Geometry of lightlike hypersurfaces of a statistical manifold

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Abstract: In this paper, we introduced the lightlike hypersurfaces of a statistical manifold. It is shown that a lightlike hypersurface of a statistical manifold is not a statistical manifold with respect to the induced connections, but the screen distribution has a canonical statistical structure. Some relations between induced geometric objects with respect to dual connections in a lightlike hypersurface of a statistical manifold are obtained. An example is presented. Induced Ricci tensors for lightlike hypersurface of a statistical manifold are computed.

Key-Words: Lightlike hypersurface, Statistical manifolds, Dual connections.

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1 Introduction

A statistical manifold, the Riemannian connection used to model the information, the fields of information geometry, as such a generalization of the Riemannian manifold equipped with a relatively new mathematics branch, uses the differential geometry tool to examine the statistical inference, information loss and prediction, [6]. In 1975, the role of differential geometry in statistics was first emphasized by [12]. Later, Amari used differential geometric tools to develop this idea, [1], [2].

A Riemannian manifold $(\widetilde{M}, \widetilde{g})$ with a Riemannian metric $\widetilde{g}$ and the Levi-Civita connection $\widetilde{D}^0$ is called a statistical manifold if there exists a pair of torsion-free connection $(\widetilde{D}, \widetilde{D}^*)$ such that the following relation satisfies for any tangent vector fields X, Y and Z on $\widetilde{M}$

$$\widetilde{g}(X, \widetilde{D}^*_Z Y) = Z\widetilde{g}(X, Y) - \widetilde{g}(\widetilde{D}_Z X, Y), \tag{1}$$

where

$$\widetilde{D}^0 = \frac{1}{2}(\widetilde{D} + \widetilde{D}^*). \tag{2}$$

In 1989, [28], initiated the study of geometry of submanifolds of statistical manifolds. He obtained Gauss-Weingarten formulas, Gauss and Codazzi equations, etc.. Later, in 2009, [14], studied hypersurfaces of a statistical manifold. Also, studied submanifolds of statistical manifolds of constant curvature, [3]. In addition to, many authors have studied on different types of statistical manifolds, [15], [26], [27].

On the other hand, lightlike geometry is one of the important research areas in differential geometry and has many applications in physics and mathematics. The geometry of lightlike submanifolds of a semi-Riemannian manifold was presented by [9], (see also, [10], [11]). Lightlike hypersurfaces in various spaces have been studied by many authors including those of [4], [5], [7] [8], [10], [13], [17], [18], [19], [21], [22], [23], [24], [25].

Motivated by these circumstances, in this paper, we initiate the study of lightlike geometry of statistical manifolds. In section 2, we present basic definitions and results about statistical manifolds and lightlike hypersurfaces. In Section 3, we show that induced connections on a lightlike hypersurface of a statistical manifold are not dual connections and a lightlike hypersurface is not statistical manifold. Moreover, we show that the second fundamental forms are not degenerate. Later, we characterize the parallelness and integrability of the screen distribution. Equivalent conditions are also obtained between the induced objects. This section concludes with an example. In section 4, we obtain formula for curvature tensors of a lightlike hypersurface of a statistical manifold. In general, in lightlike geometry, Ricci tensor is not symmetric, so we also obtain new conditions for Ricci tensor to be symmetric.

2 Preliminaries

Let $(\bar{M},\bar{g})$ be an $(m+2)$-dimensional semi-Riemannian manifold with index$(\bar{g}) = q \geq 1$. Let (M,g) be a hypersurface of $(\bar{M},\bar{g})$ with $g = \bar{g}|_M$. If the induced metric g on M is degenerate, then M is called a lightlike (null or degenerate) hypersurface ([9], [10], [11]). In this case, there exists a null vector field $\xi \neq 0$ on M such that

$$g(\xi, X) = 0, \qquad \forall\, X \in \Gamma(TM). \tag{3}$$

The radical or the null space of T_xM, at each point $x \in M$, is a subspace $Rad\, T_xM$ defined by

$$Rad\ T_xM = \{\xi \in T_xM : g_x(\xi, X) = 0,\ X \in \Gamma(TM)\}. \tag{4}$$

The dimension of $Rad\, T_xM$ is called the nullity degree of g. We recall that the nullity degree of g for a lightlike hypersurface of $(\bar{M},\bar{g})$ is 1. Since g is degenerate and any null vector being orthogonal to itself, $T_xM^{\perp}$ is also null and

$$Rad\, T_xM = T_xM \cap T_xM^{\perp}. \tag{5}$$

Since $\dim T_xM^{\perp} = 1$ and $\dim Rad\, T_xM = 1$, we have $Rad\, T_xM = T_xM^{\perp}$. We call $Rad\, TM$ a radical distribution and it is spanned by the null vector field ξ. The complementary vector bundle $S(TM)$ of $Rad\, TM$ in TM is called the screen bundle of M. We note that any screen bundle is non-degenerate. This means that

$$TM = Rad\, TM \perp S(TM), \tag{6}$$

with $\perp$ denoting the orthogonal-direct sum. The complementary vector bundle $S(TM)^{\perp}$ of $S(TM)$ in $T\bar{M}$ is called screen transversal bundle and it has rank 2. Since $Rad\, TM$ is a lightlike subbundle of $S(TM)^{\perp}$ there exists a unique local section N of $S(TM)^{\perp}$ such that

$$\bar{g}(N,N) = 0, \quad \bar{g}(\xi, N) = 1. \tag{7}$$

Note that N is transversal to M and $\{\xi, N\}$ is a local frame field of $S(TM)^{\perp}$ and there exists a line subbundle $ltr(TM)$ of $T\bar{M}$, and it is called the lightlike transversal bundle, locally spanned by N. Hence we have the following decomposition:

$$\begin{aligned} T\bar{M} &= TM \oplus ltr(TM) \\ &= S(TM) \perp Rad\, TM \oplus ltr(TM), \end{aligned} \tag{8}$$

where $\oplus$ is the direct sum but not orthogonal ([9], [10]). From the above decomposition of a semi-Riemannian manifold $\bar{M}$ along a lightlike hypersurface M, we can consider the local quasi-orthonormal field of frames of $\bar{M}$ along M given by

$$\{E_1, \ldots, E_m, \xi, N\},$$

where $\{E_1, \ldots, E_m\}$ is an orthonormal basis of $\Gamma(S(TM))$. Let $\bar{\nabla}$ is the Levi-Civita connection of $(\bar{M},\bar{g})$. In view of the splitting (8), we have the following Gauss and Weingarten formulas, respectively,

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{9}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^t_X N \tag{10}$$

for any $X, Y \in \Gamma(TM)$, where $\nabla_X Y,\ A_N X \in \Gamma(TM)$ and $h(X,Y),\ \nabla^t_X N \in \Gamma(ltr(TM))$. If we set

$$B(X,Y) = \bar{g}(h(X,Y),\xi) \quad , \quad \tau(X) = \bar{g}(\nabla^t_X N, \xi),$$

then (9) and (10) become

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \tag{11}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N, \tag{12}$$

respectively. Here, B and A are called the second fundamental form and the shape operator of the lightlike hypersurface M, respectively, [9]. Let P be the projection of $T(M)$ on $S(T(M))$. Then, for any $X \in \Gamma(TM)$, we can write

$$X = PX + \eta(X)\xi, \tag{13}$$

where η is a 1-form given by

$$\eta(X) = \bar{g}(X, N). \tag{14}$$

From (11), we have

$$(\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y), \tag{15}$$

for all $X,Y,Z \in \Gamma(TM)$, where the induced connection ∇ is a non-metric connection on M. From (6), we have

$$\nabla_X W = \nabla^*_X W + h^*(X,W) = \nabla^*_X W + C(X,W)\xi, \tag{16}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi \tag{17}$$

for all $X \in \Gamma(TM)$, $W \in \Gamma(S(TM))$, where $\nabla^*_X W$ and $A^*_\xi X$ belong to $\Gamma(S(TM))$. Here C, A^*_ξ and ∇^* are called the local second fundamental form, the local shape operator and the induced connection on $S(TM)$, respectively. We also have

$$\begin{gathered} g(A^*_\xi X, W) = B(X,W),\ g(A^*_\xi X, N) = 0 \\ B(X,\xi) = 0, \quad g(A_N X, N) = 0. \end{gathered} \tag{18}$$

Moreover, from the first and third equations of (18), we have

$$A^*_\xi \xi = 0. \tag{19}$$

The mean curvature H of M with respect to an $\{E_i\}$, $i = 1, \ldots m$, orthonormal basis of $\Gamma(S(TM))$ is defined by

$$H = \frac{1}{m}\sum_{i=1}^{m} \varepsilon_i B(E_i, E_i), \quad \varepsilon_i = g(E_i, E_i). \tag{20}$$

3 Lightlike hypersurfaces of a statistical manifold

Let $(\widetilde{M}, \widetilde{g})$ be a semi-Riemannain manifold. If there exists a torsion free connection $\widetilde{D}$ subject to the following identity

$$(\widetilde{D}_X\widetilde{g})(Y,Z) = (\widetilde{D}_Y\widetilde{g})(X,Z) \tag{21}$$

for all $X, Y, Z \in \Gamma(T\widetilde{M})$ then $\widetilde{M}$ is called statistical, [14]. For a statistical manifold $(\widetilde{M}, \widetilde{g})$, the $\widetilde{g}-$ dual of $\widetilde{D}$, denoted by $\widetilde{D}^*$, is defined by the following identity:

$$\widetilde{g}(X, \widetilde{D}^*_Z Y) = Z\widetilde{g}(X,Y) - \widetilde{g}(\widetilde{D}_Z X, Y). \tag{22}$$

It is easy to check that $\widetilde{D}^*$ is torsion free. If $\widetilde{D}^0$ is the Levi-Civita connection of $\widetilde{g}$, then we can write

$$\widetilde{D}^0 = \frac{1}{2}(\widetilde{D} + \widetilde{D}^*). \tag{23}$$

Note that a statistical manifold is represented by $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$.

Let (M, g) be a lightlike hypersurface of a statistical manifold $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$. Then, Gauss and Weingarten formulas with respect to dual connections are given by [14]

$$\widetilde{D}_X Y = D_X Y + B(X,Y)N, \tag{24}$$

$$\widetilde{D}_X N = -A^*_N X + \tau^*(X)N, \tag{25}$$

$$\widetilde{D}^*_X Y = D^*_X Y + B^*(X,Y)N, \tag{26}$$

$$\widetilde{D}^*_X N = -A_N X + \tau(X)N \tag{27}$$

for all $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltrTM)$, where $D_X Y$, $D^*_X Y$, $A_N X$, $A^*_N X \in \Gamma(TM)$ and

$$B(X,Y) = \widetilde{g}(\widetilde{D}_X Y, \xi), \quad \tau^*(X) = \widetilde{g}(\widetilde{D}_X N, \xi),$$

$$B^*(X,Y) = \widetilde{g}(\widetilde{D}^*_X Y, \xi), \quad \tau(X) = \widetilde{g}(\widetilde{D}^*_X N, \xi).$$

Here, D, D^*, B, B^*, A_N and A^*_N are called the induced connections on M, the second fundamental forms and the Weingarten mappings with respect to $\widetilde{D}$ and $\widetilde{D}^*$, respectively. Using Gauss formulas, we obtain

$$\begin{aligned} Xg(Y,Z) &= g(\widetilde{D}_X Y, Z) + g(Y, \widetilde{D}^*_X Z), \\ &= g(D_X Y, Z) + g(Y, D^*_X Z) \\ &+ B(X,Y)\eta(Z) + B^*(X,Z)\eta(Y). \end{aligned} \tag{28}$$

From the equation (28), we have the following result.

Theorem. Let (M, g) be a lightlike hypersurface of a statistical manifold $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$. Then:

(i) Induced connections D and D^* are not dual connections.

(ii) A lightlike hypersurface of a statistical manifold need not to be a statistical manifold with respect to the dual connections.

Using Gauss and Weingarten formulas in (28), we get

$$\begin{aligned} &(D_X g)(Y,Z) + (D^*_X g)(Y,Z) = B(X,Y)\eta(Z) \\ &+B(X,Z)\eta(Y) + B^*(X,Y)\eta(Z). \\ &+B^*(X,Z)\eta(Y) \end{aligned} \tag{29}$$

Proposition. Let (M, g) be a lightlike hypersurface of a statistical manifold $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$. Then the following assertions are true:

(i) Induced connections D and D^* are symmetric connections.

(ii) The second fundamental forms B and B^* are symmetric.

Proof. We know that $T^{\widetilde{D}} = 0$. Moreover,

$$\begin{aligned} T^{\widetilde{D}}(X,Y) &= \widetilde{D}_X Y - \widetilde{D}_Y X - [X,Y] \\ &= D_X Y - D_Y X - [X,Y] \\ &+ B(X,Y)N - B(Y,X)N = 0. \end{aligned} \tag{30}$$

Comparing the tangent and transversal components of (30), we obtain

$$B(X,Y) = B(Y,X), \qquad T^D = 0,$$

where T^D is the torsion tensor field of D. Thus, second fundamental form B is symmetric and induced connection D is symmetric connection.

Similarly, it can be shown that the second fundamental form B^* is symmetric and the induced connection D^* is a symmetric connection.

Let P denote the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (6). Then, we have

$$D_X PY = \nabla_X PY + \overline{h}(X, PY), \tag{31}$$

$$D_X \xi = -\overline{A}_\xi X + \overline{\nabla}^t_X \xi \tag{32}$$

for all $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, where $\nabla_X PY$ and $\overline{A}_\xi X$ belong to $\Gamma(S(TM))$, ∇ and $\overline{\nabla}^t$ are linear connections on $\Gamma(S(TM))$ and $\Gamma(RadTM)$ respectively. Here, $\overline{h}$ and $\overline{A}$ are called screen second fundamental form and screen shape operator of $S(TM)$, respectively. If we define

$$C(X, PY) = g(\overline{h}(X, PY), N), \tag{33}$$

$$\varepsilon(X) = g(\overline{\nabla}^t_X \xi, N), \ \forall X, Y \in \Gamma(TM). \quad (34)$$

One can show that

$$\varepsilon(X) = -\tau(X).$$

Therefore, we have

$$D_X PY = \nabla_X PY + C(X, PY)\xi, \quad (35)$$

$$D_X \xi = -\overline{A}_\xi X - \tau(X)\xi, \ \forall X, Y \in \Gamma(TM). \quad (36)$$

Here $C(X, PY)$ is called the local screen fundamental form of $S(TM)$.

Similarly, the relations of induced dual objects on $S(TM)$ are given by

$$D^*_X PY = \nabla^*_X PY + C^*(X, PY)\xi, \quad (37)$$

$$D^*_X \xi = -\overline{A}^*_\xi X - \tau^*(X)\xi, \ \forall X, Y \in \Gamma(TM). \quad (38)$$

Using (28), (35), (37) and Gauss-Weingarten formulas, the relationship between induced geometric objects are given by

$$B(X, \xi) + B^*(X, \xi) = 0, \ g(A_N X + A^*_N X, N) = 0, \quad (39)$$

$$C(X, PY) = g(A_N X, PY),$$
$$C^*(X, PY) = g(A^*_N X, PY). \quad (40)$$

Now, using the equation (39) we can state the following result.

Proposition. Let (M, g) be a lightlike hypersurface of a statistical manifold $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$. Then second fundamental forms B and B^* are not degenerate.

Additionally, due to $\widetilde{D}$ and $\widetilde{D}^*$ are dual connections we obtain

$$B(X, Y) = g(\overline{A}^*_\xi X, Y) + B^*(X, \xi), \quad (41)$$

$$B^*(X, Y) = g(\overline{A}_\xi X, Y) + B(X, \xi). \quad (42)$$

Using (41) and (42) we get

$$\overline{A}^*_\xi \xi + \overline{A}_\xi \xi = 0.$$

Proposition. Let (M, g) be a lightlike hypersurface of a statistical manifold $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$. Then the screen distribution $(S(TM), g, \nabla, \nabla^*)$ has a statistical structure.

Proof. From (28), for any $X, Y \in \Gamma(S(TM))$ we obtain

$$Xg(Y, Z) = g(D_X Y, Z) + g(Y, D^*_X Z).$$

Using (35) and (37) in the last equation, we get

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z).$$

Thus ∇ and ∇^* are dual connections. Moreover, the torsion tensor of $S(TM)$ with respect to ∇ is given

$$T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Using (35) in the last equation we obtain $T^\nabla = 0$. Similarly, the torsion tensor of $S(TM)$ with respect to ∇^* is equal to zero. Also, using (35) we have $(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z)$.

Proposition. Let (M, g) be a lightlike hypersurface of a statistical manifold $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$. Then the following assertions are equivalent:

(i) The screen distribution $S(TM)$ is parallel.

(ii) $C(X, Y) = 0$ for all $X, Y \in \Gamma(S(TM))$.

(iii) $C^*(X, Y) = 0$ for all $X, Y \in \Gamma(S(TM))$.

Proof. For any $X, Y \in \Gamma(S(TM))$, from Gauss-Weingarten formulas and (40), we obtain

$$g(D^*_X Y, N) = C^*(X, Y), \quad (43)$$

$$g(D_X Y, N) = C(X, Y), \quad (44)$$

Then, the proof is completed.

Proposition. Let (M, g) be a lightlike hypersurface of a statistical manifold $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$. Then the following assertions are equivalent:

(i) The screen distribution $S(TM)$ is integrable.

(ii) $C(Y, X) = C(X, Y)$ for all $X, Y \in \Gamma(S(TM))$.

(iii) $C^*(X, Y) = C^*(Y, X)$ for all $X, Y \in \Gamma(S(TM))$.

Proof. For any $X, Y \in \Gamma(S(TM))$, from Gauss-Weingarten formulas and (40), we obtain

$$g([X, Y], N) = C(X, Y) - C(Y, X). \quad (45)$$

$$g([X, Y], N) = C^*(X, Y) - C^*(Y, X). \quad (46)$$

These equations prove our assertions.

Considering ([11], [16], [20]), we can give the following definition

Definition. Let (M, g) be a hypersurface of a statistical manifold $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$.

(i) M is called totally geodesic with respect to $\widetilde{D}$ if $B = 0$.

(ii) M is called totally geodesic with respect to $\widetilde{D}^*$ if $B^* = 0$.

(iii) M is called totally tangentially umbilical with respect to $\widetilde{D}$ if $B(X, Y) = kg(X, Y)$ for all $X, Y \in \Gamma(TM)$, where k is smooth function.

(iv) M is called totally tangentially umbilical with respect to $\widetilde{D}^*$ if $B^*(X,Y) = k^*g(X,Y)$, for any $X, Y \in \Gamma(TM)$, where k^* is smooth function.

(v) M is called totally normally umbilical with respect to $\widetilde{D}$ if $A^*_N X = kX$ for any $X, Y \in \Gamma(TM)$, where k is smooth function.

(vi) M is called totally normally umbilical with respect to $\widetilde{D}^*$ if $A_N X = k^*X$ for all $X, Y \in \Gamma(TM)$, where k^* is smooth function.

In view of (36), (38), (41) and (42), we have the following proposition.

Proposition. Let (M, g) be a lightlike hypersurface of a statistical manifold $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$. Then the following assertions are equivalent:

(i) M is totally geodesic with respect to $\widetilde{D}$ (resp. M is totally geodesic with respect to $\widetilde{D}^*$).

(ii) $\overline{A}^*_\xi$ vanishes on M (resp. $\overline{A}_\xi$ vanishes on M).

(iii) $RadTM$ is a parallel distribution with respect to $\widetilde{D}$ (resp. $RadTM$ is a parallel distribution with respect to $\widetilde{D}^*$).

(iv) $B^*(X,Y) = g(\overline{A}_\xi X, Y)$ (resp. $B(X,Y) = g(\overline{A}^*_\xi X, Y)$), for all $X, Y \in \Gamma(TM)$.

Next, we have the following

Proposition. Let (M, g) be a lightlike hypersurface of a statistical manifold $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$. Then the following assertions are equivalent:

(i) M is totally geodesic with respect to $\widetilde{D}$ and $\widetilde{D}^*$.

(ii) $\overline{A}_\xi X = \overline{A}^*_\xi X = 0$ for all $X \in \Gamma(TM)$.

(iii) $D_X g + D^*_X g = 0$ for all $X \in \Gamma(TM)$.

(iv) $D_X\xi + D^*_X\xi \in \Gamma(RadTM)$ for all $X \in \Gamma(TM)$.

Proof. From (39), (41) and (42) we get the equivalence of **(i)** and **(ii)**. The equation (29) implies the equivalence of **(i)** and **(iii)**. Next, by using (36) and (38) we have the equivalence of **(ii)** and **(iv)**.

Theorem. Let (M, g) be a lightlike hypersurface of a statistical manifold $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$. Then, M is totally tangentially umbilical with respect to $\widetilde{D}$ and $\widetilde{D}^*$ if and only if

$$\overline{A}^*_\xi X + \overline{A}_\xi X = \rho X, \ \forall X \in \Gamma(TM),$$

where ρ is smooth function.

Proof. Using (41) and (42) we obtain

$$kg(X,Y) = g(\overline{A}^*_\xi X, Y) + B^*(X,\xi), \tag{47}$$

and

$$k^*g(X,Y) = g(\overline{A}_\xi X, Y) + B(X,\xi). \tag{48}$$

If we add the equations (47) and (48) side by side and using (39) we complete the proof.

Proposition. Let (M, g) be a lightlike hypersurface of a statistical manifold $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$. If M is totally normally umbilical with respect to $\widetilde{D}$ and $\widetilde{D}^*$. Then

$$C(X, PY) + C^*(X, PY) = 0, \ \forall X \in \Gamma(TM).$$

Proof. Let k and k^* be smooth functions and let $A^*_N X = kX$ and $A_N X = k^*X$, then using (39) we get $k + k^* = 0$. Thus, from (40) proof is completed.

It is known that M is screen locally conformal lightlike hypersurface of a statistical manifold $\widetilde{M}$ if

$$A_N = \varphi \overline{A}^*_\xi, \ A^*_N = \varphi^* \overline{A}_\xi, \tag{49}$$

where φ and φ^* are non-vanishing smooth functions on M. Using (40) and (49) we get the following proposition.

Proposition. Let (M, g) be a lightlike hypersurface of a statistical manifold $(\widetilde{M}, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$. Then, M is screen locally conformal if and only if

$$C(X,Y) + C^*(X,Y) = \sigma(B(X,Y) + B^*(X,Y)),$$

,for all $X, Y \in \Gamma(S(TM))$,where σ is non-vanishing smooth functions on M.

Now, we give an example.

Example. Let $(R^4_2, \widetilde{g})$ be a 4-dimensional semi-Euclidean space with signature $(-,-,+,+)$ of the canonical basis $(\partial_0, \ldots, \partial_3)$. Consider a hypersurface M of R^4_2 given by

$$x_0 = x_1 + \sqrt{2}\sqrt{x_2^2 + x_3^2}.$$

For simplicity, we set $f = \sqrt{x_2^2 + x_3^2}$. It is easy to check that M is a lightlike hypersurface whose radical distribution $RadTM$ is spanned by

$$\xi = f(\partial_0 - \partial_1) + \sqrt{2}(x_2\partial_2 + x_3\partial_3).$$

Then the lightlike transversal vector bundle is given by

$$ltr(TM) = Span\{N = \frac{1}{4f^2}\{f(-\partial_0 + \partial_1) + \sqrt{2}(x_2\partial_2 + x_3\partial_3)\}\}.$$

It follows that the corresponding screen distribution $S(TM)$ is spanned by

$$\{W_1 = \partial_0 + \partial_1, \ W_2 = -x_3\partial_2 + x_2\partial_3\}.$$

Then, by direct calculations we obtain

$$\widetilde{\nabla}_X W_1 = \widetilde{\nabla}_{W_1} X = 0,$$
$$\widetilde{\nabla}_{W_2} W_2 = -x_2\partial_2 - x_3\partial_3,$$
$$\widetilde{\nabla}_\xi \xi = \sqrt{2}\xi,\ \widetilde{\nabla}_{W_2}\xi = \widetilde{\nabla}_\xi W_2 = \sqrt{2}W_2,$$

for any $X \in \Gamma(TM)$, [11].

We define an affine connection $\widetilde{D}$ as follows

$$\begin{aligned} &\widetilde{D}_X W_1 = \widetilde{D}_{W_1} X = 0,\ \widetilde{D}_{W_2} W_2 = -2x_2\partial_2 \\ &\widetilde{D}_\xi \xi = \sqrt{2}\xi - \sqrt{2}N, \qquad (50) \\ &\widetilde{D}_{W_2}\xi = \widetilde{D}_\xi W_2 = \sqrt{2}W_2 - \sqrt{2}W_1. \end{aligned}$$

Then using (23) we obtain

$$\begin{aligned} &\widetilde{D}^*_X W_1 = \widetilde{D}^*_{W_1} X = 0,\ \widetilde{D}^*_{W_2} W_2 = -2x_3\partial_3 \\ &\widetilde{D}^*_\xi \xi = \sqrt{2}\xi + \sqrt{2}N, \qquad (51) \\ &\widetilde{D}^*_{W_2}\xi = \widetilde{D}^*_\xi W_2 = \sqrt{2}W_2 + \sqrt{2}W_1. \end{aligned}$$

Then $(R^4_2, \widetilde{g}, \widetilde{D}, \widetilde{D}^*)$ is a statistical manifold. Thus, by using Gauss formulas (24) and (26) we obtain

$$\begin{aligned} &B(X, W_1) = B(W_1, X) = 0, \\ &B(W_2, W_2) = -2\sqrt{2}x_2^2,\ B(\xi,\xi) = -\sqrt{2} \\ &B(X, W_2) = B(W_2, X) = 0, \qquad (52) \end{aligned}$$

and

$$\begin{aligned} &B^*(X, W_1) = B^*(W_1, X) = 0, \\ &B^*(W_2, W_2) = -2\sqrt{2}x_3^2,\ B^*(\xi,\xi) = \sqrt{2} \\ &B^*(X, W_2) = B^*(W_2, X) = 0. \qquad (53) \end{aligned}$$

The equations (50), (51), (52) and (53) imply that induced connections D and D^* are symmetric connections and the second fundamental forms B and B^* are symmetric. This verifies Proposition 3. Moreover, the equations $B(\xi,\xi) = -\sqrt{2}$ and $B^*(\xi,\xi) = \sqrt{2}$ show the accuracy of the Proposition 3.

Using (50), (51), (52) and (53) we get

$$\begin{aligned} &D_X W_1 = D_{W_1} X = 0,\ D_\xi \xi = \sqrt{2}\xi, \\ &D_{W_2} W_2 = \frac{\sqrt{2}x_2^2}{2f}(-\partial_0 + \partial_1) \\ &+\frac{1}{4f^2}\{(4x_2^3 - 2x_2)\partial_2 + 4x_3x_2^2\partial_3)\}, \\ &D_{W_2}\xi = D_\xi W_2 = \sqrt{2}W_2 - \sqrt{2}W_1, \quad (54) \end{aligned}$$

and

$$\begin{aligned} &D^*_X W_1 = D^*_{W_1} X = 0,\ D^*_\xi \xi = \sqrt{2}\xi, \\ &D^*_{W_2} W_2 = \frac{\sqrt{2}x_3^2}{2f}(-\partial_0 + \partial_1) \\ &+\frac{1}{4f^2}\{4x_3^2x_2\partial_2 + (4x_3^3 - 2x_3)\partial_3)\}, \\ &D^*_{W_2}\xi = D^*_\xi W_2 = \sqrt{2}W_2 + \sqrt{2}W_1. \end{aligned}$$

(55)

If we choose $X = W_2$, $Y = W_2$ and $Z = \xi$, (54) and (55) indicate that induced connections D^* and D are not dual connections. This verifies Theorem 3.

From (35) and (37), we have

$$\begin{aligned} &C(X, W_1) = C(W_1, X) = 0, \\ &C(W_2, W_2) = -\frac{\sqrt{2}}{2}(\frac{x_2}{f})^2, \\ &C(\xi, W_2) = 0 \qquad (56) \end{aligned}$$

and

$$\begin{aligned} &C^*(X, W_1) = C^*(W_1, X) = 0, \\ &C^*(W_2, W_2) = -\frac{\sqrt{2}}{2}(\frac{x_3}{f})^2, \\ &C^*(\xi, W_2) = 0. \qquad (57) \end{aligned}$$

From (56) and (57), we say that C and C^* are symmetric. Thus we have Proposition 3.

Using (54) and (55) in (35) and (37) we obtain

$$\begin{aligned} &\nabla_X W_1 = \nabla_{W_1} X = 0, \\ &\nabla_{W_2} W_2 = \frac{1}{f^2}\{(2x_2^3 - \frac{x_2}{2})\partial_2 + 2x_3x_2^2\partial_3\}, \\ &\nabla_\xi W_2 = \sqrt{2}W_2 - \sqrt{2}W_1, \qquad (58) \end{aligned}$$

and

$$\begin{aligned} &\nabla^*_X W_1 = \nabla^*_{W_1} X = 0, \\ &\nabla^*_{W_2} W_2 = \frac{1}{f^2}\{2x_3^2x_2\partial_2 + (2x_3^3 - \frac{x_3}{2})\partial_3\}, \\ &\nabla^*_\xi W_2 = \sqrt{2}W_2 + \sqrt{2}W_1. \qquad (59) \end{aligned}$$

From (58) and (59), the torsion tensors vanish with respect to ∇ and ∇^*. Furthermore, ∇ and ∇^* are dual connections. This situation verifies Proposition 3.

4 Curvature tensors of a lightlike hypersurface of a statistical manifold

We denote by $\widetilde{R}$ and $\widetilde{R}^*$ the curvature tensor of $\widetilde{D}$ and $\widetilde{D}^*$, respectively. The curvature tensors satisfy

$$\widetilde{g}(\widetilde{R}^*(X,Y)Z,W) = -\widetilde{g}(\widetilde{R}(X,Y)W,Z). \qquad (60)$$

Using Gauss-Weingarten formulas, the curvature tensors $\widetilde{R}$ and $\widetilde{R}^*$ of the connection $\widetilde{D}$ and $\widetilde{D}^*$ are given by

$$\begin{aligned} \widetilde{R}(X,Y)Z \;=\; & R(X,Y)Z - B(Y,Z)A^*_N X \\ +\; & (B(Y,Z)\tau^*(X) - B(X,Z)\tau^*(Y))N \\ +\; & ((D_X B)(Y,Z) - (D_Y B)(X,Z))N, \\ +\; & B(X,Z)A^*_N Y \qquad (61) \end{aligned}$$

and

$$\begin{aligned}\widetilde{R}^*(X,Y)Z &= R^*(X,Y)Z-B^*(Y,Z)A_NX\\ &+ (B^*(Y,Z)\tau(X)-B^*(X,Z)\tau(Y))N\\ &+ ((D^*_XB^*)(Y,Z)-(D^*_YB^*)(X,Z))N\\ &+ B^*(X,Z)A_NY, \qquad (62)\end{aligned}$$

where R and R^* are the curvature tensor with respect to D and D^*, respectively. Consider curvature tensors $\widetilde{R}$ and $\widetilde{R}^*$ of type $(0,4)$. From the above equation and the Gauss-Weingarten equations for M and $S(TM)$ we obtain

$$\begin{aligned}g(\widetilde{R}(X,Y)Z,PW) &= g(R(X,Y)Z,PW)\\ &- B(Y,Z)C^*(X,PW)\\ &+ B(X,Z)C^*(Y,PW), \qquad (63)\end{aligned}$$

$$\begin{aligned}g(\widetilde{R}^*(X,Y)Z,PW) &= g(R^*(X,Y)Z,PW)\\ &- B^*(Y,Z)C(X,PW)\\ &+ B^*(X,Z)C(Y,PW), \qquad (64)\end{aligned}$$

$$\begin{aligned}g(\widetilde{R}(X,Y)Z,\xi) &= B(Y,Z)\tau^*(X)\\ &+ (D_XB)(Y,Z)-(D_YB)(X,Z)\\ &- B(X,Z)\tau^*(Y) \qquad (65)\end{aligned}$$

$$\begin{aligned}g(\widetilde{R}^*(X,Y)Z,\xi) &= B^*(Y,Z)\tau(X)\\ &- B^*(X,Z)\tau(Y)\\ &+ (D^*_XB^*)(Y,Z)\\ &- (D^*_YB^*)(X,Z), \qquad (66)\end{aligned}$$

$$\begin{aligned}g(\widetilde{R}(X,Y)Z,N) &= g(R(X,Y)Z,N)\\ &- B(Y,Z)g(A^*_NX,N)\\ &+ B(X,Z)g(A^*_NY,N), \qquad (67)\end{aligned}$$

$$\begin{aligned}g(\widetilde{R}^*(X,Y)Z,N) &= g(R^*(X,Y)Z,N)\\ &- B^*(Y,Z)g(A_NX,N)\\ &+ B^*(X,Z)g(A_NY,N), \qquad (68)\end{aligned}$$

$$\begin{aligned}g(\widetilde{R}(X,Y)\xi,N) &= g(R(X,Y)\xi,N)\\ &- B(Y,\xi)g(A^*_NX,N)\\ &+ B(X,\xi)g(A^*_NY,N), \qquad (69)\end{aligned}$$

$$\begin{aligned}g(\widetilde{R}^*(X,Y)\xi,N) &= g(R^*(X,Y)\xi,N)\\ &- B^*(Y,\xi)g(A_NX,N)\\ &+ B^*(X,\xi)g(A_NY,N), \qquad (70)\end{aligned}$$

where

$$g(R(X,Y)\xi,N) = C(Y,\overline{A}_\xi X) - C(X,\overline{A}_\xi Y) -2d\tau(X,Y),$$

$$g(R^*(X,Y)\xi,N) = C^*(Y,\overline{A}^*_\xi X) -C^*(X,\overline{A}^*_\xi Y) - 2d\tau(X,Y).$$

Now, let M be a lightlike hypersurface of a $(m+2)$-dimensional statistical manifold $\widetilde{M}$. We consider the local quasi-orthonormal basis $\{E_i,\xi,N\}$, $i=1,\dots m$, of $\widetilde{M}$ along M, where $\{E_1,\dots,E_m\}$ is an orthonormal basis of $\Gamma(S(TM))$. Then, we obtain

$$\begin{aligned}R^{D(0,2)}(X,Y) &= \sum_{i=1}^{m}\varepsilon_i g(R(X,E_i)Y,E_i)\\ &+ \widetilde{g}(R(X,\xi)Y,N), \qquad (71)\end{aligned}$$

where ε_i denotes the causal character (∓ 1) of respective vector field E_i. Using Gauss-Weingarten equations we have

$$\begin{aligned}g(R(X,E_i)Y,E_i) &= g(\widetilde{R}(X,E_i)Y,E_i)\\ &+ B(E_i,Y)C^*(X,E_i)\\ &- B(X,Y)C^*(E_i,E_i) \qquad (72)\end{aligned}$$

Substituting this in (71), using (40) and (41) we obtain

$$\begin{aligned}R^{D(0,2)}(X,Y) &= \widetilde{Ric}(X,Y) - B(X,Y)trA^*_N\\ &+ g(A^*_NX,\overline{A}^*_\xi Y)\\ &+ g(R(X,\xi)Y,N) \qquad (73)\end{aligned}$$

where $\widetilde{Ric}(X,Y)$ is the Ricci tensor of $\widetilde{M}$ with respect to $\widetilde{D}$. Similarly, dual tensor of M with respect to D^* as follows:

$$\begin{aligned}R^{D^*(0,2)}(X,Y) &= \widetilde{Ric}^*(X,Y) - B^*(X,Y)trA_N\\ &+ g(A_NX,\overline{A}_\xi Y)\\ &+ g(R^*(X,\xi)Y,N) \qquad (74)\end{aligned}$$

From First Bianchi identities and (73) we get

$$\begin{aligned}&R^{D(0,2)}(X,Y) - R^{D(0,2)}(Y,X)\\ &= \sum_{i=1}^{m}\varepsilon_i((B(E_i,Y)C^*(X,E_i)\\ &-B(E_i,X)C^*(Y,E_i) + g(\widetilde{R}(X,Y)E_i,E_i))\\ &+g(\widetilde{R}(X,Y)\xi,N). \qquad (75)\end{aligned}$$

Therefore, $R^{D(0,2)}$ is not symmetric.

The statistical manifold $(\widetilde{M},\widetilde{g})$ is called of constant curvature c if

$$\widetilde{R}(X,Y)Z = c(Y,Z)X - g(X,Z)Y. \qquad (76)$$

Moreover, if $(\widetilde{D},\widetilde{g})$ is a statistical structure of constant c, then using (60) we can easily see that $(\widetilde{D}^*,\widetilde{g})$

is also a statistical structure of constant c . Then, using (40), (41), (69) and (76) in (75) we have

$$R^{D(0,2)}(X,Y) - R^{D(0,2)}(Y,X) \\ = C^*(X, \overline{A}^*_\xi Y) - C^*(Y, \overline{A}^*_\xi X),$$

and similarly

$$R^{D^*(0,2)}(X,Y) - R^{D^*(0,2)}(Y,X) \\ = C(X, \overline{A}_\xi Y) - C(Y, \overline{A}_\xi X).$$

Then we have the following theorem

Theorem. Let (M, g) be a lightlike hypersurface of a statistical manifold $(\widetilde{M}^{n+2}(c), \widetilde{g})$ of constant sectional curvature c. Then the following assertions are true:

(i) The tensor $R^{D(0,2)}(X,Y)$ is symmetric if and only if

$$C^*(X, \overline{A}^*_\xi Y) = C^*(Y, \overline{A}^*_\xi X).$$

(ii) The tensor $R^{D^*(0,2)}(X,Y)$ is symmetric if and only if

$$C(X, \overline{A}_\xi Y) = C(Y, \overline{A}_\xi X).$$

Thus, in view of Propositon 3, we have the following:

Corollary. Let (M, g) be a lightlike hypersurface of a statistical manifold $(\widetilde{M}^{n+2}(c), \widetilde{g})$ of constant sectional curvature c. If $S(TM)$ is parallel then the tensors $R^{D(0,2)}$ and $R^{D^*(0,2)}$ are symmetric with respect to connections D and D^*, respectively.

5 Conclusion

Neural networks are useful for solving many complex optimization problems in electromagnetic theory. In 2019, the Event Horizon Telescope (EHT) collaboration released the first image of a black hole's shadow with the help of deep learning algorithms. This image provides direct evidence for the existence of black holes and the general theory of relativity, and indirectly for the existence of lightlike geometry in the universe. A statistical manifold is the emerging branch of mathematics that generalizes the Riemannian manifold and is used to model information; and also uses differential geometry tools to study statistical inference, loss of information, and prediction. It can be applied to many fields such as statistical manifolds, neural networks, machine learning, and artificial intelligence. On the other hand, the study of lightlike manifolds is one of the most important research areas in differential geometry, with many applications in physics and mathematics, such as general relativity, electromagnetism, and black hole theory.

In this paper, we introduced a new structure on statistical manifolds. This is called lightlike hypersurface of a statistical manifold. We have characterized some tensors of lightlike hypersurfaces on statistical manifolds.

This study, which is made with a new perspective, will open the way for scientists working in the field of differential geometry and physics. Differential geometers and physicists can produce many studies by applying the different types of this structure we developed on any kind of complex manifold.

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