Accurate Average Run Length Analysis for Detecting Changes in a Long-Memory Fractionally Integrated MAX Process Running on EWMA Control Chart

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Abstract: - Numerical evaluation of the average run length (ARL) when detecting changes in the mean of an autocorrelated process running on an exponentially weighted moving average (EWMA) control chart has received considerable attention. However, accurate computation of the ARL of changes in the mean of a long-memory model with an exogenous (*X*) variable, which often occurs in practice, is challenging. Herein, we provide an accurate determination of the ARL for long-memory models such as the fractionally integrated MAX processes (FIMAX) with exponential white noise running on an EWMA control chart by using an analytical formula based on an integral equation. From a computational perspective, the analytical formula approach is accomplished by solving the solution for the integral equation obtained via the Fredholm integral equation of the second kind. Moreover, the existence and uniqueness of the solution for the analytical formula were confirmed via Banach's fixed-point theorem. Its efficacy was compared with that of the ARL derived by using the well-known numerical integral equation (NIE) technique under the same circumstances in terms of the ARL percentage accuracy and computational processing time. The percentage accuracy was 100%, which indicates excellent agreement between the two methods, and the analytical formula also required much less computational processing time. An example to illustrate the effectiveness of the proposed approach with a process involving real data running on an EWMA control chart is also provided herein. The explicit formula method offers an accurate determination of the ARL and a new approach for validating its computation, especially for long-memory scenarios running on EWMA control charts.

Key-Words: - Fractionally Integrated Moving Average with Exogenous Variable, Exponential white noise, Numerical Integral Equation Method.

Received: December 8, 2022. Revised: May 25, 2023. Accepted: June 17, 2023. Published: July 24, 2023.

1 Introduction

Control charts are critical for monitoring processes in the production and manufacturing sectors. They are divided into two categories: memory-less and memory-type charts. The first and most well-known memory-less control chart is the Shewhart control chart introduced in the 1920s, [1], which relies entirely on the present observations without consideration of past ones. This is why the Shewhart control chart is only sensitive to detecting large shifts in a process parameter. On the other hand, both the current and past data are used in the plotting statistic of memory-type charts, of which the exponentially weighted moving average (EWMA) control chart, [2], and the cumulative sum (CUSUM) control chart, [3], are the most well-known. This feature helps them to be sensitive for detecting small-to-moderate shifts in a process parameter. The CUSUM control chart is used to monitor process dispersion while the EWMA control chart is used to monitor changes in the process mean. The EWMA control chart has been widely utilized in a wide range of fields and operations, including healthcare, manufacturing, credit card fraud detection, weather monitoring, and stock exchange trading where the small process shifts may inflict significant financial penalties. For more related works on an EWMA chart, we refer to [4], [5], [6], and therein cited references.

Monitoring the performance of a process is based on a control chart and the distribution of the observations from both simple and complex processes. However, phenomena such as autocorrelation, which often occur in real situations, violate the assumption that the observations are independently and normally distributed. Thus, normal and non-normal distributions must be considered for effective process monitoring.

Autocorrelation in processes can be captured using time series models. An important category of these is the stationary process model, in which it is assumed that the process is stable around a constant mean. This type of model provides a foundation for monitoring processes involving autocorrelation. In the present research, we considered the following fundamental time series models. The conventional Box-Jenkins autoregressive (AR) integrated moving-average (MA) (ARIMA) model can be generalized as the AR fractionally integrated MA (ARFIMA) model, which enables non-integer (fractional) differencing parameter values. The ARFIMA processes contain a fractional differencing parameter (d) that is used to determine whether the model is stationary and invertible, [7], [8], [9], [10], [11]. A complete explanation of long-memory processes is provided in [12].

Numerous applications in fields such as economics, finance, environmental research, and engineering involve long-memory processes. In [13], the study used ARFIMAX models for estimating the realized volatilities in a Dow Jones Industrial Average portfolio. Although there is a relationship between econometric models and economic indicators (variables affecting economic forecasting), the exogenous variable is not affected by other variables in the system, only by external influences such as exchange, interest, and inflation rates, among others. Exogenous variables affect an econometric model when forecasting economic situations. If the forecasting model includes an exogenous variable for economic forecasting and other fields, the model is usually more accurate than the one without it. The EWMA control chart has often been used with long-memory processes involving time series, [14], [15].

The error in a time series model (also called white noise) is defined as the difference between the actual and estimated values. This should be minimized to maximize the accuracy of the model. It is not always the case that the white noise (also called Gaussian white noise) created by autocorrelated data follows a normal distribution. Considering non-Gaussian white noise has been effective in studying many phenomena, such as wind speed, oxygen concentration, and water flow rate. Numerous academicians have concentrated on time series models using non-Gaussian white noise, with exponentially distributed white noise being especially interesting, [16], [17].

Evaluating the performance of the EWMA control chart can be made based on the average run length (ARL), which is the average number of consecutive points in a process that falls within the control limits before an out-of-control signal is given. ARL_0 denotes the in-control ARL value, whereas ARL_1 denotes the out-of-control ARL value. ARL_0 should be the largest value, while ARL_1 should be the smallest value for measuring the performance of charts. The ARL can be computed via Monte Carlo simulation, the Markov Chain approach, or the integral equation technique. There are two types of integral equation techniques: using an analytical formula and the numerical integral equation (NIE) technique. Many researchers have calculated the ARL through the solution of an integral equation. In [18], the authors derived analytical formulas for the ARL for MA(q) processes with exponential white noise running on EWMA and CUSUM control charts. Recently, [19], the authors used the integral equation technique to provide an analytical formula for the ARL of a stationary MAX process running on an EWMA control chart. Finally, in [20], the author derived the ARL for a long-memory seasonality SFIMAX model with exponential white noise running on a CUSUM control chart using analytical formulas. The existence and uniqueness of a solution for the analytical formula of the ARL can be proved by using Banach's fixed-point theorem, [21], [22]. As mentioned above, the research has applied the NIE technique to verify the accuracy of an analytical formula, which is an accepted method for evaluating the performance of control charts.

The main aim of the present study is to derive an analytical formula to accurately compute the ARL for a long-memory FIMA model focusing on an exogenous (X) variable with exponential white noise running on an EWMA control chart and compare its efficacy with that using the well-established NIE method. In addition, the analytical formula for detecting changes in the mean is applied to processes involving real data.

The rest of the article is as follows. In Section 2, we provide brief outlines of the $\text{FIMAX}(d,q,r)$ model with exponential white noise and the EWMA control chart. The ARLs obtained by using the analytical formula and NIE techniques are also provided. In Section 3, a performance comparison of the proposed analytical formula with the NIE technique is provided. An example of a process involving real data is also presented to illustrate the effectiveness of the proposed technique. Section 4 offers conclusions on the study. Finally, the existence and uniqueness of the ARL computation were confirmed via Banach's fixed-point theorem, the details of which are shown in Appendix A.

2 Materials and Methods

Here, brief outlines of the $\text{FIMAX}(d,q,r)$ model with exponential white noise and the EWMA control chart along with ARL computations derived by using the analytical formula and NIE techniques are provided.

2.1 Preliminaries

The long memory fractional integration $\text{MAX}(d,q,r)$, (or $\text{FIMAX}(d,q,r)$) model was chosen for this study because it is stationary (as most processes are in practice) and contains both fractionally integrated and MA components with an exogenous (X) variable. Hence, the effect of each parameter can be examined. In addition, we consider white noise with an exponential distribution.

2.1.1 The Long-Memory FIMAX(d,q,r) Model with Exponential White Noise

Let Y_t be a sequence of a long-memory $\text{FIMAX}(d,q,r)$ model where d is the fractional integration parameter, q is the order of the MA process, and r is the explanatory variable order in the model, [13]. The latter can be written as

$$(1-B)^d(Y_t-\mu)=\Theta(B)\varepsilon_t+\sum_{j=1}^{r}\beta_j X_{jt}, \qquad (1)$$

where μ is the mean of $\{Y_t\}$ $\Theta(B)=1-\theta_1 B-\theta_2 B^2-...-\theta_q B^q$ comprising MA polynomials on backward-shift operator B ($\theta_1,\theta_2,...,\theta_q$ are the coefficients for the MA polynomials), X_{jt} are explanatory variables, β_j are unknown parameters, and ε_t is a white noise process assumed to be exponentially distributed as $\varepsilon_t \sim Exp(v)$ when shift parameter $v>0$. To determine whether the process is long-memory, the fractional (d) can take on non-integer values in the range $(0,0.5)$; this fractional order of integration gives rise to the long-memory FIMAX model, [11].

Since the fractional difference operator $(1-B)^d$ is defined by the expansion

$$(1-B)^d=\sum_{k=0}^{\infty}\binom{d}{k}(-B)^k$$

$$=1-dB-\frac{d(1-d)}{2}B^2-\frac{d(1-d)(2-d)}{6}B^3-..., \qquad (2)$$

for any real value of d, the fractionally integrated white noise process can be defined as

$$(1-B)^d Y_t=\varepsilon_t,$$

$$Y_t-dY_{t-1}-\frac{d(1-d)}{2}Y_{t-2}-\frac{d(1-d)(2-d)}{6}Y_{t-3}-...=\varepsilon_t, \qquad (3)$$

Note that $B^kY_t=Y_{t-k}$ for order k.

Therefore, equations (1) and (3) can be rearranged to satisfy the generalized form of the FIMAX model as follows:

$$Y_t=\mu+\varepsilon_t-\theta_1\varepsilon_{t-1}-\theta_2\varepsilon_{t-2}-...-\theta_q\varepsilon_{t-q}$$
$$+\beta_1X_{1t}+\beta_2X_{2t}+...+\beta_rX_{rt}$$
$$+dY_{t-1}+\frac{d(1-d)}{2}Y_{t-2}+\frac{d(1-d)(2-d)}{6}Y_{t-3}+...,$$

or

$$Y_t=\mu+\varepsilon_t-\sum_{i=1}^{q}\theta_i\varepsilon_{t-j}+\sum_{j=1}^{r}\beta_jX_{jt}$$
$$+dY_{t-1}+\frac{d(1-d)}{2}Y_{t-2}+\frac{d(1-d)(2-d)}{6}Y_{t-3}+..., \qquad (4)$$

where $|\theta_i|<1;\, i=1,2,...,q$ are MA coefficients and $\beta_j;\, j=1,2,...,r$ are coefficients depending on variable r. The initial value of a long-memory $\text{FIMAX}(d,q,r)$ model must satisfy $Y_{t-1},Y_{t-2},Y_{t-3},...,$ and $X_{1t},X_{2t},...,X_{rt}$ = 1. For exponential white noise, the initial value of ε_t is 1. By using this fact, we can apply the generalized form of the $\text{FIMAX}(d,q,r)$ model in equation (4) to the EWMA control chart.

2.1.2 The EWMA Control Charts for Long-Memory FIMAX(d,q,r) Model with Exponential White Noise

The EWMA control chart is exceptional at rapidly detecting small-to-moderate shifts in a process parameter it suitably assigns weights to both the current and the past observations. The EWMA control statistic (Z_t) for monitoring a shift in the process mean is given by

$$Z_t=\begin{cases} Z_0, & t=0 \\ (1-\lambda)Z_{t-1}+\lambda Y_t, & t=1,2,..., \end{cases} \qquad (5)$$

where the initial value $Z_0=Y_0$ (the target process mean), Y_t is the sequence of the $\text{FIMAX}(d,q,r)$ process with exponential white noise and λ is the smoothing parameter (or weighting parameter) satisfying $\lambda\in(0,1]$. In general, a large value (close to one) of the smoothing constant is suitable for detecting a large shift while a small value ($\lambda\in[0.05,0.25]$) is recommended for detect a small shift, [23].

Note that, when λ is large (close to one), a relatively lower weight is given to older data, leading to a short-memory process on the EWMA control chart. Indeed, $\lambda = 1$ is equivalent to the Shewhart control chart. Meanwhile, as the value of λ approaches zero, more weight is given to the older observations than the most recent ones. Thus, for very small values of λ, the EWMA control chart becomes more like the CUSUM control chart, in which observations are weighted equally, [24].

The value of λ and the control limits of the EWMA control chart have a strong impact on its performance. Thus, their values should be carefully chosen by the user to bestow the chart with desirable properties for both in-control and out-of-control situations.

The upper control limit (UCL), control limit (CL), and lower control limit (LCL) of the EWMA control chart are respectively defined as

$$UCL = \mu_0 + L\sigma\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2t}\right]},$$
$$CL = \mu_0,$$
$$LCL = \mu_0 - L\sigma\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2t}\right]}, \quad (6)$$

where μ_0 and σ are the process mean and standard deviation, respectively, and L is the design parameter for the EWMA control chart, the value of which depends on the choice of the smoothing constant λ and the desired value of the in-control ARL. For the EWMA control chart statistic Z_t, an out-of-control signal occurs whenever $Z_t > \text{UCL}$ or $Z_t < \text{LCL}$.

2.2 Computation of the ARL for a Long-Memory FIMAX(*d*, *q*, *r*) Model with Exponential White Noise on a One-Sided EWMA Control Chart

To evaluate the performance of the EWMA control chart in terms of the ARL of a long-memory $\text{FIMAX}(d,q,r)$ model running on it, we derived it using both the analytical formula and NIE techniques based on integral equations while focusing on the upper-sided EWMA. The successive values of the EWMA statistic generated by the long-memory $\text{FIMAX}(d,q,r)$ process in equation (4) can be expressed as

$$Z_t = (1-\lambda)Z_{t-1} + \lambda\mu + \lambda\varepsilon_t - \lambda\sum_{i=1}^{q}\theta_i\varepsilon_{t-j} + \lambda\sum_{j=1}^{r}\beta_j X_{jt} + d\lambda Y_{t-1} + \frac{d(1-d)}{2}\lambda Y_{t-2} + \frac{d(1-d)(2-d)}{6}\lambda Y_{t-3} + ..., \quad (7)$$

where the initial value for monitoring with the EWMA statistic is $Z_0 = \varphi;\ 0 < \varphi < H.$

Let τ_H be the stopping time for detecting when the out-of-control process on an upper-sided EWMA control chart exceeds the given predetermined threshold for the first time; i.e.,

$$\tau_H = \inf\{t > 0;\ Z_t > H\ \}, \quad (8)$$

where H is the predetermined UCL of the EWMA control chart. If Z_t is in the range $0 < Z_t < H$, then the process is in control, which can be defined as

$$((1-\lambda)\varphi - \lambda\begin{pmatrix}\mu - \sum_{i=1}^{q}\theta_i\varepsilon_{t-j} + \sum_{j=1}^{r}\beta_j X_{jt} + dY_{t-1} \\ + \frac{d(1-d)}{2}Y_{t-2} + \frac{d(1-d)(2-d)}{6}Y_{t-3} + ...\end{pmatrix})/\lambda < \varepsilon_t <$$
$$\left(H - (1-\lambda)\varphi - \lambda\begin{pmatrix}\mu - \sum_{i=1}^{q}\theta_i\varepsilon_{t-j} + \sum_{j=1}^{r}\beta_j X_{jt} + dY_{t-1} \\ + \frac{d(1-d)}{2}Y_{t-2} + \frac{d(1-d)(2-d)}{6}Y_{t-3} + ...\end{pmatrix}\right)/\lambda$$

or $l < \varepsilon_t < u.$

2.2.1 Derivation of the ARL as an Analytical Formula based on an Integral Equation

Here, the analytical formula is derived as the solution to an integral equation.

Let $\text{L}(\varphi)$ denote the ARL of a long-memory $\text{FIMAX}(d,q,r)$ model with initial value $Z_0 = \varphi$ running on an EWMA control chart; i.e., $\text{ARL} = \text{L}(\varphi) = E_\infty(\tau_H).$ Function $\text{L}(\varphi)$ can be written in the form

$$\text{L}_p(\varphi) = \{1 - P(l < \varepsilon < u)\}\int_l^u (1 + \text{L}((1-\lambda)\varphi + \lambda\begin{pmatrix}\mu - \sum_{i=1}^{q}\theta_i\varepsilon_{t-j} + \sum_{j=1}^{r}\beta_j X_{jt} + dY_{t-1} \\ + \frac{d(1-d)}{2}Y_{t-2} + \frac{d(1-d)(2-d)}{6}Y_{t-3} + ...\end{pmatrix})f(\varepsilon)d\varepsilon$$
$$= 1 + \int_l^u \text{L}((1-\lambda)\varphi + \lambda\begin{pmatrix}\mu - \sum_{i=1}^{q}\theta_i\varepsilon_{t-j} + \sum_{j=1}^{r}\beta_j X_{jt} + dY_{t-1} \\ + \frac{d(1-d)}{2}Y_{t-2} + \frac{d(1-d)(2-d)}{6}Y_{t-3} + ...\end{pmatrix})f(\varepsilon)d\varepsilon.$$

$\omega = (1-\lambda)\varphi + \lambda\begin{pmatrix}\mu - \sum_{i=1}^{q}\theta_i\varepsilon_{t-j} + \sum_{j=1}^{r}\beta_j X_{jt} + dY_{t-1} \\ + \frac{d(1-d)}{2}Y_{t-2} + \frac{d(1-d)(2-d)}{6}Y_{t-3} + ...\end{pmatrix}$ can be used to change the integral variable. Thereby, we can obtain the integral equation as

$$\mathrm{L}_p(\varphi) = 1 + \frac{1}{\lambda}\int_0^H \mathrm{L}(\omega) f(\frac{\omega-(1-\lambda)\varphi}{\lambda} - \mu + \sum_{i=1}^{q}\theta_i\varepsilon_{t-j} - \sum_{j=1}^{r}\beta_j X_{jt} - dY_{t-1} - \frac{d(1-d)}{2}Y_{t-2} - \frac{d(1-d)(2-d)}{6}Y_{t-3} + ...)d\omega \quad (9)$$

$$\mathrm{L}_p(\varphi) = 1 + \frac{1}{\lambda}\int_0^H \mathrm{L}(\omega)\left(\frac{1}{v}\exp\left\{-\frac{\omega}{\lambda v}\right\} \times\exp\left\{\frac{(1-\lambda)\varphi}{\lambda v} + \begin{pmatrix}\mu-\sum_{i=1}^{q}\theta_i\varepsilon_{t-j}+\sum_{j=1}^{r}\beta_j X_{jt}+dY_{t-1}\\ +\frac{d(1-d)}{2}Y_{t-2}+\frac{d(1-d)(2-d)}{6}Y_{t-3}+...\end{pmatrix}\frac{1}{\lambda}\right\}\right)d\omega$$

Accordingly, the integral equation is derived from the Fredholm integral equation of the second kind as follows:

$$\mathrm{L}_p(\varphi) = 1 + \frac{1}{\lambda}\int_0^H \mathrm{L}(\omega)\left(\frac{1}{v}\exp\left\{-\frac{\omega}{\lambda v}\right\} \times\exp\left\{\frac{(1-\lambda)\varphi}{\lambda v} + \begin{pmatrix}\mu-\sum_{i=1}^{q}\theta_i\varepsilon_{t-j}+\sum_{j=1}^{r}\beta_j X_{jt}+dY_{t-1}\\ +\frac{d(1-d)}{2}Y_{t-2}+\frac{d(1-d)(2-d)}{6}Y_{t-3}+...\end{pmatrix}\frac{1}{v}\right\}\right)d\omega$$

$$\mathrm{L}_p(\varphi) = 1 + \frac{1}{\lambda}\int_0^H \mathrm{L}(\omega)\left(\exp\left\{-\frac{\omega}{\lambda v}\right\} \times\exp\left\{\frac{(1-\lambda)\varphi}{\lambda v} + \begin{pmatrix}\mu-\sum_{i=1}^{q}\theta_i\varepsilon_{t-j}+\sum_{j=1}^{r}\beta_j X_{jt}+dY_{t-1}\\ +\frac{d(1-d)}{2}Y_{t-2}+\frac{d(1-d)(2-d)}{6}Y_{t-3}+...\end{pmatrix}\frac{1}{v}\right\}\right)d\omega \quad (10)$$

Equation (10) corresponds to the analytical formula, which is defined as

$$\mathrm{L}_p(\varphi) - \frac{1}{\lambda}\int_0^H \mathrm{L}(\omega)\left(\exp\left\{-\frac{\omega}{\lambda v}\right\} \times\exp\left\{\frac{(1-\lambda)\varphi}{\lambda v} + \begin{pmatrix}\mu-\sum_{i=1}^{q}\theta_i\varepsilon_{t-j}+\sum_{j=1}^{r}\beta_j X_{jt}+dY_{t-1}\\ +\frac{d(1-d)}{2}Y_{t-2}+\frac{d(1-d)(2-d)}{6}Y_{t-3}+...\end{pmatrix}\frac{1}{v}\right\}\right)d\omega = 1 \quad (11)$$

where $k(\varphi,g) = \frac{1}{\lambda}\int_0^H \mathrm{L}(\omega)\left(\exp\left\{-\frac{\omega}{\lambda v}\right\}\times\exp\left\{\frac{(1-\lambda)\varphi}{\lambda v} + \begin{pmatrix}\mu-\sum_{i=1}^{q}\theta_i\varepsilon_{t-j}+\sum_{j=1}^{r}\beta_j X_{jt}+dY_{t-1}\\ +\frac{d(1-d)}{2}Y_{t-2}+\frac{d(1-d)(2-d)}{6}Y_{t-3}+...\end{pmatrix}\frac{1}{v}\right\}\right)d\omega$ is a kernel function, $\mathrm{L}:[0,H]\to\mathbb{R}$ as an unknown function, and the mapping T is defined as

$$T\left(\mathrm{L}_p(\varphi)\right) = 1 + \frac{1}{\lambda}\int_0^H \mathrm{L}(\omega)\left(\exp\left\{-\frac{\omega}{\lambda v}\right\} \times\exp\left\{\frac{(1-\lambda)\varphi}{\lambda v} + \begin{pmatrix}\mu-\sum_{i=1}^{q}\theta_i\varepsilon_{t-j}+\sum_{j=1}^{r}\beta_j X_{jt}+dY_{t-1}\\ +\frac{d(1-d)}{2}Y_{t-2}+\frac{d(1-d)(2-d)}{6}Y_{t-3}+...\end{pmatrix}\frac{1}{v}\right\}\right)d\omega \quad (12)$$

This existence and uniqueness of the ARL computation were confirmed via Banach's fixed-point theorem, the details of which are shown in Appendix A.

Theorem 1 (Banach's fixed-point theorem, see, [25]).
Let $M=(M,\vartheta)$ be a complete metric space, then the mapping $T: M\to M$ is said to be a contraction mapping on M if there exists real number $\ell;\ 0\le\ell<1.$, such that

$$\vartheta\left(T(\mathrm{L}_1),T(\mathrm{L}_2)\right)\le\ell\,\vartheta\left(\mathrm{L}_1,\mathrm{L}_2\right) \text{ for } \mathrm{L}_1,\ \mathrm{L}_2\in M.$$

Subsequently, T has a precisely unique fixed point (e.g. unique $\mathrm{L}(.)\in M$ such that $T(\mathrm{L})=\mathrm{L}$). Let

$$Q(\varphi) = \exp\left\{\frac{(1-\lambda)\varphi}{\lambda v} + \begin{pmatrix}\mu-\sum_{i=1}^{q}\theta_i\varepsilon_{t-j}+\sum_{j=1}^{r}\beta_j X_{jt}+dY_{t-1}\\ +\frac{d(1-d)}{2}Y_{t-2}+\frac{d(1-d)(2-d)}{6}Y_{t-3}+...\end{pmatrix}\frac{1}{v}\right\}d\omega,$$

$0\le\varphi\le H.$

Consequently,

$$\mathrm{L}_p(\varphi) = 1 + \frac{Q(\varphi)}{\lambda v}\int_0^H \mathrm{L}(\omega).\exp\left\{-\frac{\omega}{\lambda v}\right\}d\omega,\quad 0\le\varphi\le H.$$

Let $\eta = \int_0^H \mathrm{L}(\omega).\exp\left\{-\frac{\omega}{\lambda v}\right\}d\omega,$ so we have

$$\mathrm{L}_p(\varphi) = 1 + \frac{Q(\varphi)}{\lambda v}.\eta$$

By solving constant $\eta = \int_0^H \mathrm{L}(\omega).\exp\left\{-\frac{\omega}{\lambda v}\right\}d\omega,$ we obtain

$$\eta = \int_0^H \left(1+\frac{Q(\omega)}{\lambda v}\eta\right).\exp\left\{-\frac{\omega}{\lambda v}\right\}d\omega$$

$$\eta = \int_0^H \exp\left\{-\frac{\omega}{\lambda v}\right\}d\omega + \frac{\eta}{\lambda v}\int_0^H Q(\omega)\exp\left\{-\frac{\omega}{\lambda v}\right\}d\omega$$

$$\eta = \int_0^H \exp\left\{-\frac{\omega}{\lambda v}\right\}d\omega + \frac{\eta}{\lambda v}\int_0^H \exp\left\{\frac{(1-\lambda)\omega}{\lambda v} + \begin{pmatrix}\mu-\sum_{i=1}^{q}\theta_i\varepsilon_{t-j}+\sum_{j=1}^{r}\beta_j X_{jt}+dY_{t-1}\\ +\frac{d(1-d)}{2}Y_{t-2}+\frac{d(1-d)(2-d)}{6}Y_{t-3}+...\end{pmatrix}\frac{1}{v}.\exp\left\{-\frac{\omega}{\lambda v}\right\}d\omega\right.$$

$$\eta = \lambda v\left(1-\exp\left\{-\frac{H}{\lambda v}\right\}\right) + \frac{\eta}{\lambda}\exp\left\{\frac{1}{v}\begin{pmatrix}\mu-\sum_{i=1}^{q}\theta_i\varepsilon_{t-j}+\sum_{j=1}^{r}\beta_j X_{jt}+dY_{t-1}\\ +\frac{d(1-d)}{2}Y_{t-2}+\frac{d(1-d)(2-d)}{6}Y_{t-3}+...\end{pmatrix}\right\}.\left(1-\exp\left\{-\frac{H}{v}\right\}\right)$$

$$\eta - \frac{\eta}{\lambda}\exp\left\{\frac{1}{v}\begin{pmatrix}\mu-\sum_{i=1}^{q}\theta_i\varepsilon_{t-j}+\sum_{j=1}^{r}\beta_j X_{jt}+dY_{t-1}\\ +\frac{d(1-d)}{2}Y_{t-2}+\frac{d(1-d)(2-d)}{6}Y_{t-3}+...\end{pmatrix}\right\}.\left(1-\exp\left\{-\frac{H}{v}\right\}\right) = \lambda v\left(1-\exp\left\{-\frac{H}{\lambda v}\right\}\right)$$

Hence, $\eta = \lambda v\left(1-\exp\left\{-\frac{H}{\lambda v}\right\}\right) \Big/ \left[1-\frac{1}{\lambda}\right.$

$$\left.\times\exp\left\{\frac{1}{v}.\begin{pmatrix}\mu-\sum_{i=1}^{q}\theta_i\varepsilon_{t-j}+\sum_{j=1}^{r}\beta_jX_{jt}+dY_{t-1}\\ +\frac{d(1-d)}{2}Y_{t-2}+\frac{d(1-d)(2-d)}{6}Y_{t-3}+...\end{pmatrix}\right\}.\left(1-\exp\left\{-\frac{H}{v}\right\}\right)\right] \quad (13)$$

Substitute a constant η into the Equation $L_p(\varphi) = 1 + \frac{Q(\varphi)}{\lambda v}.\eta$, then $L_p(\varphi)$ as

$$Q(\varphi) = \exp\left\{\frac{(1-\lambda)\varphi}{\lambda v}+\begin{pmatrix}\mu-\sum_{i=1}^{q}\theta_i\varepsilon_{t-j}+\sum_{j=1}^{r}\beta_jX_{jt}+dY_{t-1}\\ +\frac{d(1-d)}{2}Y_{t-2}+\frac{d(1-d)(2-d)}{6}Y_{t-3}+...\end{pmatrix}\frac{1}{v}\right\}$$

where $0\le\varphi\le H$.

$$L_p(\varphi) = 1+\left[\exp\left\{\frac{(1-\lambda)\varphi}{\lambda v}+\begin{pmatrix}\mu-\sum_{i=1}^{q}\theta_i\varepsilon_{t-j}+\sum_{j=1}^{r}\beta_jX_{jt}+dY_{t-1}\\ +\frac{d(1-d)}{2}Y_{t-2}+\frac{d(1-d)(2-d)}{6}Y_{t-3}+...\end{pmatrix}.\frac{1}{v}\right\}\right.$$
$$\left.\times\lambda\left(1-\exp\left\{-\frac{H}{\lambda v}\right\}\right)\right]\left[\lambda\left(1-\frac{1}{\lambda}\left(1-\exp\left\{-\frac{H}{v}\right\}\right.\right.\right.$$
$$\left.\left.\left.\times\exp\left\{\begin{pmatrix}\mu-\sum_{i=1}^{q}\theta_i\varepsilon_{t-j}+\sum_{j=1}^{r}\beta_jX_{jt}+dY_{t-1}\\ +\frac{d(1-d)}{2}Y_{t-2}+\frac{d(1-d)(2-d)}{6}Y_{t-3}+...\end{pmatrix}.\frac{1}{v}\right\}\right)\right)\right]^{-1}$$

Thereby, $L_p(\varphi)$ can be written in the form

$$L_p(\varphi) = 1-\left[\lambda\left(1-\exp\left\{-\frac{H}{\lambda v}\right\}\right).\exp\left\{\frac{(1-\lambda)\varphi}{\lambda v}\right\}\right]$$
$$\left[\left(1-\exp\left\{-\frac{H}{v}\right\}-\lambda\exp\left\{-\frac{1}{v}\begin{pmatrix}\mu-\sum_{i=1}^{q}\theta_i\varepsilon_{t-j}+\sum_{j=1}^{r}\beta_jX_{jt}+dY_{t-1}\\ +\frac{d(1-d)}{2}Y_{t-2}+\frac{d(1-d)(2-d)}{6}Y_{t-3}+...\end{pmatrix}\right\}\right)\right]^{-1} \quad (14)$$

According to equation (14), when the process is in control, the parameter v can be replaced with v_0. Subsequently, the analytical formula for the in-control ARL becomes

$$ARL_0 = 1-\left[\lambda\left(1-\exp\left\{-\frac{H}{\lambda v_0}\right\}\right).\exp\left\{\frac{(1-\lambda)\varphi}{\lambda v_0}\right\}\right]$$
$$\left[\left(1-\exp\left\{-\frac{H}{v_0}\right\}-\lambda\exp\left\{-\frac{1}{v_0}\begin{pmatrix}\mu-\sum_{i=1}^{q}\theta_i\varepsilon_{t-j}+\sum_{j=1}^{r}\beta_jX_{jt}+dY_{t-1}\\ +\frac{d(1-d)}{2}Y_{t-2}+\frac{d(1-d)(2-d)}{6}Y_{t-3}+...\end{pmatrix}\right\}\right)\right]^{-1} \quad (15)$$

On the contrary, for the out-of-control process, the parameter v can be replaced with v_1. Therefore, the analytical formula for the out-of-control ARL can be written as

$$ARL_1 = 1-\left[\lambda\left(1-\exp\left\{-\frac{H}{\lambda v_1}\right\}\right).\exp\left\{\frac{(1-\lambda)\varphi}{\lambda v_1}\right\}\right]$$
$$\left[\left(1-\exp\left\{-\frac{H}{v_1}\right\}-\lambda\exp\left\{-\frac{1}{v_1}\begin{pmatrix}\mu-\sum_{i=1}^{q}\theta_i\varepsilon_{t-j}+\sum_{j=1}^{r}\beta_jX_{jt}+dY_{t-1}\\ +\frac{d(1-d)}{2}Y_{t-2}+\frac{d(1-d)(2-d)}{6}Y_{t-3}+...\end{pmatrix}\right\}\right)\right]^{-1} \quad (16)$$

This ARL derived from the analytical formula shows that the calculation scheme can be easily performed.

2.2.1 The Approximate ARL by using the NIE Technique based on an Integral Equation

The NIE technique is usually used to verify the accuracy of an analytical formula. It is based on the solution for the integral equation, [26], in equation (9). The composite midpoint Rule is applied to divide domain interval $[0,H]$ into m sub-grids of equal length; i.e.,

$$\int_0^H L(\omega)f\Big(\frac{\omega-(1-\lambda)\varphi}{\lambda}-\mu+\sum_{i=1}^{q}\theta_i\varepsilon_{t-j}-\sum_{j=1}^{r}\beta_jX_{jt}-dY_{t-1}$$
$$-\frac{d(1-d)}{2}Y_{t-2}-\frac{d(1-d)(2-d)}{6}Y_{t-3}+...\Big)d\omega\approx\sum_{j=1}^{m}w_jf(a_j) \quad (17)$$

We can then substitute equation (17) into equation (9) to obtain a linear system of equations. Thereby, the approximate ARL calculated by using the NIE technique can be written in the form

$$L_N(\varphi)\approx 1+\frac{1}{\lambda}+\sum_{j=1}^{m}w_jL_N(a_j)f\left(\frac{a_j-(1-\lambda)\varphi}{\lambda}-\begin{pmatrix}\mu-\sum_{i=1}^{q}\theta_i\varepsilon_{t-j}+\sum_{j=1}^{r}\beta_jX_{jt}+dY_{t-1}\\ +\frac{d(1-d)}{2}Y_{t-2}+\frac{d(1-d)(2-d)}{6}Y_{t-3}+...\end{pmatrix}\right) \quad (18)$$

with a set of constant weights $w_j=\frac{H}{m}$, and $a_j=\frac{H}{m}\left(j-\frac{1}{2}\right)$; $j=1,2,\ldots,m.$

2.3 Algorithms to establish the in-control and out-of-control ARL values

Algorithms were constructed to determine the control limits and obtain results for the out-of-control ARL.

2.3.1 Construction of the control limits

Algorithm 1: The analytical formula derived by using the Mathematica program to establish the in-control ARL value

Step-1: Solve the generalized form of the long-memory FIMAX(0.05, 1, 1) model with exponential white noise defined as Y_t:
- (1.1) MA coefficients $\theta_1 = \pm 0.8, \pm 0.4, \pm 0.2.$
- (1.2) Exogenous variable coefficient $\beta_1 = 1.$
- (1.3) The mean of the exponential parameter values $(\varepsilon_t \sim Exp(\nu))$ for the in-control process $(\nu = \nu_0)$ = 1.

Step-2: Compute the EWMA statistic:
- (2.1) Smoothing parameter λ = 0.01.
- (2.2) Compute the proposed EWMA statistic (Z_t) for the long-memory FIMAX mode given in equation (5).

Step-3: Compute decision interval H in conjunction with λ by utilizing equation (15) so that the attained in-control ARL is close to or equal to 500 corresponding to the specific shift size $(\delta) = 0$.

Step-4: Repeat Steps 2 and 3 for λ = 0.05 and 0.10

Step-5: Repeat Steps 1–4 for long-memory FIMAX(0.2, 1, 1) and FIMAX(0.40, 1, 1).

2.3.2 Computation of the out-of-control ARL

Algorithm 2: Analytical formula derived from Mathematica program for a shift in the process mean from ν_0 to ν_1, where $\nu_1 = (1+\delta)\nu_0$

Step-1: Repeat Algorithm 1, Steps 1 and 2 to solve the generalized form of a long-memory FIMAX(0.05, 1, 1) process running on an EWMA control chart.

Step-2: Compute the out-of-control ARL for changes in the process mean:
- (2.1) Take the value of the control coefficient (H,λ) from the output of Algorithm 1.
- (2.2) Computation of the out-of-control ARL corresponding to (H,λ) a shift size of 0.01 by utilizing equation (16).

Step-3: Record the computational time for the first out-of-control ARL signal from the control limits.

Step-4: Repeat Steps 2 and 3 for δ = 0.05, 0.25, 0.50, 0.75, or 1.00.

Step-5: Repeat Steps 1-4 for long-memory FIMAX(0.2, 1, 1) and FIMAX(0.40, 1, 1).

3 Results and Discussion

The details and results of a comparative study of the performances of the proposed analytical formula with the NIE technique are provided in this section. An example of a process involving real data to illustrate the effectiveness of the proposed technique is also offered.

Table 1. The values of H for various $\text{FIMAX}(d, q, r)$ models and values θ_1 for in-control $\text{ARL} = 500$.

Coefficient parameters			λ		
d	θ_1	β_1	0.01	0.05	0.10
0.05	0.80	0.10	2.50211521E-13	2.57482300E-07	1.12643400E-02
	0.40	0.10	1.67720900E-13	1.72595400E-07	7.41667000E-03
	0.20	0.10	1.37319000E-13	1.41309200E-07	6.03307000E-03
	-0.20	0.10	9.20490000E-14	9.47223000E-08	4.00579000E-03
	-0.40	0.10	7.53640000E-14	7.75521000E-08	3.26830000E-03
	-0.80	0.10	5.05179300E-14	5.19847000E-08	2.17959000E-03
0.20	0.80	0.10	1.97060000E-13	2.02783300E-07	8.76905000E-03
	0.40	0.10	1.32090000E-13	1.35929700E-07	5.79698000E-03
	0.20	0.10	1.08148495E-13	1.11289800E-07	4.72230000E-03
	-0.20	0.10	7.24938000E-14	7.45998000E-08	3.14201000E-03
	-0.40	0.10	5.93510000E-14	6.10771000E-08	2.56548000E-03
	-0.80	0.10	3.97871000E-14	4.09412000E-08	1.71278400E-03
0.40	0.80	0.10	1.52550000E-13	1.56983000E-07	6.72397000E-03
	0.40	0.10	1.02258000E-13	1.05228900E-07	4.45961000E-03
	0.20	0.10	8.37213700E-14	8.61542000E-08	3.63712200E-03
	-0.20	0.10	5.61205000E-14	5.77508000E-08	2.42414100E-03
	-0.40	0.10	4.59500000E-14	4.72823000E-08	1.98057000E-03
	-0.80	0.10	3.08010000E-14	3.16943000E-08	1.32350000E-03

The performance metric for the comparison is

$$\%\,\text{Accuracy} = 100 - \left| \frac{L_p(\varphi) - L_N(\varphi)}{L_p(\varphi)} \right| \times 100\%, \quad (19)$$

where $L_p(\varphi)$ and $L_N(\varphi)$ are the ARL values obtained by using the analytical formula and NIE techniques, respectively. A value greater than 95% means that the proposed formula provided an out-of-control ARL value close to that for the NIE technique, which indicates good agreement between them.

The first task was to compute the value of decision interval H in conjunction with the selection of λ so that the in-control ARL is close to the target value (500 in this case, which is commonly used in the statistical process monitoring) The values for H for various models and values of the coefficient parameter using Algorithm 1 are reported in Table 1.

3.1 Performance Comparison

Using the models and parameter values in Table 1, we computed the out-of-control ARL obtained by using the analytical formula and NIE techniques for FIMAX(0.05, 1, 1), FIMAX(0.20, 1, 1), and FIMAX(0.40, 1, 1) models running on an EWMA control chart, the results for which are reported in Table 2, Table 3, and Table 4. The first row of each cell in the tables shows the out-of-control ARL values using the analytical formula and NIE techniques corresponding to shift magnitudes of 0.01, 0.05, 0.25, 0.50, 0.75, or 1.00 (in that order) and the second row shows the computational time (seconds). For each value of the smoothing parameter (λ) (see first column) and MA coefficient (see second column), the best-performing technique is indicated in bold. Moreover, similar performances of the two techniques are indicated in the percentage accuracy (Acc%) column by gray shading.

The results suggest that the out-of-control ARL values calculated by using the analytical formula are close to those approximated by using the NIE technique. As expected, as the shift size was increased, the sensitivity (i.e., the out-of-control ARL values) of both techniques also increased. In particular, both techniques showed great sensitivity by quickly detecting small-to-moderate shifts in the process mean $(0 < \delta \le 0.5)$ but not moderate-to-large and large shifts $(0.50 < \delta \le 1.00)$.

The precision values of the proposed analytical formula compared to the NIE technique in terms of percentage accuracy are reported in Table 2, Table 3, and Table 4.

It can be seen that the percentage accuracy results were 100% in all cases, implying good agreement between the two methods and that the proposed analytical formula is very accurate.

The computational times for calculating the out-of-control ARL values only took a fraction of a second with the analytical formula compared to 3–120 seconds with the NIE technique. As θ was decreased, the computational time increased inversely with the out-of-control ARL value.

It can be seen that the lowest out-of-control ARL values occurred with the following long-memory models: FIMAX(0.40, 2, 1), FIMAX (0.20, 1, 1), and FIMAX(0.05, 1, 1).

The out-of-control ARL values for FIMAX(0.4, 1, 1) with different values of coefficient parameter θ_1 are shown in Figure 1. The results reveal that out-of-control ARL values tended to decrease rapidly when the magnitude of the shift was small $(\delta \le 0.25)$, followed by small-to-moderate shifts $(0.25 < \delta \le 0.50)$ for all cases. The green line for λ = 0.01 indicates the lowest out-of-control ARL value. Meanwhile, the out-of-control ARL values were for all shift sizes and levels of λ.

In summary, the analytical formula performed exceptionally well in detecting small-to-moderate changes in the mean of a long-memory FIMAX model running on the EWMA control chart. Its accuracy was confirmed by comparison with the well-established NIE technique. Moreover, it could compute out-of-control ARL values much more quickly than with the NIE technique.

3.2 Application of the Proposed Technique to Processes Involving Real Data

For this demonstration, we used movements in the gold futures price, [27], with the UDS/THB exchange rate, [28], as the exogenous variable. As the USD/THB exchange rate increased (i.e., the Thai baht depreciates), the price of gold decreased, and vice versa. The dataset covers the period from September 1, 2001, to January 1, 2023, and comprises 257 daily observations. We tested whether the dataset can fit a long-memory process and the distribution of the white noise by utilizing the statistical software packages Eviews and SPSS, respectively.

DOI: 10.37394/23206.2023.22.58

Table 2. Out-of-control ARL values are computed by using the analytical formula and NIE technique for FIMAX($d = 0.05, 1, 1$) running on an EWMA control chart when the in-control ARL is 500.

λ	θ_1	δ																	
		0.01		Acc %	0.05		Acc %	0.25		Acc %	0.50		Acc %	0.75		Acc %	1.00		Acc %
		$L_p(u)$	$L_N(u)$		$L_p(u)$	$L_N(u)$		$L_p(u)$	$L_N(u)$		$L_p(u)$	$L_N(u)$		$L_p(u)$	$L_N(u)$		$L_p(u)$	$L_N(u)$	
0.01	0.8	361.892	361.892	100%	105.922	105.922	100%	1.701	1.701	100%	1.009	1.009	100%	1.000	1.000	100%	1.000	1.000	100%
		(0.001)	(3.95)		(0.001)	(4.25)		(0.001)	(5.96)		(0.001)	(7.22)		(0.001)	(9.10)		(0.001)	(11.53)	
	0.4	360.462	360.462	100%	103.943	103.943	100%	1.647	1.647	100%	1.007	1.007	100%	1.000	1.000	100%	1.000	1.000	100%
		(0.001)	(14.02)		(0.001)	(15.58)		(0.001)	(17.42)		(0.001)	(19.12)		(0.001)	(21.45)		(0.001)	(22.88)	
	0.2	359.749	359.749	100%	102.966	102.966	100%	1.622	1.622	100%	1.007	1.007	100%	1.000	1.000	100%	1.000	1.000	100%
		(0.001)	(24.11)		(0.001)	(25.69)		(0.001)	(27.45)		(0.001)	30.05)		(0.001)	(31.11)		(0.001)	(32.98)	
	-0.2	358.338	358.338	100%	101.046	101.046	100%	1.574	1.574	100%	1.006	(1.006	100%	1.000	1.000	100%	1.000	1.000	100%
		(0.001)	(34.05)		(0.001)	(36.48)		(0.001)	(39.81)		(0.001)	41.5)		(0.001)	(43.14)		(0.001)	(44.80)	
	-0.4	357.632	357.632	100%	100.098	100.098	100%	1.551	1.551	100%	1.005	(1.005	100%	1.000	1.000	100%	1.000	1.000	100%
		(0.001)	(46.85)		(0.001)	(49.13)		(0.001)	(52.1)		(0.001)	(53.98)		(0.001)	(54.22)		(0.001)	(57.68)	
	-0.8	**356.218**	**356.218**	100%	**98.225**	**98.225**	100%	**1.509**	**1.509**	100%	**1.004**	**1.004**	100%	**1.000**	**1.000**	100%	**1.000**	**1.000**	100%
		(0.001)	(58.46)		(0.001)	(60.22)		(0.001)	(63.59)		(0.001)	(65.87)		(0.001)	(67.23)		(0.001)	(69.98)	
0.05	0.8	412.819	412.819	100%	198.975	198.975	100%	11.0908	11.0908	100%	1.7243	1.7243	100%	1.108	1.108	100%	1.025	1.03	100%
		(0.001)	)4.11)		(0.001)	(5.782)		(0.001)	(7.42)		(0.001)	(9.094)		(0.001)	(10.79)		(0.001)	(12.55)	
	0.4	411.191	411.191	100%	195.239	195.239	100%	10.3149	10.3149	100%	1.6339	1.6339	100%	1.091	1.091	100%	1.021	1.021	100%
		(0.001)	(14.23)		(0.001)	(15.89)		(0.001)	(17.61)		(0.001)	(19.42)		(0.001)	(21.11)		(0.001)	(22.87)	
	0.2	410.379	410.379	100%	193.398	193.398	100%	9.9598	9.9598	100%	1.593	1.593	100%	1.083	1.083	100%	1.019	1.019	100%
		(0.001)	(24.68)		(0.001)	(26.36)		(0.001)	(27.98)		(0.001)	(29.65)		(0.001)	(31.45)		(0.001)	(33.15)	
	-0.2	408.762	408.762	100%	193.398	189.768	100%	9.262	9.262	100%	1.519	1.519	100%	1.070	1.070	100%	1.015	1.015	100%
		(0.001)	(34.81)		(0.001)	(36.48)		(0.001)	(39.81)		(0.001)	(41.5)		(0.001)	(43.14)		(0.001)	(44.8)	
	-0.4	407.955	407.955	100%	187.979	187.979	100%	8.938	8.938	100%	1.486	1.486	100%	1.064	1.064	100%	1.014	1.014	100%
		(0.001)	(48.23)		(0.001)	(49.93)		(0.001)	(51.56)		(0.001)	(53.2)		(0.001)	(54.87)		(0.001)	(56.56)	
	-0.8	406.347	406.347	100%	184.451	184.451	100%	8.327	8.327	100%	1.425	1.425	100%	1.054	1.054	100%	1.011	1.011	100%
		(0.001)	(58.39)		(0.001)	(60.05)		(0.001)	(65.01)		(0.001)	(66.67)		(0.001)	(68.36)		(0.001)	(70.00)	
0.10	0.8	455.816	455.816	100%	320.169	320.169	100%	75.9308	75.9308	100%	21.441	21.441	100%	8.885	8.885	100%	4.789	4.789	100%
		(0.001)	(71.74)		(0.001)	(73.45)		(0.001)	(75.11)		(0.001)	(78.47)		(0.001)	(80.12)		(0.001)	(81.87)	
	0.4	453.823	453.823	100%	313.505	313.505	100%	69.6153	69.615	100%	18.668	18.668	100%	7.543	7.543	100%	4.051	4.051	100%
		(0.001)	(83.53)		(0.001)	(85.25)		(0.001)	(86.9)		(0.001)	(88.56)		(0.001)	(90.19)		(0.001)	(91.89)	
	0.2	452.842	452.842	100%	310.269	310.269	100%	66.7045	66.7045	100%	17.445	17.445	100%	6.970	6.970	100%	3.743	3.743	100%
		(0.001)	(93.64)		(0.001)	(95.30)		(0.001)	(96.97)		(0.001)	(98.8)		(0.001)	(100.47)		(0.001)	(102.11)	
	-0.2	450.903	450.903	100%	303.957	303.957	100%	61.3078	61.3078	100%	15.274	15.274	100%	5.982	5.982	100%	3.223	3.223	100%
		(0.001)	(103.75)		(0.001)	(105.50)		(0.001)	(107.14)		(0.001)	(110.45)		(0.001)	(112.28)		(0.001)	(113.97)	
	-0.4	449.944	449.944	100%	300.872	300.872	100%	58.8005	58.8005	100%	14.307	14.307	100%	5.555	5.555	100%	3.003	3.003	100%
		(0.001)	(115.69)		(0.001)	(119.11)		(0.001)	(120.83)		(0.001)	(122.5)		(0.001)	(124.15)		(0.001)	(125.84)	
	-0.8	448.033	448.033	100%	294.827	294.827	100%	54.1259	54.1259	100%	12.579	12.579	100%	4.813	4.813	100%	2.629	2.629	100%
		(0.001)	(127.62)		(0.001)	(129.26)		(0.001)	(130.9)		(0.001)	(132.59)		(0.001)	(134.41)		(0.001)	(136.11)	

Note: The numerical results in parentheses are computational times in seconds

Table 3. Out-of-control ARL values are computed by using the analytical formula and NIE technique for FIMAX($d = 0.2, 1, 1$) running on an EWMA control chart when the in-control ARL is 500.

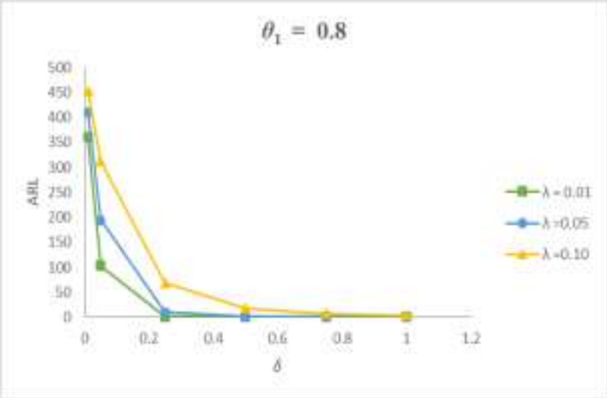
λ	θ_1	δ																	
		0.01		Acc %	0.05		Acc %	0.25		Acc %	0.50		Acc %	0.75		Acc %	1.00		Acc %
		$L_p(u)$	$L_N(u)$		$L_p(u)$	$L_N(u)$		$L_p(u)$	$L_N(u)$		$L_p(u)$	$L_N(u)$		$L_p(u)$	$L_N(u)$		$L_p(u)$	$L_N(u)$	
0.01	0.8	361.039 (0.001)	361.039 (5.44)	100%	104.738 (0.001)	104.738 (7.15)	100%	1.668 (0.001)	1.668 (9.25)	100%	1.007 (0.001)	1.007 (11.25)	100%	1.000 (0.001)	1.000 (12.15)	100%	1.000 (0.001)	1.000 (13.25)	100%
	0.4	359.611 (0.001)	359.611 (15.22)	100%	102.777 (0.001)	102.777 (16.25)	100%	1.617 (0.001)	1.617 (19.56)	100%	1.006 (0.001)	1.006 (20.15)	100%	1.000 (0.001)	1.000 (21.55)	100%	1.000 (0.001)	1.000 (24.12)	100%
	0.2	358.909 (0.001)	358.909 (25.2)	100%	101.814 (0.001)	101.814 (27.55)	100%	1.592 (0.001)	1.592 (29.1)	100%	1.006 (0.001)	1.006 (30.56)	100%	1.000 (0.001)	1.000 (31.22)	100%	1.000 (0.001)	1.000 (32.96)	100%
	-0.2	357.493 (0.001)	357.493 (34.96)	100%	99.912 (0.001)	99.912 (37.15)	100%	1.547 (0.001)	1.547 (38.2)	100%	1.005 (0.001)	1.005 (39.68)	100%	1.000 (0.001)	1.000 (41.24)	100%	1.000 (0.001)	1.000 (42.56)	100%
	-0.4	356.775 (0.001)	356.775 (44.15)	100%	98.972 (0.001)	98.972 (45.89)	100%	1.525 (0.001)	1.525 (48.05)	100%	1.005 (0.001)	1.005 (49.78)	100%	1.000 (0.001)	1.000 (51.13)	100%	1.000 (0.001)	1.000 (52.98)	100%
	-0.8	**355.385** (0.001)	**355.385** (54.15)	100%	**97.132** (0.001)	**97.132** (55.39)	100%	**1.485** (0.001)	**1.485** (57.15)	100%	**1.004** (0.001)	**1.004** (59.45)	100%	**1.000** (0.001)	**1.000** (62.78)	100%	**1.000** (0.001)	**1.000** (63.85)	100%
0.05	0.8	411.846 (0.001)	411.846 (5.79)	100%	196.736 (0.001)	196.736 (7.48)	100%	10.620 (0.001)	10.620 (9.07)	100%	1.669 (0.001)	1.669 (10.68)	100%	1.097 (0.001)	1.097 (12.34)	100%	1.023 (0.001)	1.023 (13.95)	100%
	0.4	410.223 (0.001)	410.223 (15.54)	100%	193.043 (0.001)	193.043 (17.15)	100%	9.881 (0.001)	9.881 (18.79)	100%	1.585 (0.001)	1.585 (20.43)	100%	1.082 (0.001)	1.082 (22.04)	100%	1.018 (0.001)	1.018 (23.65)	100%
	0.2	409.413 (0.001)	409.413 (25.23)	100%	191.223 (0.001)	191.223 (26.84)	100%	9.532 (0.001)	9.532 (28.45)	100%	1.548 (0.001)	1.548 (30.15)	100%	1.075 (0.001)	1.075 (31.79)	100%	1.017 (0.001)	1.017 (33.54)	100%
	-0.2	407.799 (0.001)	407.799 (35.2)	100%	187.634 (0.001)	187.634 (36.79)	100%	8.876 (0.001)	8.876 (38.39)	100%	1.479 (0.001)	1.479 (40.11)	100%	1.063 (0.001)	1.063 (41.75)	100%	1.014 (0.001)	1.014 (43.55)	100%
	-0.4	406.994 (0.001)	406.994 (44.96)	100%	185.865 (0.001)	185.865 (46.56)	100%	8.568 (0.001)	8.568 (48.26)	100%	1.448 (0.001)	1.448 (49.85)	100%	1.058 (0.001)	1.058 (51.45)	100%	1.012 (0.001)	1.012 (53.14)	100%
	-0.8	405.389 (0.001)	405.389 (54.74)	100%	182.377 (0.001)	182.377 (56.34)	100%	7.986 (0.001)	7.986 (57.95)	100%	1.392 (0.001)	1.392 (59.57)	100%	1.049 (0.001)	1.049 (61.18)	100%	1.010 (0.001)	1.010 (62.84)	100%
0.10	0.8	454.619 (0.001)	454.619 (64.43)	100%	316.157 (0.001)	316.157 (66.02)	100%	72.076 (0.001)	72.076 (67.63)	100%	19.729 (0.001)	19.729 (69.34)	100%	8.050 (0.001)	8.050 (70.92)	100%	4.327 (0.001)	4.327 (72.52)	100%
	0.4	452.652 (0.001)	452.652 (74.12)	100%	309.648 (0.001)	309.648 (75.71)	100%	66.157 (0.001)	66.157 (77.31)	100%	17.219 (0.001)	17.219 (78.88)	100%	6.865 (0.001)	6.865 (80.56)	100%	3.687 (0.001)	3.687 (82.15)	100%
	0.2	451.681 (0.001)	451.681 (83.76)	100%	306.477 (0.001)	306.477 (85.35)	100%	63.418 (0.001)	63.418 (86.96)	100%	16.108 (0.001)	16.108 (88.56)	100%	6.357 (0.001)	6.357 (90.23)	100%	3.418 (0.001)	3.418 (91.82)	100%
	-0.2	449.758 (0.001)	449.758 (93.45)	100%	300.278 (0.001)	300.278 (95.05)	100%	58.328 (0.001)	58.328 (96.65)	100%	14.128 (0.001)	14.128 (98.24)	100%	5.477 (0.001)	5.477 (99.85)	100%	2.963 (0.001)	2.963 (101.48)	100%
	-0.4	448.805 (0.001)	448.805 (103.09)	100%	297.244 (0.001)	297.244 (104.66)	100%	55.958 (0.001)	55.958 (106.37)	100%	13.245 (0.001)	13.245 (108.09)	100%	5.095 (0.001)	5.095 (109.84)	100%	2.770 (0.001)	2.770 (111.41)	100%
	-0.8	446.910 (0.001)	446.910 (113.02)	100%	291.291 (0.001)	291.291 (114.63)	100%	51.533 (0.001)	51.533 (116.29)	100%	11.662 (0.001)	11.662 (118.02)	100%	4.431 (0.001)	4.431 (119.63)	100%	2.441 (0.001)	2.441 (121.23)	100%

Note: The numerical results in parentheses are computational times in seconds

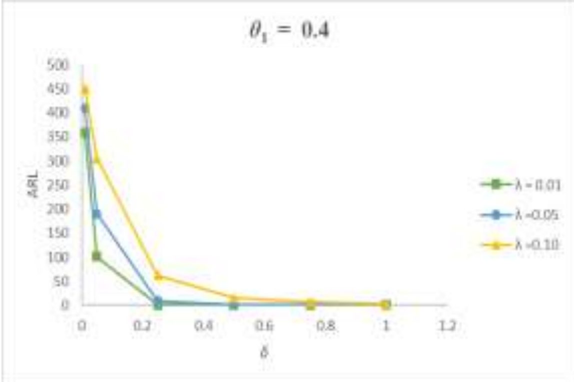
Table 4. Out-of-control ARL values are computed by using the analytical formula and NIE technique for FIMAX($d=0.4,1,1$) running on an EWMA control chart when the in-control ARL is 500.

λ	θ_1	δ = 0.01		Acc %	0.05		Acc %	0.25		Acc %	0.50		Acc %	0.75		Acc %	1.00		Acc %
		$L_p(u)$	$L_N(u)$		$L_p(u)$	$L_N(u)$		$L_p(u)$	$L_N(u)$		$L_p(u)$	$L_N(u)$		$L_p(u)$	$L_N(u)$		$L_p(u)$	$L_N(u)$	
0.01	0.8	360.125 (0.001)	360.125 (4.05)	100%	103.479 (0.001)	103.479 (5.11)	100%	1.635 (0.001)	1.635 (7.56)	100%	1.007 (0.001)	1.007 (9.15)	100%	1.000 (0.001)	1.000 (11.23)	100%	1.000 (0.001)	1.000 (13.25)	100%
	0.4	358.709 (0.001)	358.709 (14.53)	100%	101.546 (0.001)	101.546 (16.32)	100%	1.586 (0.001)	1.586 (18.23)	100%	1.006 (0.001)	1.006 (20.54)	100%	1.000 (0.001)	1.000 (21.31)	100%	1.000 (0.001)	1.000 (24.90)	100%
	0.2	358.005 (0.001)	358.005 (25.65)	100%	100.595 (0.001)	100.595 (28.78)	100%	1.563 (0.001)	1.563 (29.15)	100%	1.005 (0.001)	1.005 (32.56)	100%	1.000 (0.001)	1.000 (33.25)	100%	1.000 (0.001)	1.000 (34.25)	100%
	-0.2	356.595 (0.001)	356.595 (36.44)	100%	98.712 (0.001)	98.712 (37.56)	100%	1.519 (0.001)	1.519 (38.98)	100%	1.005 (0.001)	1.005 (40.25)	100%	1.000 (0.001)	1.000 (42.56)	100%	1.000 (0.001)	1.000 (43.98)	100%
	-0.4	355.895 (0.001)	355.895 (45.35)	100%	97.792 (0.001)	97.792 (46.89)	100%	1.499 (0.001)	1.499 (48.63)	100%	1.004 (0.001)	1.004 (50.09)	100%	1.000 (0.001)	1.000 (53.13)	100%	1.000 (0.001)	1.000 (55.36)	100%
	-0.8	**354.496** (0.001)	**354.496** (57.46)	100%	**95.962** (0.001)	**95.962** (59.15)	100%	**1.461** (0.001)	**1.461** (61.34)	100%	**1.004** (0.001)	**1.004** (62.78)	100%	**1.000** (0.001)	**1.000** (64.01)	100%	**1.000** (0.001)	**1.000** (66.95)	100%
0.05	0.8	410.806 (0.001)	410.806 (4.12)	100%	194.364 (0.001)	194.364 (5.76)	100%	10.139 (0.001)	10.139 (10.48)	100%	1.614 (0.001)	1.614 (12.12)	100%	1.087 (0.001)	1.087 (13.72)	100%	1.020 (0.001)	1.020 (15.26)	100%
	0.4	409.187 (0.001)	409.187 (16.22)	100%	190.716 (0.001)	190.716 (18.45)	100%	9.437 (0.001)	9.437 (20.07)	100%	1.538 (0.001)	1.508 (21.67)	100%	1.073 (0.001)	1.073 (23.31)	100%	1.016 (0.001)	1.016 (24.90)	100%
	0.2	408.379 (0.001)	408.379 (26.50)	100%	188.920 (0.001)	188.920 (28.11)	100%	9.1065 (0.001)	9.1065 (29.69)	100%	1.5028 (0.001)	1.5028 (31.40)	100%	1.0674 (0.001)	1.0674 (33.08)	100%	1.0147 (0.001)	1.0147 (34.76)	100%
	-0.2	406.769 (0.001)	406.769 (36.78)	100%	185.372 (0.001)	185.372 (38.01)	100%	8.483 (0.001)	8.483 (39.61)	100%	1.440 (0.001)	1.440 (41.29)	100%	1.057 (0.001)	1.057 (42.98)	100%	1.012 (0.001)	1.012 (44.62)	100%
	-0.4	405.966 (0.001)	405.966 (45.12)	100%	183.625 (0.001)	183.625 (47.8)	100%	8.190 (0.001)	8.190 (49.47)	100%	1.412 (0.001)	1.412 (51.09)	100%	1.052 (0.001)	1.052 (52.78)	100%	1.011 (0.001)	1.011 (54.37)	100%
	-0.8	404.366 (0.001)	404.366 (57.63)	100%	180.179 (0.001)	180.179 (59.38)	100%	7.637 (0.001)	7.637 (60.97)	100%	1.360 (0.001)	1.360 (62.55)	100%	1.044 (0.001)	1.044 (64.21)	100%	1.009 (0.001)	1.009 (65.88)	100%
0.10	0.8	453.357 (0.001)	453.357 (4.34)	100%	311.964 (0.001)	311.964 (6.03)	100%	68.217 (0.001)	68.217 (7.68)	100%	18.076 (0.001)	18.076 (9.31)	100%	7.264 (0.001)	7.264 (10.95)	100%	3.900 (0.001)	3.900 (12.76)	100%
	0.4	451.410 (0.001)	451.410 (14.40)	100%	305.598 (0.001)	305.598 (16.04)	100%	62.676 (0.001)	62.676 (17.68)	100%	15.812 (0.001)	15.812 (19.32)	100%	6.223 (0.001)	6.223 (21.01)	100%	3.348 (0.001)	3.348 (22.68)	100%
	0.2	450.447 (0.001)	450.447 (26.11)	100%	302.489 (0.001)	302.489 (27.73)	100%	60.104 (0.001)	60.104 (29.7)	100%	14.807 (0.001)	14.807 (31.14)	100%	5.774 (0.001)	5.774 (32.82)	100%	3.115 (0.001)	3.115 (34.50)	100%
	-0.2	448.538 (0.001)	448.538 (36.15)	100%	296.401 (0.001)	296.401 (37.82)	100%	55.314 (0.001)	55.314 (39.43)	100%	13.009 (0.001)	13.009 (41.07)	100%	4.995 (0.001)	4.995 (42.71)	100%	2.720 (0.001)	2.720 (44.34)	100%
	-0.4	447.589 (0.001)	447.589 (46.03)	100%	293.417 (0.001)	293.417 (47.65)	100%	53.080 (0.001)	53.080 (49.34)	100%	12.205 (0.001)	12.205 (50.96)	100%	4.656 (0.001)	4.656 (52.62)	100%	2.552 (0.001)	2.552 (54.29)	100%
	-0.8	445.705 (0.001)	445.705 (56.06)	100%	287.558 (0.001)	287.558 (57.76)	100%	48.903 (0.001)	48.903 (59.48)	100%	10.763 (0.001)	10.763 (61.21)	100%	4.066 (0.001)	4.066 (63.04)	100%	2.265 (0.001)	2.265 (64.78)	100%

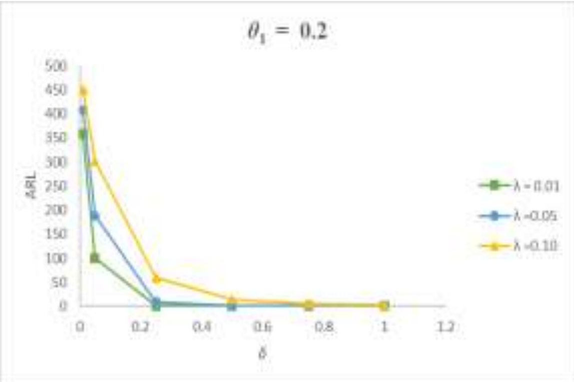
Note: The numerical results in parentheses are computational times in seconds



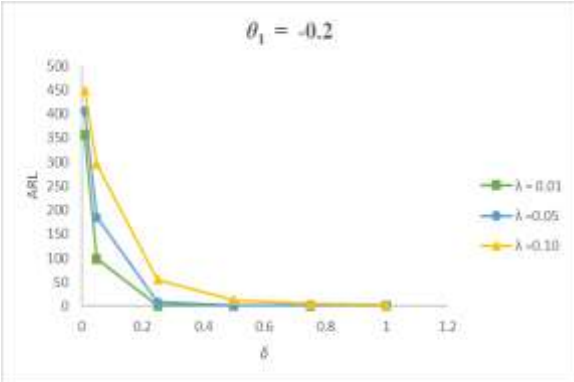
(a) $\theta_1 = 0.8$



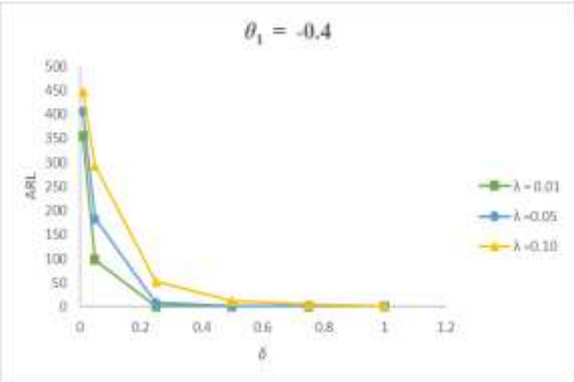
(b) $\theta_1 = 0.4$



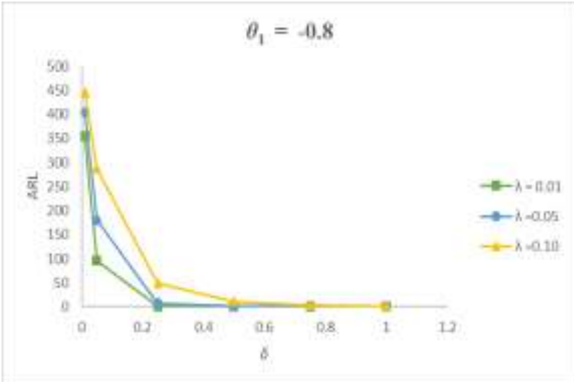
(c) $\theta_1 = 0.2$



(d) $\theta_1 = -0.2$



(e) $\theta_1 = -0.4$



(f) $\theta_1 = -0.8$

Fig. 1: Shifts in the mean of the FIMAX(0.4, 1, 1) model with various values of coefficient parameter θ_1 running on an EWMA control chart calculated using the analytical formula.

Table 5. The statistical results for the gold futures price dataset with the UDS/THB exchange rate as the exogenous variable.

Parameters:	Coefficient	t-Statistic	Prob.
MA(1)	-47.2713	-9.15545	0.00*
d	0.499999	722.704	0.00*
UDS/THB	0.495184	9.055184	0.00*
R-squared			0.981843
Adjusted R-squared		0.981700	
Testing whether the white noise is exponentially distributed.			
Exponential Parameter (v)			39.577325
Kolmogorov-Smirnov			0.692
Asymptotic Significance (2-Sided)			0.725

*A significance level of 0.05.

As reported in Table 5, the dataset is a valid fit for a long-memory FIMAX model since all of the parameters had *p*-values less than 0.05. The exponential parameter (v) of the dataset provided a Kolmogorov-Smirnov value of 0.692. The corresponding p-values based on asymptotic significance (2-sided) were 0.725, suggesting that the long-memory FIMAX(0.499999, 1, 1) model was a suitable fit. Testing whether the white noise fits an exponential distribution also yielded a p-value less than 0.05. Thus, the process running on an EWMA control chart was long-memory

FIMAX(0.499999, 1, 1) with coefficients $\hat{\theta}_1 = -47.2713$ and $\hat{\beta}_1 = 0.495184$.

The model is given by

$$Y_t = \varepsilon_t + 47.2713\varepsilon_{t-1} + 0.495184X_{1t} + 0.499999Y_{t-1} + 0.124999999Y_{t-2} + 0.062500042Y_{t-3}, \quad (20)$$

where $\varepsilon_t \sim Exp(v = 39.577325)$

To apply the analytical formula, we fitted the dataset based on Equation (20). The EWMA control limit (H) was computed equal when in-control ARL = 500 using Algorithm 1 for smoothing parameter λ = 0.50, 0.51, 0.53, or 0.55. Thereby, we used

$$\mathrm{L}_p(\varphi) = 1 - \left\lfloor \lambda\left(1 - \exp\left\{-\frac{H}{\lambda v}\right\}\right).\exp\left\{\frac{(1-\lambda)\varphi}{\lambda v}\right\} \right\rfloor \left[\left(1 - \exp\left\{-\frac{H}{v}\right\} - \lambda \exp\left\{-\frac{1}{v}\begin{pmatrix} 47.2713\varepsilon_{t-1} + 0.495184X_{1t} \\ +0.499999Y_{t-1} + 0.124999Y_{t-2} \\ +0.062500042Y_{t-3} \end{pmatrix}\right\}\right)\right]^{-1}. \quad (21)$$

and

$$\mathrm{L}_N(\varphi) \approx 1 + \frac{1}{\lambda} + \sum_{j=1}^{m} w_j \mathrm{L}_N(a_j) f\left(\frac{a_j - (1-\lambda)\varphi}{\lambda} - \begin{pmatrix} 47.2713\varepsilon_{t-1} + 0.495184X_{1t} \\ +0.499999Y_{t-1} + 0.124999Y_{t-2} \\ +0.062500042Y_{t-3} \end{pmatrix}\right), \quad (22)$$

to compute the out-of-control ARL on an EWMA control chart using the analytical formula and NIE techniques, respectively. The results are reported in Table 6 and Figure 2.

Table 6. The out-of-control ARL results using the analytical formula and NIE techniques for the FIMAX(0.499999,1,1) model with exponential white noise for real data running on an EWMA control chart when the in-control ARL is 500.

δ	λ	0.50	0.51	0.53	0.55
	H	46.07293	49.1854	55.64264	62.64577
0.01	$\mathrm{L}_p(u)$	474.049	475.216	478.525	483.826
		(0.001)	(0.001)	(0.001)	(0.001)
	$\mathrm{L}_N(u)$	474.049	475.216	478.525	483.826
		(29.54)	(30.54)	(31.01)	(30.23)
0.05	$\mathrm{L}_p(u)$	384.181	388.151	399.714	419.384
		(0.001)	(0.001)	(0.001)	(0.001)
	$\mathrm{L}_N(u)$	384.181	388.151	399.714	419.384
		(30.74)	(30.89)	(31.34)	(31.56)
0.25	$\mathrm{L}_p(u)$	148.991	152.218	161.907	180.098
		(0.001)	(0.001)	(0.001)	(0.001)
	$\mathrm{L}_N(u)$	148.991	152.218	161.907	180.098
		(33.98)	(34.62)	(34.68)	(35.18)
0.50	$\mathrm{L}_p(u)$	59.627	60.697	63.769	69.366
		(0.001)	(0.001)	(0.001)	(0.001)
	$\mathrm{L}_N(u)$	59.627	60.697	63.769	69.366
		(39.28)	(40.04)	(41.34)	(42.58)
0.75	$\mathrm{L}_p(u)$	30.222	30.671	31.874	33.5921
		(0.001)	(0.001)	(0.001)	(0.001)
	$\mathrm{L}_N(u)$	30.222	30.671	31.874	33.5921
		(40.56)	(42.35)	(42.72)	(42.94)
1.00	$\mathrm{L}_p(u)$	18.045	18.292	18.907	19.901
		(0.001)	(0.001)	(0.001)	(0.001)
	$\mathrm{L}_N(u)$	18.045	18.292	18.907	19.901
		(41.58)	(42.01)	(42.14)	(42.98)
5.00	$\mathrm{L}_p(u)$	1.916	1.943	1.996	2.054
		(0.001)	(0.001)	(0.001)	(0.001)
	$\mathrm{L}_N(u)$	1.916	1.943	1.996	2.054
		(52.91)	(53.24)	(53.77)	(54.56)

Note: *The numerical results in parentheses are computational times in seconds*

For the real data, results suggest that the out-of-control ARL values calculated using the analytical formula equal those approximated using the NIE technique, indicating good agreement between the two methods and that the proposed analytical formula was very accurate. We compared the out-of-control ARL results versus δ for δ = 0.01, 0.05, 0.25, 0.50, 0.75, 1.00, or 5.00 in the long-memory FIMAX model. The findings indicate that the out-of-control ARL values tended to decline rapidly when detecting small-to-moderate shifts in the process mean and monotonically as δ was increased for all smoothing parameter values.

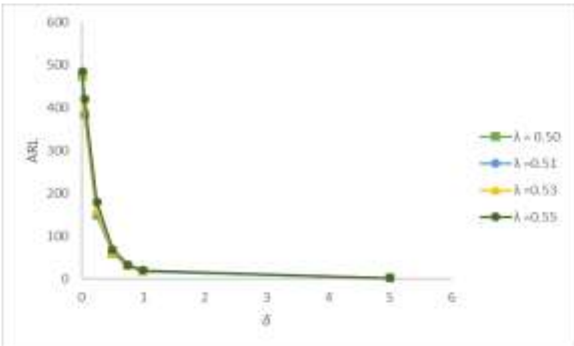
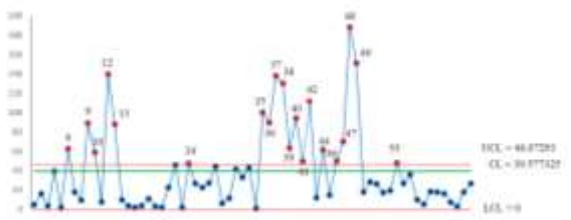


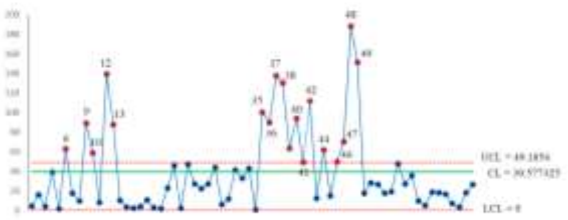
Fig. 2: Graphical representation of the out-of-control ARL results for the FIMAX(0.499999,1,1) model with exponential white noise for real data running on an EWMA control chart when the in-control ARL is 500.

Moreover, when the smoothing parameter value was increased, detection became slower and the out-of-control ARL was larger. The smallest smoothing parameter value (λ = 0.50) provided the best detection performance for all values of δ considered. The computational times for calculating the out-of-control ARL values took a fraction of a second with the analytical formula compared to 30–55 seconds with the NIE technique. The result corresponds to the computational in Table 2, Table 3, and Table 4.

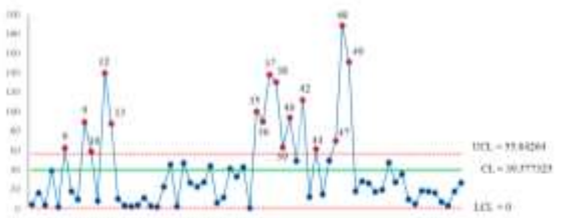
The EWMA control chart running the in the FIMAX model with real data is graphically displayed in Figure 3. It can be seen that in all cases of the smoothing parameter value tested, the process remained under statistical control for the first five observations. The total number of out-of-control signals for λ = 0.50, 0.51,0.53, or 0.55 were 20, 18, 16, and 14 points, respectively. Hence, the model with the smallest value of λ performed the best.



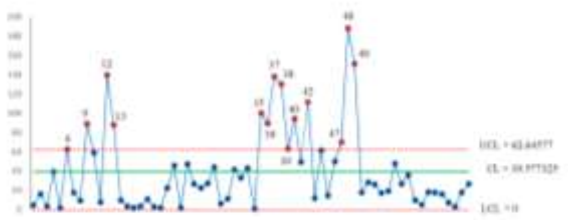
$(a)\ \lambda=0.50$



$(b)\ \lambda=0.51$



$(c)\ \lambda=0.53$



$(d)\ \lambda=0.55$

Fig. 3: Graphical representation of the out-of-control ARL results for the FIMAX(0.499999,1,1) model with exponential white noise for real data running on an EWMA control chart when the in-control ARL is 500 for various smoothing parameter values: $(a)\ \lambda=0.50$, $(b)\ \lambda=0.51$, $(c)\ \lambda=0.53$, and $(d)\ \lambda=0.55$.

4 Conclusions

We provided a new technique to accurately compute the ARL for a long-memory FIMAX(d,q,r) model with exponential white noise running on the EWMA control chart using an analytical formula based on an integral equation. Its performance was measured against that of the well-established NIE technique. For all the control chart smoothing parameter values of the chart and FIMAX scenarios tested, the proposed analytical formula provided out-of-control ARL values close to those with the NIE technique. For clarity, we have verified the accuracy of the analytical formula with the NIE method as the percentage accuracy. The percentage accuracy results were 100% in all cases, implying good agreement between the two techniques and that the proposed analytical formula is very accurate and quick. Therefore, using the analytical formula as an alternative approach for deriving the ARL for a shift in the mean of this scenario is plausible.

To demonstrate the practicability of the proposed analytical formula, we applied it to a process involving the gold futures price and exchange rates over a specific time period. The out-of-control ARL values show that the analytical formula approached performed very well in all of the scenarios tested and that it is a good alternative to using the analytical formula for this endeavor. In addition, this analytical formula can be extended to develop commercial packages to evaluate ARL to analyze and control the manufacturing process or other aspects. Future work of this study could be extended to other control charts that have been developed, such as modified EWMA, modified CUSUM, and enhanced EWMA.

Acknowledgment:
The author gratefully acknowledges the editor and referees for their valuable comments and suggestions which greatly improve this paper. This research was funded by King Mongkut's University of Technology North Bangkok, Contract No. KMUTNB-66-BASIC-05.

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Appendix A

Theorem 2: $L_p(\varphi)$, the ARL obtained from the analytical formula based on an integral equation for a long-memory FIMAX model on an EWMA control chart, exists and is unique.

Proof: To prove the existence of the integral equation applied to derive the ARL. Let T be a contraction in complete metric space (M,ϑ), $\mathbf{C}[0,H]$ be a set of all continuous functions on interval $[0,H]$, and the in-control ARL be an arbitrary but fixed element in M. Define a sequence of iterates $\{L_n\}_{n\geq 0}$ in M that satisfies $L_{n+1} = T(L_n)$, for all $n \geq 1$. Consider

$$\vartheta(L_{n+1},L_n) = \vartheta\left(T(L_n),T(L_{n-1})\right) \leq \ell\,\vartheta\left(T(L_n),T(L_{n-1})\right), 0 \leq \ell < 1.$$

Continuing inductively, we obtain

$$\vartheta(L_{n+1},L_n) \leq \ell\,\vartheta(L_n,L_{n-1}) \leq \ell^2\vartheta(L_{n-1},L_{n-2}) \leq ... \leq \ell^n\vartheta(L_1,L_0).$$

By repeatedly applying the triangle inequality to this formula when $n \geq m$ we arrive at

$$\vartheta(L_n,L_m) \leq \vartheta(L_n,L_{n-1}) + ... + \vartheta(L_{m+1},L_m),$$

Thus, it follows that

$$\vartheta(L_n,L_m) \leq (\ell^{n-1} + \ell^{n-2} + ... + \ell^m)\vartheta(L_1,L_0).$$

Taking the property of the sum of a geometric series in ℓ, we obtain

$$\vartheta(L_n,L_m) \leq \frac{\ell^n}{1-\ell}\vartheta(L_1,L_0).$$

where $\frac{\ell^n}{1-\ell}$ as $n \to \infty$. So, $\{L_n\}_{n\geq 0}$ is a Cauchy sequence and $\lim_{n\to\infty} T(L_n) = L$.

Hence, the existing continuous function $L:[0,H] \to \mathbb{R}$ satisfies the integral equation.

Proof: To prove the uniqueness of the integral equation applied to derive the ARL. Let L_1 and L_2 be two arbitrary functions for $\mathbf{C}[0,H]$. The common term for the complete metric space is $\left(\mathbf{C}[0,H], \|.\|_\infty\right)$. That is to say, a set of continuous functions of the ARL defined on $[0,H]$, and $\mathbf{C}[0,H]$ becomes norm space if

$$\| L\|_\infty = \sup_{\upsilon\in[0,b]}\left|\int_0^b k(\varphi,g)dg\right|,$$

for all functions $k(\varphi,g) \in \mathbf{C}[0,H]$, where $k(\varphi,g)$ is

$$k(\varphi,g) = \frac{1}{\lambda}\int_0^H L(\omega)\left(\exp\left\{-\frac{\omega}{\lambda v}\right\}\times\exp\left\{\frac{(1-\lambda)\varphi}{\lambda v} + \begin{pmatrix}\mu - \sum_{i=1}^{q}\theta_i\varepsilon_{t-j} + \sum_{j=1}^{r}\beta_j X_{jt} + dY_{t-1} \\ + \frac{d(1-d)}{2}Y_{t-2} + \frac{d(1-d)(2-d)}{6}Y_{t-3} + ...\end{pmatrix}\frac{1}{v}\right\}\right)d\omega$$

The kernel function of the integral equation used to define the ARL is

$$\|T(L_1) - T(L_2)\|_\infty = \sup_{\varphi\in[0,H]}\left|\int_0^H k(\varphi,g)\left(L_1(g) - L_2(g)\right)dg\right|$$

Hence, we obtain $\|T(L_1) - T(L_2)\|_\infty \leq \ell\|L_1 - L_2\|_\infty$

where $\ell = \sup_{\varphi\in[0,H]}\left|\int_0^H k(\varphi,g)\,dg\right| < 1.$

Applying Banach's fixed point theorem leads to contraction mapping. Therefore, T is a unique continuous function that satisfies the integral equation in equation (12)

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The author contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

This research was funded by King Mongkut's University of Technology North Bangkok, Contract No. KMUTNB-66-BASIC-05.

Conflict of Interest

The author has no conflict of interest to declare.