table 4, Table 5, Table 6 and Table 7 with different values of μ, λ, η and n. According to the tables, the estimated MSE and AL decrease in most of the situations when n increases. We note that the estimation method works sufficiently well for estimating the parameters μ, λ and η

6 Applications

In this section, we exhibit applications to two real-world datasets to demonstrate the applicability and importance of the KTL distribution. The goodness-of-fit statistics for the KTL distribution and other competing distributions are examined, and the MLEs of their parameters are shown.

We will also compare the fits of the KTL distribution to other models: TL, unit exponential Pareto (UExP), [42], exponential Pareto (ExP), [43], the unit-Weibull (UW), [44], Kumaraswamy (K), [45], Marshall-Olkin Kumaraswamy (MOK), [46], Marshall-Olkin extended Topp Leone (MOETL), [47], unit-Gompertz (UGo), [48], unit generalized log Burr XII (UGLB), [49], Topp Leone (ToLe), [50], and unit gamma Gompertz (UGGo), [51], distribution.

To examine the fit of all competitive models, Cramer-von Mises (CV), Anderson-Darling (A-D), and Kolmogorov Smirnov (KS) statistics with their p-values (PV) were used.

6.1 The rate of COVID-19 recovery in Turkey

We're curious in studying on the pace of COVID-19 in Turkey recovery during the epidemic. In Turkey, the daily ratio of total recoveries to total confirmed cases was calculated. From March 27 to April 20, a total of 25 observations were made, [42]. The information is as follows: 0.0074, 0.0095, 0.0113, 0.015, 0.018, 0.0212, 0.0229, 0.0231, 0.0328, 0.0385, 0.0439, 0.0464, 0.0483, 0.0507, 0.0515, 0.0568, 0.0605, 0.0648, 0.0737, 0.0818, 0.0955, 0.1099, 0.127, 0.1388, 0.1476.

Table 8 proposes ten different distributions for comparison with the KTL. It is observed that the KTL achieves the minimal value in all metrics of goodness of fit, with a big value of PV. This demonstrates its applicability and efficiency for modelling the dataset, as opposed to the other competing distributions.Figure 3 shows the fitted KTL PDF, CDF, and P-P plots of the data set The rate of COVID-19 recovery in Turkey.

We consider the KTL model to analyze this dataset compared with the other competitive models. The MLEs of the parameters of each fitted distribution with their stranded errors (SE) and the goodness of fit statistics are reported in Table 8. According to the table, it is clear that all the models work quite well for analyzing this dataset since the PV of those models are greater than 0.05. However, the KTL model provides the best fit among all competitive models since it has the smallest values of CV, A-D and KS statistics, as well as having the largest PV.

6.2 The rate of COVID-19 recovery in France

The World Health Organization verified that the first death from COVID-19 occurred in France. This data set comprises 38 observations estimated as the daily ratio of total recoveries to the cumulative number of confirmed cases and different cumulative numbers of confirmed death cases in France from 1 January to 7 February 2022. a total of 38 observations were made, [42]. The information is as follows: 0.195, 0.2338, 0.2368, 0.1073, 0.1592, 0.2784, 0.0689, 0.1791, 0.1121, 0.1865, 0.2631, 0.0716, 0.1411, 0.1477, 0.1874, 0.0853, 0.0922, 0.1711, 0.1962, 0.2146, 0.1041, 0.1524, 0.1811, 0.0643, 0.2698, 0.1245, 0.176, 0.2363, 0.0712, 0.1361, 0.1386, 0.3316, 0.077, 0.1367, 0.1549, 0.2178, 0.0951, 0.1346

Table 9 shows the MLEs estimation of the parameters and the goodness-of-fit metrics. Table 9 shows that the KTL achieves the lowest values of the goodness-of-fit metrics when compared to the other nine recommended models. Figure 4 displays the histogram with fitted pdf, fitted CDF, and P-P plot of the KTL model.

Table 3: Simulation results for the KTL distribution at $\mu = \lambda = \eta$=0.3

n	parameter	MLE	MSE	90% LL	90% UL	90% AL	95% LL	95% UL	95% AL
100	μ	0.3055	0.0034	0.1690	0.4420	0.2730	0.1428	0.4682	0.3253
	λ	0.2758	0.0098	-0.0509	0.6025	0.6534	-0.1134	0.6651	0.7785
	η	0.3063	0.0007	0.2337	0.3788	0.1450	0.2198	0.3927	0.1728
200	μ	0.3284	0.0072	0.2206	0.4362	0.2156	0.2000	0.4569	0.2569
	λ	0.3912	0.0536	0.0766	0.7059	0.6293	0.0163	0.7662	0.7499
	η	0.3026	0.0012	0.2522	0.3530	0.1008	0.2425	0.3626	0.1201
300	μ	0.3194	0.0015	0.2382	0.4007	0.1625	0.2226	0.4162	0.1936
	λ	0.3173	0.0070	0.1148	0.5198	0.4051	0.0760	0.5586	0.4826
	η	0.3050	0.0003	0.2643	0.3458	0.0815	0.2565	0.3536	0.0971

Table 4: Simulation results for the KTL distribution at $\mu = \lambda = \eta$=0.5

n	parameter	MLE	MSE	90% LL	90% UL	90% AL	95% LL	95% UL	95% AL
100	μ	0.5552	0.0870	0.2923	0.8182	0.5260	0.2419	0.8686	0.6267
	λ	0.5444	0.3956	-0.0483	1.1372	1.1855	-0.1618	1.2507	1.4125
	η	0.5435	0.0190	0.3795	0.7076	0.3280	0.3481	0.7390	0.3908
200	μ	0.5016	0.0074	0.3613	0.6420	0.2807	0.3344	0.6689	0.3345
	λ	0.4869	0.0261	0.1460	0.8278	0.6818	0.0807	0.8931	0.8124
	η	0.5212	0.0053	0.4203	0.6222	0.2020	0.4009	0.6416	0.2406
300	μ	0.5165	0.0107	0.3910	0.6420	0.2510	0.3670	0.6661	0.2991
	λ	0.5870	0.0460	0.2328	0.9411	0.7083	0.1650	1.0089	0.8439
	η	0.4949	0.0018	0.4177	0.5721	0.1544	0.4029	0.5869	0.1840

Table 5: Simulation results for the KTL distribution at μ=0.8 and $\lambda = \eta$=0.3

n	parameter	MLE	MSE	90% LL	90% UL	90% AL	95% LL	95% UL	95% AL
100	μ	0.8310	0.1532	0.2126	1.4493	1.2367	0.0942	1.5678	1.4735
	λ	0.4102	0.3108	-0.1986	1.0190	1.2177	-0.3152	1.1356	1.4508
	η	0.3166	0.0018	0.2490	0.3842	0.1352	0.2360	0.3971	0.1611
200	μ	0.8261	0.0494	0.4259	1.2864	0.8605	0.3435	1.3688	1.0253
	λ	0.2882	0.0151	0.0278	0.5286	0.5008	-0.0202	0.5766	0.5967
	η	0.3154	0.0009	0.2680	0.3628	0.0948	0.2589	0.3719	0.1130
300	μ	0.8171	0.0696	0.5496	1.3047	0.7551	0.4773	1.3770	0.8997
	λ	0.3010	0.0347	0.1055	0.6165	0.5110	0.0566	0.6655	0.6089
	η	0.3069	0.0005	0.2701	0.3437	0.0736	0.2630	0.3507	0.0877

Table 6: Simulation results for the KTL distribution at μ=0.8 and $\lambda = \eta$=0.5

n	parameter	MLE	MSE	90%			95%		
				LL	UL	AL	LL	UL	AL
100	μ	0.8975	0.0665	0.3814	1.4136	1.0323	0.2825	1.5125	1.22995
	λ	0.6950	0.1692	-0.0599	1.4499	1.5098	-0.2045	1.5944	1.79892
	η	0.4708	0.0031	0.3563	0.5854	0.2291	0.3343	0.6073	0.272959
200	μ	0.7585	0.0159	0.4925	1.0246	0.5321	0.4415	1.0755	0.634012
	λ	0.5209	0.0347	0.1353	0.9065	0.7712	0.0614	0.9803	0.918915
	η	0.5060	0.0032	0.4158	0.5962	0.1805	0.3985	0.6135	0.215007
300	μ	0.8595	0.0124	0.6053	1.1136	0.5083	0.5567	1.1623	0.605648
	λ	0.5041	0.0190	0.2093	0.7988	0.5895	0.1529	0.8552	0.702393
	η	0.5208	0.0012	0.4456	0.5961	0.1506	0.4311	0.6105	0.179386

Table 7: Simulation results for the KTL distribution at μ=0.9, λ=0.6 and η=0.4

n	parameter	MLE	MSE	90%			95%		
				LL	UL	AL	LL	UL	AL
100	μ	1.0393	0.1903	0.2677	1.8110	1.5433	0.1199	1.9587	1.8388
	λ	0.8659	0.3727	-0.2526	1.9844	2.2370	-0.4667	2.1986	2.6654
	η	0.4007	0.0018	0.3099	0.4915	0.1816	0.2925	0.5089	0.2164
200	μ	0.9218	0.0417	0.5210	1.3226	0.8016	0.4443	1.3993	0.9550
	λ	0.7150	0.1307	0.1462	1.2839	1.1378	0.0372	1.3929	1.3556
	η	0.4005	0.0017	0.3367	0.4642	0.1275	0.3245	0.4764	0.1519
300	μ	0.8874	0.0173	0.5927	1.1820	0.5893	0.5363	1.2384	0.7021
	λ	0.6478	0.0227	0.2536	1.0420	0.7884	0.1782	1.1175	0.9393
	η	0.4010	0.0011	0.3491	0.4528	0.1036	0.3392	0.4627	0.1235

Table 8: The CV, A-D, KS, PV, MLEs, and SEs for the data set of The rate of COVID-19 recovery in Turkey

Models	CV	A-D	KS	PV		MLEs (SEs)
KTL	0.025	0.187	0.083	0.954	μ	13.234 (3583)
					η	1.757 (65.008)
					λ	2.473 (821.933)
TL	0.126	1.426	0.151	0.620	λ	18.633 (3.727)
UExP	0.030	0.229	0.101	0.941	α	1.342 (0.209)
					β	0.115 (0.054)
					λ	2.063 (1.069)
UW	0.065	0.386	0.136	0.692	α	0.005 (0.003)
					β	4.160 (0.418)
K	0.031	0.231	0.102	0.933	α	1.416 (0.230)
					β	50.941 (31.323)
MOK	0.036	0.237	0.113	0.872	α	0.138 (0.159)
					β	1.876 (0.346)
					λ	47.547 (52.990)
UGo	0.124	0.726	0.160	0.493	α	0.017 (0.013)
					β	1.146 (0.180)
MOETL	0.056	0.348	0.106	0.915	α	0.006 (0.005)
					β	2.066 (0.298)
UGLB	0.038	0.251	0.099	0.947	α	1.839 (3.079)
					β	2.724 (1.134)
					λ	3.605 (1.664)
UGGo	0.039	0.255	0.219	0.156	α	1.288 (0.283)
					β	29.959 (14.070)
					λ	0.805 (0.400)
ExP	0.030	0.231	0.103	0.930	α	1.432 (0.224)
					β	0.117 (0.909)
					λ	2.501 (27.807)

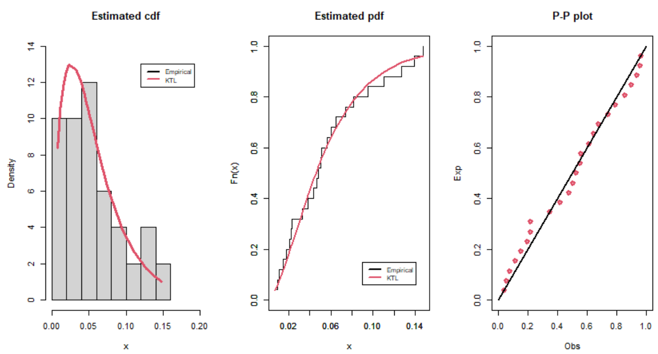

Figure 3: Estimated pdf, CDF and P-P plots of competitive model for the data set of The rate of COVID-19 recovery in Turkey

The MLEs of the parameters of each fitted distributions with their stranded errors (SE), and the goodness of fit statistics are reported in Table 9. From the table, it is found that the UEPD, MOETL, UGIBXI and K models provide sufficient results for analyzing this data set besides KTL model. However, the KTL distribution is the best among all tested models.

Table 9: The CV, A-D, KS, PV, MLEs, and SEs for the data set of The rate of COVID-19 recovery in France

Models	CV	A-D	KS	PV		MLEs (SEs)
KTL	0.023	0.200	0.0613	0.999	μ	5.707 (9.199)
					η	4.395 (2.897)
					λ	7.846 (19.41)
TL	0.742	14.901	0.333	0.001	λ	6.398 (1.415)
UExP	0.037	0.297	0.072	0.981	α	2.172 (0.265)
					β	0.354 (0.112)
					λ	2.640 (0.503)
UW	0.084	0.546	0.120	0.599	α	0.028 (0.015)
					β	4.869 (0.597)
K	0.567	0.231	0.071	0.949	α	2.658 (0.339)
					β	91.92 (50.57)
MOK	0.170	1.041	0.154	0.295	α	0.149 (0.114)
					β	3.814 (0.900)
					λ	191.13 (88.89)
UGo	0.17	1.041	0.154	0.295	α	0.007 (0.004)
					β	2.319 (0.229)
MOETL	0.064	0.453	0.077	0.965	α	0.004 (0.002)
					β	4.270 (0.410)
UGLB	0.043	0.321	0.079	0.957	α	2.093 (1.282)
					β	3.081 (1.080)
					λ	2.204 (0.828)
UGGo	0.063	0.437	0.153	0.301	α	2.523 (0.426)
					β	157.24 (77.15)
					λ	1.385 (0.709)

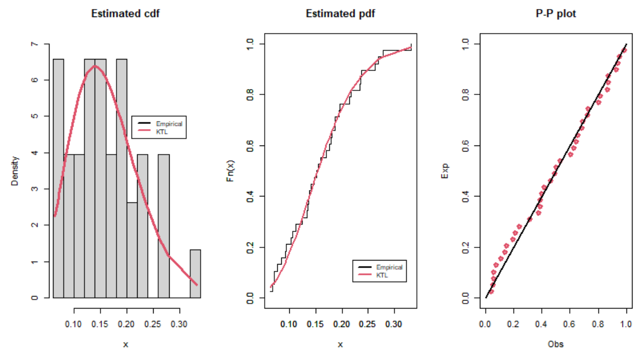

Figure 4: Estimated pdf, CDF and P-P plots of competitive model for the data set of The rate of COVID-19 recovery in France

7 Conclusion

In this paper, we proposed a new generalization of the Truncated Lomax Distribution using the tractable features of the Kumaraswamy generated family. Several of its statistical properties are studied. The maximum likelihood method is used to estimate the value of the parameters. It is found that the estimation of the model parameters performs quite well. Finally, the flexibility of the introduced family was illustrated by a real dataset, which showed that the Kumaraswamy truncated Lomax Distribution is better than every other lifetime models used in this study.

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