On $\mathscr{F}-$flat structures in vector bundles over foliated manifolds

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Abstract: We give the definition of the families of $\mathscr{F}-$flat structures and $\mathscr{F}-$flat connections in vector bundles over $\mathscr{F}-$foliated manifolds. Essential: existence of a $\mathscr{F}-$flat structure is equivalent to the existence of a $\mathscr{F}-$flat connection. Let $\{\overset{\lambda}{\xi}\}$ be a family of subbundles of a vector bundle ξ. There exists a family of $\mathscr{F}-$flat structure $\{\overset{\lambda}{\Lambda}\}$ in ξ, relative at $\overset{\lambda}{\xi}$, if and only if exists a family of $\mathscr{F}-$flat connections $\{\overset{\lambda}{\nabla}\}$ in ξ (Theorem III.5). $\mathscr{F}-$flat structures (Theorem III.1), and integrable $\mathscr{F}-$flat structures (Theorem III.5), are considered. Finally, integrable $\Gamma-$structure and $\mathscr{F}-$flat structures on total space of a vector bundle are presented (Theorem IV.1).

Key-Words: $\mathscr{F}-$flat structure, $\mathscr{F}-$flat connection, integrable $\mathscr{F}-$flat structure.

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1 Introduction

The notion of foliation of manifold is of great interest for geometers. It is the basis of some results regarding the decomposition of tangent bundle of the foliated manifold into the tangent bundle to the leaves and the transverse bundle. Many authors have dealt with this topic from different point of view. The tangent bundle can be structured in various ways, [1], [2], [6].

In, [5] the foliations studied are induced by geometric structures. In our paper, on the contrary, we use the $\mathscr{F}-$flat structures to obtain (affine) geometric structures on the leaves. In other work, [6], the leaves have remarkable structures, that is piecewise-linear, differentiable or analytic structure.

The origin of the present work can be found in, [4]. The notion of $\mathscr{F}-$flat structures was introduced by us, [3]. In present paper we obtain some interesting characteristic results regarding the links between $\mathscr{F}-$flat structures and $\mathscr{F}-$flat connections for paracompact manifolds. We also prove an existence theorem of $\mathscr{F}-$flat structures and formula for connection which define a $\mathscr{F}-$flat structure.

Here the word "foliation" means a foliated atlas and a decomposition of a manifold M into connected submanifolds of dimension p. We suppose that the manifolds, foliations, maps are $C^{\infty}-$differentiable ($C^{\infty}-$diff.) on the morphisms of vector bundles are of constant rank. We use terms "fiber bundle with structure group", or "vector bundle". The $\mathscr{F}-$flat structures and $\mathscr{F}-$flat connections are defined in vector bundles over foliated manifolds, for which the transition functions are constant along the leaves of foliation $\mathscr{F}$. Suppose that the leaf topology admits a countable base.

Convention: $i, i', j, j', k, k', ... = 1, 2, ..., p$; $\widehat{i}, \widehat{i}', \widehat{j}, \widehat{j}', \widehat{k}, \widehat{k}', ... = p+1, p+2, ...$; $a, b, c, ... = 1, 2, ..., m$ (or $m+n$). $m=$dim$M$.

We use the classical summation convention for indices.

2 Families of $\mathscr{F}-$flat structures ($\mathscr{F}$.f.s.) and $\mathscr{F}-$flat connections ($\mathscr{F}$.f.c.) on vector bundles

The principal tool of this section is to present some relations between $\mathscr{F}$.f.s. $\overset{\lambda}{\Lambda}$ and $\mathscr{F}$.f.c. $\overset{\lambda}{\nabla}$ defined in vector bundles over foliated manifolds. Let M be a $C^{\infty}-$diff., paracompact, $\mathscr{F}-$foliated manifold, where $\mathscr{F} = \{(U_d, \Psi_\alpha)\}_{\alpha\in I} = \{(U_d, x^k, x^{\widehat{k}})\}_{\alpha\in I}$. Let $\xi = (E, \pi, M)$ be a vector bundle over M; E is total space of ξ, $n=$its projection, and $R^u=$fiber of ξ.

Denote: TM is tangent bundle of M, and $T\mathscr{F}$ is the tangent bundle of $\mathscr{F}$. Here (x^k) are coordinates in a leaf of $\mathscr{F}$, $\widehat{x} = (x^{\widehat{k}})$ =secondary coordinates. Consider a family of subgroups $\{\overset{\lambda}{G}\}$ of $GL(n, R)$, $\lambda = 1, 2, ...$ and $r=$rangξ. The set of all sections of a vector bundle (;) is denoted $\Gamma(;)$. Let $\overset{\lambda}{\xi} = (\overset{\lambda}{E}, \overset{\lambda}{\Pi}, M)$ be the subbundle of ξ with structure group $\overset{\lambda}{G}$; $\overset{\lambda}{E}$ is total space of $\overset{\lambda}{\xi}$, $\overset{\lambda}{\Pi}$ =projection, and R^u =fiber of $\overset{\lambda}{\xi}$. Denote $\overset{\lambda}{A}_{\alpha,\beta} : U_\alpha \cap U_\beta \to \overset{\lambda}{G}$ the transition functions of ξ which define $\overset{\lambda}{\xi}$, $\alpha, \beta \in I$. We use sections of ξ relative to ξ, i.e. $s_d : U_d \to \overset{\lambda}{E}/U_\alpha$. Consider the open covering $\{U_\alpha\}_{\alpha\in I}$ of M and s_α, s_β two local frames fields of $\overset{\lambda}{\xi}$ on U_α, U_β, respectively.

Definition 2.1. A set $\overset{\lambda}{\Lambda} = \{(s_\alpha, \overset{\lambda}{A}_{\alpha\beta})/s_\alpha = \overset{\lambda}{A}_{\alpha\beta} s_\beta\}$ defines an $\mathscr{F}-$flat structure in vector bundle ξ relative to $\overset{\lambda}{\xi}$ if the transition functions $\overset{\lambda}{A}_{\alpha\beta}$ are constant along the leaves of $\mathscr{F}$, i.e. $\overset{\lambda}{A}_{\alpha\beta}(x) = \overset{\lambda}{A}_{\alpha\beta}(x^{\widehat{k}})$, $x = \Psi_\alpha(x^k, x^{\widehat{k}}) = \Psi_\beta^{-1}(x^{k'}, x^{\widehat{k'}})$. The vector bundle ξ endowed with an $\mathscr{F}-$flat structure $\overset{\lambda}{\Lambda}$ is called an $\mathscr{F}-$flat vector bundle relative to $\overset{\lambda}{\xi}$.

Let $\overset{\lambda}{\nabla}$ be a connection of $\overset{\lambda}{\xi}$.

Definition 2.2. The frame fields $\sigma_\alpha, \sigma_\beta$ of $\overset{\lambda}{\xi}$ are parallel at the connection $\overset{\lambda}{\nabla}$ along leaves of $\mathscr{F}-$flat (p. a. l. $\mathscr{F}$) if $\overset{\lambda}{\nabla}_X \sigma_\alpha = \overset{\lambda}{\nabla}_X \sigma_\beta = 0, \forall X \in \Gamma(T\mathscr{F})$.

Lemma 2.3. *Let $\overset{\lambda}{\nabla}$ be a connection on $\overset{\lambda}{\xi}$, $h = (U_\alpha; x^k, x^{\widehat{k}}) \in \mathscr{F}$, and $\sigma_\alpha, \sigma_\beta$ frame fields of $\overset{\lambda}{\xi}$ (p.a.l.$\mathscr{F}$), and $\sigma_\alpha = B_{\alpha\beta}\sigma_\beta$ on $U_\alpha \cap U_\beta$.*
If $\overset{\lambda}{\nabla}_X \sigma_\alpha = \overset{\lambda}{\nabla}_X \sigma_\beta = 0$, then:
$B_{\alpha\beta}$ are constant along the leaves of $\mathscr{F}$.

Proof. Let ω, ω' be the connection forms of $\overset{\lambda}{\nabla}$ with respect to $\sigma_\alpha, \sigma_\beta$, respectively. Then $\omega(X) = B_{\alpha\beta}^{-1}\omega'(X)B_{\alpha\beta} + B_{\alpha\beta}^{-1} dB_{\alpha\beta}(X)$, and $dB_{\alpha\beta}(X) = 0$, for $X = \dfrac{\partial}{\partial x^k} \in \Gamma(T\mathscr{F})$. Hence $\dfrac{\partial B_{\alpha\beta}}{\partial x^k} = 0$.

Using this lemma, we are able to study some properties of $\mathscr{F}$-f.s. of ξ relative to $\overset{\lambda}{\xi}$.

Definition 2.4. The connection $\overset{\lambda}{\nabla}$ of $\overset{\lambda}{\xi}$ is $\mathscr{F}$-flat along the leaves of $\mathscr{F}$ if its curvature $\Omega(\overset{\lambda}{\nabla})$ satisfies the condition $\Omega(\overset{\lambda}{\nabla}(X,Y)) = 0, \forall X, Y \in \Gamma(T\mathscr{F})$.

The following result justifies the denomination of "$\mathscr{F}$-flat structure".

Theorem 2.5. *Consider a vector bundle $\xi = (E, \pi, M)$ over a paracompact, $\mathscr{F}$-foliated manifold M. Let $\overset{\lambda}{\xi}$ be a subbundle of ξ. There exists an $\mathscr{F}$-flat structure $\overset{\lambda}{\Lambda}$ in $\overset{\lambda}{\xi}$ if and only if exists an $\mathscr{F}$-flat connection $\overset{\lambda}{\nabla}$ in $\overset{\lambda}{\xi}$.*

Proof. Consider, for λ arbitrary fixed, an $\mathscr{F}$-flat connection $\overset{\lambda}{\nabla}$ in $\overset{\lambda}{\xi}$, and $\widetilde{s}_\alpha = (\widetilde{s}_a^b)$ a frame field of E/U_α, $a, b = 1, 2, ..., n$. We determine a frame field $s_d = \overset{\lambda}{A}_{d\beta} \cdot \widetilde{s}_\beta$, $s_\alpha = (s_a^b)$, $\overset{\lambda}{A}_{\alpha\beta} = (A_a^b)$ such that $\overset{\lambda}{\nabla}_X s_\alpha = 0$, $X \in \Gamma(T\mathscr{F})$ where $\overset{\lambda}{A}_{\alpha\beta}$ is an unknown matrix. Denote $\overset{\lambda}{\omega} = (\overset{\lambda}{\omega}{}_a^b)$ the connection form of $\overset{\lambda}{\nabla}$ relative to $\widetilde{s}_\alpha = (\widetilde{s}_a^b)$, where $\overset{\lambda}{\omega}{}_a^b = \overset{\lambda}{\Gamma}{}_{ak}^b dx^k + \overset{\lambda}{\Gamma}{}_{a\widehat{k}}^b dx^{\widehat{k}}$. On $U_\alpha \cap U_\beta \neq \emptyset$, we have: $\overset{\lambda}{\nabla}_{\frac{\partial}{\partial q^k}} s_\alpha = \overset{\lambda}{\nabla}_{\frac{\partial}{\partial_q^k}} (\overset{\lambda}{A}{}_a^b \widetilde{s}_b) = \left(\frac{\partial \overset{\lambda}{A}{}_a^b}{\partial x^k} + \overset{\lambda}{A}{}_a^c \overset{\lambda}{\Gamma}{}_{ck}^b\right)\widetilde{s}_b = 0$. Therefore, $(\overset{\lambda}{A}{}_a^b)$ satisfies the equations

$$\frac{\partial \overset{\lambda}{A}{}_a^b}{\partial x^k} + \overset{\lambda}{A}{}_a^c \overset{\lambda}{\Gamma}{}_{ck}^b = 0. \tag{2.1}$$

In this system, $x = (x^k)$ are independent variables and $\widetilde{x} = (x^{\widetilde{k}})$ are parameters. We transform this system in the Pfaff system $d\overset{\lambda}{A}{}_c^b + \overset{\lambda}{A}{}_a^c \Gamma_{ck}^b dx^k = 0$, where d is the exterior differentiation operator. Using Frobenius theorem, [7], and $\det(\overset{\lambda}{A}{}_a^b) \neq 0$, we obtain the following compatibility conditions:

$$\frac{\partial \overset{\lambda}{\Gamma}{}_{ai}^b}{\partial x^k} - \frac{\partial \overset{\lambda}{\Gamma}{}_{ak}^b}{\partial x^i} + \overset{\lambda}{\Gamma}{}_{ai}^c \overset{\lambda}{\Gamma}{}_{ck}^b - \overset{\lambda}{\Gamma}{}_{ak}^c \overset{\lambda}{\Gamma}{}_{ci}^b = 0. \tag{2.2}$$

On the other hand, $\Omega(\overset{\lambda}{\nabla})\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^i}\right) = 0$, where $\Omega(\overset{\lambda}{\nabla})$ denotes the curvature of $\overset{\lambda}{\nabla}$. These relations coincide with the relation (2.2). Hence exists: $\overset{\lambda}{A}_\alpha\beta = (\overset{\lambda}{A}{}_a^b)$ and $s_\alpha = (s_a)$, where $\overset{\lambda}{\nabla}_X s_\alpha = \overset{\lambda}{\nabla}_X s_\beta = 0$, $X \in \Gamma(T\mathscr{F})$. Then, using the lemma 2.3, the set $\overset{\lambda}{\Lambda} = \{(s_\alpha, \overset{\lambda}{A}_{\alpha\beta})\}_{\alpha\beta\in I}$ is an $\mathscr{F}$p.s. in $\overset{\lambda}{\xi}$.

Conversely: let $\overset{\lambda}{\Lambda} = \{(s_\alpha, \overset{\lambda}{A}_{\alpha\beta})\}_{\alpha\beta\in I}$ be an $\mathscr{F}$.f.s. in $\overset{\lambda}{\xi}$, where $\overset{\lambda}{A}_{\alpha\beta}(x) = \overset{\lambda}{A}_{\alpha\beta}(x^{\overset{\lambda}{k}})$, $x \in U_\alpha \cap U_\beta$. Over U_α define an operator $\overset{\lambda}{\nabla}$ on E/U_α, hence:

$\overset{\lambda}{\nabla}{}^{\alpha}_X s_\alpha = 0$, $X \in \Gamma(T\mathscr{F})$, $s_\alpha \in \overset{\lambda}{\Lambda}$. Extends $\overset{\lambda}{\nabla}{}^{\alpha}_X$ to $s = \lambda^a s_a$: $\overset{\lambda}{\nabla}{}^{\alpha}_X s = X(\lambda^a)s_a$, $\lambda^a : U_d \to R$ are functions. Now, let $\{a_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Then, we define: $(\overset{\lambda}{\nabla}{}^{\alpha}_X s)(x) = \sum_\alpha a_\alpha(x)(\overset{\lambda}{\nabla}{}^{\alpha}_X s)(x)$, $\forall x \in M$. The operator $\overset{\lambda}{\nabla}{}^{\alpha}$ satisfies the condition from the definition of a connection, for $X \in \Gamma(T\mathscr{F})$. Let $T^\perp\mathscr{F}$ be a subbundle of TM complementary of $T\mathscr{F}$, $TM = T\mathscr{F} \oplus T^\perp\mathscr{F}$. Now, define a connection in $\overset{\lambda}{\xi}$, $D : \Gamma(TM) \times \Gamma(\overset{\lambda}{E}) \to \Gamma(\overset{\lambda}{E})$ by the relation $D_Z s = \overset{\lambda}{\nabla}{}^{\alpha}_X s + \overline{\nabla} \overset{s}{X}{}^{\perp}$, where $\overline{\nabla}$ is a fixed connection in $\overset{\lambda}{\xi}$, $Z = X + X^\perp \in \Gamma(TM)$, $X \in \Gamma(T\mathscr{F})$, $X^\perp \in \Gamma(T^\perp\mathscr{F})$. The connection D is $\mathscr{F}$-flat. Indeed, $D_X s = \overset{\lambda}{\nabla}{}^{\alpha}_X s$, and $D_X s_\alpha = \Theta_\alpha(X)s_\alpha = 0$, where $\Theta_\alpha = (\Theta^b_a)$ is the matrix of connection D relative to s_α and $\Omega(D)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}\right) = 0$; $\Omega(D)$ denote the curvature of D. Hence, D corresponds to $\overset{\lambda}{\Lambda}$. The theorem is proved.

Use precedent notations.

Proposition 2.6. *Let $\mathscr{R}$ be the principal frame bundle of ξ. Then exists a $\mathscr{F}$f.s. $\overset{\lambda}{\Lambda}$ in ξ if and only if there exists a subbundle $\overset{\lambda}{\mathscr{R}}$ of $\mathscr{R}$.*

Proof. The total space of $\overset{\lambda}{\mathscr{R}}$ is

$$\overset{\lambda}{E} = \bigcup_{x \in M} \{s_\alpha(x) \in \overset{\lambda}{\Lambda}(x)\}_{x \in I},$$

where $\overset{\lambda}{\Lambda}(x) = (s_\alpha(x), A_{\alpha,\beta}(x))_{\alpha,\beta \in I}$.

The projection $\overset{\lambda}{\pi}$ of $\overset{\lambda}{\mathscr{R}}$ is $\overset{\lambda}{\pi} : \overset{\lambda}{E} \to M$, $\overset{\lambda}{\pi}(s_\alpha(x)) = x$, $x \in M$.

The reciprocal results using the method to demonstrate the precedent theorem.

3 Remarkable $\mathscr{F}$-flat structures

The aim of this Section is to highlight the link between the integrability of Γ-structures and $\mathscr{F}-$f.s. This intergrability can be achieved using a special atlas of differentiable structure of manifold.

These remarkable $\mathscr{F}-$f.s. show interest in the total space of a vector bundle.

3.1 Families of $\mathscr{F}$-flat structures and tensor fields

Let $\mathscr{T}^1_2(M)$ be the set of $C^\infty-$diff tensor fields of type $\binom{1}{2}$ defined on M. Consider a family of connections $\overset{\alpha}{\nabla} = D + \alpha t$, where D is a given connection on M, $t \in \mathscr{T}^1_2(M)$ and $\alpha \in R$. Connection D and t are symmetric along the leaves of $\mathscr{T}$ if the torsion of D, $T(D)(X,Y) = 0$ and $t(X,Y) = t(Y,X)$, $\forall X, Y \in \Gamma(T\mathscr{T})$. Denote $\Omega(\overset{\alpha}{D})^b_{k\ell a}$, $\Omega(D)^b_{k\ell a} t^b_{ka}$ the components of curvatures $\Omega(\overset{\alpha}{D}), \Omega(D)$ and t, along the leaves of $\mathscr{F}$, relative to chart $h = (U; x^k, z^{\widehat{k}})$. Using precedent notations, we have

Theorem 3.1. *Let M be a $C^\infty-$diff., paracompact, and $\mathscr{F}-$foliated manifold.*

1. If D and t are symmetric along the leaves of $\mathscr{F}$, then $\overset{\alpha}{\nabla}$ is a symmetric connection along the leaves of $\mathscr{F}$.

2. If D defines an $\mathscr{F}$-flat structure λ on M, then $\overset{\alpha}{\nabla}$ defines an $\mathscr{F}$-flat structure $\overset{\alpha}{\nabla}$ on M if and only if

$$\frac{\partial t^b_{\ell a}}{\partial x^k} - \frac{\partial t^b_{ka}}{\partial x^\ell} + t^c_{\ell a}\Gamma^b_{kc} - t^c_{ka}\Gamma^b_{\ell c} = 0,\ k, \ell = 1, 2, ..., p.$$

Proof. 1. This affirmation is clear from the relation $T(\overset{\alpha}{\nabla})(X,Y) = T(D)(X,Y) + [t(X,Y) - t(Y,X)]$, $X, Y \in \Gamma(T\mathscr{F})$.

2. Using the definition of curvatures $\Omega(\overset{\alpha}{\nabla}), \Omega(D)$ along the leaves of $\mathscr{F}$, we obtain the relations

$$\Omega(\overset{\alpha}{\nabla})\left(\frac{\partial}{\partial k^k}, \frac{\partial}{\partial x^\ell}\right)\frac{\partial}{\partial x^a} = \Gamma(\overset{\alpha}{\nabla})^b_{k\ell a}\frac{\partial}{\partial x^b} +$$

$$t\left(\frac{\partial t^b_{\ell a}}{\partial x^k} - \frac{\partial t^b_{kc}}{\partial x^\ell} + t^c_{\ell a}\Gamma^b_{kc} - \frac{\partial t^b_{kc}}{\partial y^\ell} t^c_{ka}\Gamma^b_{\ell c}\right)\frac{\partial}{\partial x^b},$$

$\frac{\partial}{\partial x^b} \in \Gamma(TM/U)$, $\frac{\partial}{\partial x^k}$, $\frac{\partial}{\partial x^\ell} \in \Gamma(T\mathscr{F}/U)$, $k, \ell = 1, 2, ..., p$; $a, b, c = 1, 2, ..., m$.

Remark (*). The Theorem 2.5 gives, for $\xi = TM = \overset{\lambda}{\xi}$, the result: there exists an $\mathscr{F}$.f.s in TM if and only if there exists an $\mathscr{F}$.f.c. in TM.

Now, the affirmation 2 results from the precedent remark.

3.2 Integrable $\mathscr{F}$-flat structures

The following is a consequence of Lemma 2.3 for $\xi = TM = \overset{\lambda}{\xi}$.

Lemma 3.2. *Let $\sigma_\alpha, \sigma_\beta$, $\delta_\alpha = B_{\alpha\beta}\delta_\beta$ be two arbitrary frame fields of $TM/U_\alpha \cap U_\beta$, $\sigma_\alpha = B_{\alpha\beta}\sigma_\beta$*

on $U_\alpha \cap U_\beta$, and ∇ a connection of TM. If $\nabla_{\frac{\partial}{\partial x^k}}\sigma_\alpha = \nabla_{\frac{\partial}{\partial x^k}}\sigma_\beta = 0$, then $B_{\alpha\beta}$ are independent from (x^k).

Definition 3.3. We say that an $\mathscr{F}$-flat structure Λ on TM is integrable if Λ is defined by the family $\left\{\frac{\partial}{\partial x^a}\right\}$ of natural frames and Jacobian matrices $J_{\alpha\beta} = \left\{\frac{\partial x^{a'}}{\partial x^a}\right\}$, $\alpha, \beta \in I$.

Let $\{\Gamma^a_{bc}\}$ be the coefficients of a connection ∇ relative to the chart $h = (U; x^k, x^{\widehat{k}})$.

Definition 3.4. An arbitrary vector field $t = t^a \frac{\partial}{\partial x^a}$ on M is parallel relative to ∇, along the leaves of $\mathscr{F}$ (p.a.l.$\mathscr{F}$) if covariant derivative of t in connection ∇, along the leaves of $\mathscr{F}$, is null: $t^a_{|k} = \frac{\partial t^a}{\partial x^k} + \Gamma^a_{kb}t^b = 0$, $j, k = 1, 2, ..., p$; $a, b = 1, 2, ..., m$.

Theorem 3.5. *Let M be a C^∞-diff., paracompact, $\mathscr{F}$-foliated manifold and ∇ a connection on M. Consider an arbitrary C^∞-diff. vector field t on M, $t(x) \neq 0$, $\forall x \in M$. Then, there exists an $\mathscr{F}$-flat structure Λ on TM if t is parallel relative to ∇, along the leaves of $\mathscr{F}$. Moreover, in precedent conditions, ∇ is $\mathscr{F}$-flat.*

Proof. *Step I.* Consider $t = t^a \frac{\partial}{\partial x^a}$ and $t^a_{|k} = 0$ ($t^a_{|k}$ denotes covariant derivative). Then: $\frac{\partial t^a}{\partial x^k} = -\Gamma^a_{kb}t^b$, and

$$\frac{\partial^2 t^a}{\partial x^k \partial x^i} = \frac{\partial^2 t^a}{\partial x^i \partial x^k} \leftrightarrow$$

$$\left(\frac{\partial \Gamma^a_{jb}}{\partial x^k} - \frac{\partial \Gamma^a_{kb}}{\partial x^j} + \Gamma^a_{kc}\Gamma^c_{jb} - \Gamma^a_{jc}\Gamma^c_{kb}\right) t^b = 0.$$

Because t is arbitrary, $t(x) \neq 0$, $x \in M$, precedent relations give

$$\frac{\partial \Gamma^a_{jb}}{\partial x^k} - \frac{\partial \Gamma^a_{kb}}{\partial x^j} + \Gamma^a_{kc}\Gamma^c_{jb} - \Gamma^a_{jc}\Gamma^c_{kb} = 0. \quad (3.1)$$

Step II. Let $\left(\frac{\partial}{\partial z^b}\right)$ be an arbitrary frame field of TM/U. We prove that there exists a frame field $\left(\frac{\partial}{\partial y^a}\right)$ parallel in connection ∇ along the leaves of $\mathscr{F}$. Indeed, let $\frac{\partial}{\partial y^a} = A^b_a \frac{\partial}{\partial z^k}$ be a frame field (p.a.l.$\mathscr{F}$), where $A = (A^b_a)$ is a unknown matrix, that satisfies the conditions $\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial y^a} = 0$. Therefore, $\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial y^a} = \nabla_{\frac{\partial}{\partial x^k}} \left(A^b_a \frac{\partial}{\partial z^b}\right) = \left(\frac{\partial A^b_a}{\partial x^k} + A^c_a \Gamma^b_{kc}\right) \frac{\partial}{\partial z^b} = 0$. Hence, $A_{\alpha\beta} = (A^b_a)$ satisfies the equations

$$\frac{\partial A^b_a}{\partial x^k} + A^c_a \Gamma^b_{kc} = 0. \quad (3.2)$$

To study this system we use Fröbenius theorem, [7]. The result coincides with the relations (3.1). Therefore, the system (3.2) is compatible. Hence, there exists: $A_{\alpha\beta}(x) = (A^b_a(x^{\widehat{k}}))$ and $\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial y^a} = 0$.

Now, we use Lemma 3.2. Consequently, there exists on TM, the $\mathscr{F}$-flat structure $\Lambda = \left\{\left(\frac{\partial}{\partial y^a}, A_{\alpha\beta}(x^{\widehat{k}})\right)\right\}_{\alpha\beta \in I}$.

The second statement follows from the definition of curvature $\Omega(\nabla)$ along the leaves of $\mathscr{F}$. Indeed, let $\widetilde{s}_\alpha = (\widetilde{s}_b)$ be a frame field of TM/U. Then:

$$\Omega(\nabla)\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j}\right)\widetilde{s}_b =$$

$$= \nabla_{\frac{\partial}{\partial x^k}}(\Gamma^b_{ja}\widetilde{s}_b) - \nabla_{\frac{\partial}{\partial x^k}}(\Gamma^b_{ka}\widetilde{s}_b) =$$

$$= \left(\frac{\partial \Gamma^b_{ja}}{\partial x^k} - \frac{\partial \Gamma^b_{ka}}{\partial x^j} + \Gamma^c_{ja}\Gamma^b_{kc} - \Gamma^c_{ka}\Gamma^b_{jc}\right)\widetilde{s}_b = 0.$$

This proves the theorem.

3.3 $\mathscr{F}$-flat structures and vector fields

Lemma 3.6. *Let M be a C^∞ diff, paracompact, $\mathscr{F}$-foliated manifold, t a C^∞-diff. vector field on M, $t(x) \neq 0$, $\forall x \in M$. A symmetric connection ∇ on M is $\mathscr{F}$-flat if and only if the second covariant derivations of t, in connection ∇, and along the leaves of $\mathscr{F}$, coincide; i.e. $t^a_{|ij} = t^a_{ji}$, $i, j = 1, 2, ..., p$; $a, b = 1, 2, ..., m$.*

Proof. Expression of t in the chart $h = \left(U; x^i, x^{\widehat{i}}\right)$ is $t = t^a \frac{\partial}{\partial x^a}$. Now, we use the definitions of curvature and torsion of a connection (along the leaves of $\mathscr{F}$). Partial covariant derivations of t in connection ∇, along the leaves of $\mathscr{F}$, satisfies the relations:

$$\begin{aligned} t^a_{|i} &= \frac{\partial t^a}{\partial x^i} + \Gamma^a_{i\ b}t^b, \text{ and} \\ t^a_{|ij} - t^a_{|ji} &= \Omega(\nabla)^a_{ijb}t^b - T(\nabla)^b_{ij}t^a_{|b} \end{aligned} \quad (3.3)$$

Since ∇ is symmetric, $T^a_{bc}(\nabla) = 0$, and hence $T(\nabla)^a_{ij} = 0$. Therefore, the relations (3.3) implies:

$$t^a_{|ij} = t^a_{|ji} = \Omega(\nabla)^a_{ijb}t^b.$$

Then, precedent relations prove Lemma 3.6.

Using precedent results and Remark (*) one obtain

Theorem 3.7. *Let M be a C^∞-diff, paracompact, $\mathscr{F}$-foliated manifold. Let t be a C^∞-diff. vector field, $t(x) \neq 0$, $\forall x \in M$. If ∇ is an $\mathscr{F}$−symmetric connection on M, then the following affirmations are equivalent:*

1) ∇ is an $\mathscr{F}$-flat connection;

2) ∇ determines an $\mathscr{F}$-flat structure on M;

3) Mixed covariant derivations of t in connection ∇, along the leaves of $\mathscr{F}$, coincide.

Proof. It is clear, from the Lemma 3.6, that 1)$\leftrightarrow$2). From the Remark (*) and Lemma 3.6 follows that 2)$\leftrightarrow$3). This proves the theorem.

4 Integrable $\mathscr{F}$-flat structures on the total space of a vector bundle over an $\mathscr{F}$-foliated manifold

Define a differentiable structure on the total space E of $\xi = (E, \pi, M)$ in the following way. Consider a trivializing atlas $A_1 = \{(U_\alpha, \varphi_\alpha, R^m)\}_{\alpha\in I}$ of E and $\mathscr{F} = \{(U_\alpha, \psi_\alpha)\}_{\alpha\in I} = \left\{\left(U_\alpha; x^k, x^{\widehat{k}}\right)\right\}_{\alpha\in I}$. Then, the atlas of E is $A = \left\{\left(\pi^{-1}U_\alpha, h_\alpha\right)\right\}_{\alpha\in I} = \left\{\left(U_\alpha; x^k, x^{\widehat{k}}, y^a\right)\right\}_{x\in I}$ where $h_\alpha : \pi^{-1}U_\alpha \to R^m \times R^n$, $h_\alpha(u) = \left(\psi_\alpha(\pi(u)), \varphi_{\alpha,\pi(u)}(u)\right)$, $u \in \pi^{-1}U_\alpha \subset E$. The coordinates change for A is: $x^{k'} = x^{k'}\left(x^k, x^{\widehat{k}}\right)$, $x^{\hat{k}'} = x^{\hat{k}'}\left(x^{\hat{k}}\right)$, $y^{a'} = M_a^{a'}(x)y^a$, $x = \psi_\alpha^{-1}\left(x^k, x^{\widehat{k}}\right) = \psi_\beta^{-1}\left(x^{k'}x^{\hat{k}'}\right)$, where $(M_a^{a'}(x))$ is a field-matrices that describes the precedent coordinates change in $\pi^{-1}(x)$, $(M_a^{a'}(x)) \in GL(n, R)$.

An $\mathscr{F}$-flat structure Λ on E is integrable if Λ is defined by the family of frames $\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^{\widehat{k}}}, \frac{\partial}{\partial y^a}\right)$, $a = 1, 2, ..., n$.

Denote $\Gamma = \left\{\begin{pmatrix} \alpha & \beta & 0 \\ 0 & \gamma & 0 \\ 0 & \delta & \varepsilon \end{pmatrix}\right\}$ where α, β, γ are $p \times p$, $(n-p) \times (n-p)$, $n \times n$ real matrices, respectively. We remark that Γ is a subgroup of $GL(m+n, R)$.

Theorem 4.1. *Let M be a C^∞-diff., paracompact, $\mathscr{F}$-foliated manifold and $\xi = (E, \pi, M)$ a vector bundle over M. Suppose that E has a differentiable structure defined by the atlas A. Then:*

1) The atlas $\mathscr{F} = \{(U_\alpha, x^k, x^{\widehat{k}})\}_{\alpha\in I}$ defines an integrable $\mathscr{F}$-flat structure Λ on TM if and only if $\mathscr{F}$ induces a locally affine structure on the leaves of $\mathscr{F}$. Moreover, in this case,

$$\Lambda = \left\{\left(\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^{\widehat{k}}}\right); \overset{1}{J}_{\alpha\beta}(x)\right)\right\}_{\alpha,\beta\in I}, \; x \in U_\alpha \cap U_\beta,$$

where $\overset{1}{J}_{\alpha\beta}(x) = \begin{pmatrix} A_i^{i'} & \frac{\partial b^{i'}(x^{\widehat{k}})}{\partial x^{\widehat{i}}} \\ 0 & \frac{\partial x^{\widehat{i}'}}{\partial x^{\widehat{i}}} \end{pmatrix}(x)$, $\left(A_i^{i'}\right)$ *is a constant $p \times p$ matrix,* $\det\left(A_i^{i'}\right) \neq 0$ *and* $b^{i'}\left(x^{\widehat{r}}\right)$ *are arbitrary real functions on $U_\alpha \cap U_\beta$.*

2) The atlas $\mathscr{F}$ defines an integrable $\mathscr{F}$-flat structure $\Lambda = \left\{\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^{\widehat{k}}}, \frac{\partial}{\partial y^a}\right); \overset{2}{J}_{\alpha\beta}(x)\right\}_{\alpha,\beta\in I}$ *on TM if and only if the atlas A defines an integrable Γ-structure on the manifold E, where* $\overset{2}{J}_{\alpha\beta}(u) \in \Gamma$,

$$\overset{2}{J}_{\alpha\beta}(u) = \begin{pmatrix} A_i^{i'} & \frac{\partial b^{i'}}{\partial x^{\widehat{i}}} & 0 \\ 0 & \frac{\partial x^{\widehat{k}'}}{\partial x^{\widehat{k}}} & 0 \\ 0 & \frac{\partial \widetilde{g}_a^{a'}(x^{\widehat{k}})}{\partial x^{\widehat{k}}} & \widetilde{g}_a^{a'}(x^{\widehat{k}}) \end{pmatrix}(u),$$

$u \in \pi^{-1}(U_\alpha \cap U_\beta)$, $\widetilde{g}_a^{a'}(x^{\widehat{k}})$ *are some real functions,* $\det(\widetilde{g}_a^{a'}(x^{\widehat{k}})) \neq 0$.

Proof. 1) We determine $\overset{1}{J}_{\alpha\beta}(x)$. To obtain $\left(x^{i'}\right)$ consider the system P.D.E. $\frac{\partial x^{i'}}{\partial x^i} = A_i^{i'}(\widehat{x})$, $\frac{\partial x^{i'}}{\partial x^{\widehat{i}}} = B_{\widehat{i}}^{i'}(\widehat{x})$, $\widehat{x} = \left(x^{\widehat{i}}\right)$, where $A_i^{i'}(\widehat{x})$, $B_{\widehat{i}}^{i'}(\widehat{x})$ are arbitrary real functions. The solution of the first equations is $x^{i'} = A_i^{i'}(\widehat{x}) + b^{i'}(\widehat{x})$, where $b^{i'}(\widehat{x})$ denote arbitrary real functions. Now, we require that the functions $(x^{i'})$ to verify the last equations:

$$\frac{\partial A_i^{i'}}{\partial x^{\widehat{i}}} + \frac{\partial b^{i'}}{\partial x^{\widehat{i}}} = B_{\widehat{i}}^{i'}(\widehat{x}),$$

$i, i', j, j' = 1, 2, ..., p; \widehat{i}, \widehat{i}' ... = p+1, p+2, ..., m$. The integrability conditions for this system of P.D.E. are

$$\frac{\partial^2 x^{i'}}{\partial x^i \partial x^{\widehat{j}}} = \frac{\partial A_i^{i'}}{\partial x^{\widehat{j}} \partial x^i} = \frac{\partial^2 x^{i'}}{\partial x^{\widehat{j}} \partial x^i} = \frac{\partial B_{\widehat{i}}^{i'}}{\partial x^i} = 0.$$

Therefore, $A_i^{i'}(\widehat{x})$ =constant. Using precedent relations, we have $\frac{\partial b^{i'}}{\partial x^{\widehat{i}}} = B_{\widehat{i}}^{i'}(\widehat{x})$, and hence $b^{i'}$ depends only on $\widehat{x} = \left(x^{\widehat{i}}\right)$. The coordinate transformations are: $x^{i'} = A_i^{i'} x^i + b^{i'}(\widehat{x})$, $x^{\widehat{i}'} = x^{\widehat{i}'}(x^{\widehat{i}})$. Hence $(x^{i'})$ defines an affine structure on a leaf of $\mathscr{F}$. Therefore

$$\Lambda = \left\{\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\widehat{i}}}\right); \overset{1}{J}_{\alpha\beta}(x)\right\}_{\alpha,\beta\in I},$$

where

$$\overset{1}{J}_{\alpha\beta}(x) = \begin{pmatrix} A_i^{i'} & \frac{\partial b^{i'}(\widehat{x})}{\partial x^{\widehat{i}}} \\ 0 & \frac{\partial x^{\widehat{i}'}}{\partial x^{\widehat{i}}} \end{pmatrix}(x).$$

Λ is an integrable $\mathscr{F}$-flat structure on TM.

Conversely: Let F be an arbitrary leaf of $\mathscr{F}$ defined by the equations $x^{\widehat{i}}$ =constant. Using precedent notations, the locally affine structure on F is given by $x^{i'} = A_i^{i'} x^i + b^{i'}$. Consequently:

$$\Lambda = \left\{ \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\hat{i}}} \right); \overset{1}{J}_{\alpha\beta}(x) \right\},$$

$$\overset{1}{J}_{\alpha\beta}(x) = \begin{pmatrix} A_i^{i'} & \frac{\partial b^{i'}}{\partial x^{\hat{i}}} \\ 0 & \frac{\partial x^{\hat{i}'}}{\partial x^{\hat{i}}} \end{pmatrix}(x).$$

2) Remark: If $\mathscr{F}$ defines an $\mathscr{F}$.f.s. on TM given by $\overset{1}{J}_{\alpha\beta}(x)$, then $\overset{1}{J}_{\alpha\beta}(x)$ is a submatrix of $\overset{2}{J}_{\alpha\beta}(u)$, $\pi(u) = x \in U_\alpha \cap U_\beta$. Hence, the problem is to determine all the elements of $\overset{2}{J}_{\alpha\beta}(u)$ in the hypothesis that $\overset{2}{J}_{\alpha\beta}(u)$ is independent from (x^i). To obtain the last line of $\overset{2}{J}_{\alpha\beta}(u)$, we use the coordinate transformations:

$$\begin{aligned} & x^{k'} = A_k^{k'} x^k + b^{k'}(x^{\hat{k}}),\ x^{\hat{k}'} = x^{\hat{k}'}(x^{\hat{k}}), \\ & y^{a'} = M_a^{a'}(x) y^a \\ & x = \psi_\alpha^{-1}(x^k, x^{\hat{k}}) = \psi_\beta^{-1}(x^{k'}, x^{\hat{k}'}). \end{aligned} \tag{4.1}$$

By an abuse of notation, we write $M_a^{a'}(x^k, x^{\hat{k}})$, for $(M_a^{a'} \circ \Psi_\alpha^{-1})(x^k, x^{\hat{k}})$.

We obtain the system of P.D.E.:

$$\begin{aligned} & \frac{\partial y^{a'}}{\partial x^k} = \frac{\partial (M_a^{a'}(x^k, x^{\hat{k}}))}{\partial x^k} y^a, \\ & \frac{\partial y^{a'}}{\partial x^{\hat{k}}} = \frac{\partial M_a^{a'}(x^k, x^{\hat{k}})}{\partial x^{\hat{k}}} y^a, \\ & \frac{\partial y^{a'}}{\partial y^a} = M_a^{a'}(x^k, x^{\hat{k}}). \end{aligned} \tag{4.2}$$

The solution of the system (4.2) do not depend on (x^k) if and only if there exist some functions $f_{a_i}^{a'}(x^{\hat{k}})$, $g_{a^{\hat{i}}}^{a'}(x^{\hat{k}})$ so that

$$\frac{\partial M_a^{a'}}{\partial x^i} = f_{a_i}^{a'}(x^{\hat{k}}),\ \frac{\partial M_a^{a'}}{\partial x^{\hat{k}}} = g_{a_{\hat{i}}}^{a'}(x^{\hat{k}}), \tag{4.3}$$

$i, j, k = 1, 2, ..., p; \hat{i}, \hat{j}, \hat{k} = p+1, p+2, ..., n, a, a' = 1, 2, ..., m.$

Solve the system (4.3). Obtain $M_a^{a'} = f_{ai}^{a'}(x^{\hat{k}}) x^i + \tilde{g}_a^{a'}(x^{\hat{k}})$, where $\tilde{g}_a^{a'}(x^{\hat{k}})$ are arbitrary real functions, $\det(\tilde{g}_a^{a'}(x^{\hat{k}})) \neq 0$. Now we want that $M_a^{a'}$ to verify the last equations (4.3):

$$\frac{\partial M_a^{a'}}{\partial x^{\hat{i}}} = \frac{\partial f_{a_i}^{a'}(x^{\hat{k}})}{\partial x^{\hat{i}}} + \frac{\partial \tilde{g}_a^{a'}(x^{\hat{k}})}{\partial x^{\hat{i}}} = g_{a^{\hat{j}}}^{a'}(x^{\hat{k}}). \tag{4.4}$$

For the integrability conditions we use (4.3) and (4.4):

$$\frac{\partial^2 M_a^{a'}}{\partial x^i \partial x^{\hat{j}}} = \frac{\partial f_{a_i}^{a'}}{\partial x^{\hat{j}}} = \frac{\partial^2 M_a^{a'}}{\partial x^{\hat{j}} \partial x^i} = \frac{\partial g_{a_{\hat{i}}}^{a'}}{\partial x^i} = 0.$$

Hence $f_{a_i}^{a'}$ =constant, and therefore $\frac{\partial \tilde{g}_a^{a'}}{\partial x^{\hat{i}}} = g_{a_{\hat{i}}}^{a'}(x^{\hat{k}})$, i.e. $\tilde{g}_a^{a'}$ do not depend on (x^i).

Coordinate change of A are:

$$\begin{aligned} & x^{k'} = A_i^{k'} x^i + b^{k'}(x^{\hat{k}}),\ x^{\hat{k}'} = x^{\hat{k}'}(x^{\hat{k}}), \\ & y^{a'} = (f_{a_i}^{a'} x^i + \tilde{g}_a^{a'}(x^{\hat{k}})) y^a. \end{aligned} \tag{4.5}$$

The last elements of $\overset{2}{J}_{\alpha\beta}(u)$ are:

$$\frac{\partial y^{a'}}{\partial x^k} = \frac{\partial}{\partial x^k}(f_{ak}^{a'} x^k + \tilde{g}_a^{a'}) y^a = f_{ak}^{a'} y^a,$$

$$\frac{\partial y^{a'}}{\partial x^{\hat{k}}} = \frac{\partial (\tilde{g}_a^{a'}(x^{\hat{k}}))}{\partial x^{\hat{k}}} y^a,$$

$$\frac{\partial y^{a'}}{\partial y^a} = f_{ak}^{a'} x^k + \tilde{g}_a^{a'}(x^{\hat{k}}).$$

The matrix $\overset{2}{J}_{\alpha\beta}(u)$ defines an $\mathscr{F}$.f.s. if $\overset{2}{J}_{\alpha\beta}(x) = \overset{2}{J}_{\alpha\beta}(x^{\hat{k}})$. Therefore, $f_{ak}^{a'} = 0$.

The last element of $\overset{2}{J}_{\alpha\beta}(u)$ are: $\left(0, \frac{\partial \tilde{g}_a^{a'}(x^{\hat{k}})}{\partial x^{\hat{k}}}, \tilde{g}_a^{a'}(x^{\hat{k}})\right)$ and hence $\overset{2}{J}_{\alpha\beta}(u) \in \Gamma$. Therefore, A defines an integrable Γ-structure on E.

Conversely: If there exists the matrix $\overset{2}{J}_{\alpha\beta}(u)$ from theorem, then the existence of $\overset{2}{J}_{\alpha\beta}(u)$ determines the existence of $\overset{1}{J}_{\alpha\beta}(u)$, $x = \pi(u) \in U_\alpha \cap U_\beta$. The affirmation follows.

Conclusions: The F-flat structures are based on families of local frames linked to each other by constant matrices along the leaves of foliation.

These structures are usefully in the study of vector field on Riemannian foliated manifolds. The remarkable structures included in the work are of interest for the total space of a vector bundle.

It would be interesting to develop the study of these structures on analytic complex manifold.

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