Investigating Stochastic Dynamics of the Gilpin-Ayala Model in Dispersed Polluted Environments

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Abstract: This research delves into the analysis of a stochastic Gilpin-Ayala model operating within an anxious environment, encompassing the phenomenon of diffusion between two distinct and specified geographical regions that are the subjects of investigation. Initially, we rigorously formulate the essential criteria for ascertaining the survival or extinction of the species. Furthermore, we furnish empirical substantiation for the presence of a stable distribution. A significant milestone of our study involves the discernment and comprehensive delineation of the pivotal determinants that intricately regulate extinction dynamics and persistence within the framework of pollution parameters. This outcome underscores the pronounced impact of pollution on ecological dynamics and affirms the necessity of incorporating pollution parameters into the purview of environmental investigations. This revelation demonstrates that in the absence of pollution, the conventional criteria governing extinction and persistence closely parallel those witnessed in unpolluted environments, thus validating the robustness of our mathematical analysis. A series of numerical depictions are introduced to validate and provide empirical support for the acquired results.

Key-Words: Stochastic Models, White Noise, Environmental Pollution, Extinction, Persistence, Stationary Distribution, Lyapunov Function, Threshold

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1 Introduction

Investigating ecological systems in the presence of environmental contaminants has become a crucial and highly regarded field of scientific research. There is a growing concern regarding the impact of pollution on biodiversity and the overall health of ecosystems. Within this realm, it is particularly significant to comprehend the behavior of populations in environments that are dispersed with pollutants. To address this, the Gilpin-Ayala model has emerged as an invaluable tool for studying the effects of pollutants on population dynamics. This model allows researchers to gain valuable insights into the interplay between species interactions and environmental stressors. Population ecology is a specialized branch of ecology that focuses on studying the intricate dynamics of species populations and their interactions with the environment. However, due to the accelerated growth of industries and agriculture, a wide range of toxic substances are being discharged into the atmosphere, posing significant risks to the survival of exposed organisms. The proliferation of species in the natural realm is intricately interconnected with the influence of environmental noise (refer to [1], [2], [3], [4], [5], [6], [7], [8], [9], for further details). Consequently, several esteemed authors have delved into stochastic population models to understand the dynamics within polluted environments (see, for instance, [10], [11]). Notably, [12] conducted an in-depth investigation into a stochastic model that specifically examines the effects of toxins on a single-species system.

$$\begin{cases} dx = x(t)\left[r_0 - k_0 x(t) - l_1 c_0(t)\right] dt \\ \qquad +\sigma_1 x(t) dB_1(t), \\ dc_0 = \left[-(g+m)c_0(t) + k c_e(t)\right] dt, \\ dc_e = \left[-h c_e(t) + u(t)\right] dt, \end{cases} \tag{1}$$

where $x(t)$ represents the population size at time t, the parameter $r_0 > 0$ corresponds to the inherent growth rate of the population in the absence of any toxicant. Furthermore, $l_1 > 0$ indicates the response rate of the population to the pollutant present in the organism. The variables c_0 and c_e denote the concentrations of noxious compounds within the organism and the surrounding environment, respectively. Additionally, B_1 denotes a conventional standard Brownian motion, delineated within the confines of a comprehensive probability space denoted as $(\Omega, \mathcal{F}, \mathbb{P})$. The parameter σ_1 signifies the magnitude of the white noise intensity, while $k > 0$ signifies the rate at which the organism absorbs the toxicant from its surrounding environment. The parameters $g > 0$ and $m > 0$ represent the egestion and depuration rates of the toxicant within the organism, respectively. Similarly, the parameter $h > 0$ represents the rate at which the toxicant undergoes environmental loss. Furthermore, $u(t)$ is a non-negative, bounded, and continuous function defined over the interval $[0, +\infty)$, representing the exogenous rate at which toxicants are introduced into the environment. The authors [12] have effectively ascertained the threshold governing species persistence or extinction and concurrently established requisite conditions for the stochastic permanence of the population. Moreover, in a related study, [13], the authors examined a scenario where both the parameters l_1 and k_0 in the model (1) are also subject to random fluctuations. By applying the Gilpin-Ayala model, they derived the following stochastic single-species model, taking into account the effects of the toxicant.

$$\begin{cases} dx = x(t)[r_0 - k_0 x^{\theta_1}(t) - l_1 c_0(t)]dt \\ \qquad +\sigma_1 x(t)dB_1(t) + \sigma_2 c_0(t)x(t)dB_1(t) \\ \qquad +\sigma_1 x^{1+\theta_1}(t)dB_3(t), \\ dc_0 = [-(g+m)c_0(t) + kc_e(t)]dt, \\ dc_e = [-hc_e(t) + u(t)]dt. \end{cases} \tag{2}$$

However, species dispersal is a widely recognized occurrence in the natural world. It frequently transpires within patches of ecological environments, mainly due to the influence of human activities and industries on the environment. Various factors have partitioned reproductive and population-centric domains and other habitats into discrete patches. These factors encompass the spatial distribution of industrial facilities and the pollution of the air, soil, and water bodies. This issue has been extensively studied and documented (refer to [14], [15], [16], [17]). The present study aims to examine the implications of the dispersal phenomenon. To achieve this, we have developed a stochastic diffusion system comprising two patches affected by a toxicant.

$$\begin{cases} dx_1 &= \Big[x_1\Big(r_1 - k_1 x_1^{\theta_1} - l_1 c_0(t)\Big) \\ & \quad +\varepsilon_{12}(x_2 - x_1)\Big]dt + \sum_{i=1}^{n}\Big[\alpha_{1i}x_1 \\ & \quad +\beta_{1i}x_1^{1+\theta_1} + \gamma_{1i}x_1 c_0(t)\Big]dB_i, \\ dx_2 &= \Big[x_2\Big(r_2 - k_2 x_2^{\theta_2} - l_2 c_0(t)\Big) \\ & \quad +\varepsilon_{21}(x_1 - x_2)\Big]dt + \sum_{i=1}^{n}\Big[\alpha_{2i}x_2 \\ & \quad +\beta_{2i}x_2^{1+\theta_2} + \gamma_{2i}x_1 c_0(t)\Big]dB_i, \\ dc_0 &= [-(g+m)c_0(t) + kc_e(t)]dt, \\ dc_e &= [-hc_e(t) + u(t)]dt. \end{cases} \tag{3}$$

Within this framework, x_i signifies the population density for a particular species residing in the ith patch. Similarly, the r_i and k_i denote the growth rate and self-competition coefficient that regulate the population dynamics within those above the ith patch. Furthermore, we assign the symbol $\varepsilon_{i,j} > 0$ to represent a diffusion coefficient. This represents a non-negative parameter that characterizes species migration from the jth patch to the ith patch (with the condition $i \neq j$). We assume that the net migration of individuals from the jth patch to the ith patch is directly proportional to the disparity in population densities $(x_i - x_j)$ between these patches. This assumption aligns with the commonly accepted framework in similar research endeavors (refer to [18], [19], for more details). The vectors $\alpha_i = (\alpha_{1i}, \alpha_{2i}, ..., \alpha_{ni})$ and $\beta_i = (\beta_{1i}, \beta_{2i}, ..., \beta_{ni})$ represent the magnitudes of white noise signals at positions r_i and k_i, respectively. In order to capture the correlation between the noises at r_i and k_i, we utilize a vector $B(t) = (B_1(t), B_2(t), ..., B_n(t))^T$, which denotes an n-dimensional Brownian motion. In this study, we consider a comprehensive probability space denoted as $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, where Ω represents the sample space, $\mathcal{F}$ denotes the sigma-algebra of events, and $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration satisfying the standard conditions. Throughout our analysis, we use the customary inner product $\langle ., . \rangle$ and the Euclidean norm $|.|$ defined on $\mathbb{R}^n$. The paper is structured as follows: Section 2 addresses the issue of determining the existence and uniqueness of a positive solution for the system described in equation (1). In Section 3, a rigorous survival analysis is conducted to establish sufficient conditions for various ecological outcomes, encompassing scenarios of extinction, non-persistence in the mean, weak persistence, and stochastic permanence of the species. Section 4 further contributes to the field by demonstrating the existence of a stationary distribution, building upon and improving the previously proposed sufficient condition by [19]. Finally, the main findings of the study are illustrated through numerical simulations, offering visual insights into the observed phenomena.

2 Existence and Uniqueness

With reference to [12], we have the following lemma.

Lemma 2.1 *If*

$$\limsup_{t\to\infty} u(t) \leq h \quad \textit{and} \quad 0 < k \leq g + m,$$

then, for each $t \geq 0$,

$$0 \leq c_e(t) < 1, \quad 0 \leq c_0(t) < 1.$$

Therefore, we assume that

$$0 < k \leq g + m \quad \text{and} \quad \limsup_{t\to\infty} u(t) \quad \leq \quad h.$$

It is imperative to acknowledge that the final two equations in a model (1) exhibit linearity concerning $c_0(t)$ and $c_e(t)$. Consequently, deriving their explicit solutions becomes straightforward. Subsequently, our focus narrows down to exclusively addressing the initial two equations within a model (1), namely:

$$\begin{cases} dx_1 = \Big[x_1\Big(r_1 - l_1c_0(t) - k_1x_1^{\theta_1}\Big) \\ \qquad +\varepsilon_{12}(x_2-x_1)\Big]dt + \sum_{i=1}^{n}\Big(\alpha_{1i}x_1 \\ \qquad +\beta_{1i}x_1^{1+\theta_1} + \gamma_{1i}x_1c_0(t)\Big)dB_i, \\ dx_2 = \Big[x_2\Big(r_2 - l_2c_0(t) - k_2x_2^{\theta_2}\Big) \\ \qquad +\varepsilon_{21}(x_1-x_2)\Big]dt + \sum_{i=1}^{n}\Big(\alpha_{2i}x_2 \\ \qquad +\beta_{2i}x_2^{1+\theta_2} + \gamma_{2i}x_1c_0(t)\Big)dB_i. \end{cases} \tag{4}$$

The population densities, denoted as x_1 and x_2, possess biological significance, requiring them to be nonnegative. To address this requirement, we will analyze system (4) within a specific region

$$\mathbb{R}^2_+ = \{(x_1,x_2)|x_i>0,\ i=1,2\}.$$

We will now demonstrate that the set $\mathbb{R}^2_+$ is a positive invariant set.

Theorem 2.2 *For each $(x_1(0),x_2(0)) \in \mathbb{R}^2_+$, there exists a unique solution $(x_1(t),x_2(t))$ to system (4) for $t \geq 0$. Furthermore, the solution will remain within $\mathbb{R}^2_+$ with a probability 1.*

Proof 1 *Since all the coefficients in system (4) exhibit local Lipschitz continuity. Therefore, for any initial value $(x_1(0),x_2(0)) \in \mathbb{R}^2_+$, there exists a unique local solution $(x_1(t),x_2(t))$ for $t \in [0,\tau_e[$, where τ_e represents the explosion time (refer to [20], [21], [22], [23], for more details). To prove that this solution is global, it is necessary to prove that $\tau_e = \infty$. Let $k_0 \geq 0$ be such that*

$$x_i(0) \in \left[\frac{1}{k_0}, k_0\right], \quad i=1,2.$$

For each integer $k \geq k_0$, we define stopping times as follows

$$\tau_k = \inf\left\{t \in [0,\tau_e];\ x_i(t) \notin \left]\frac{1}{k_0}, k\right[,\ i=1,2\ \right\}.$$

Set $\tau_\infty = \lim_{k\to+\infty} \tau_k$, thus $\tau_\infty \leq \tau_e$ a.s..
Assume that τ_∞ finite, then there are two constants $T > 0$ and $\varepsilon \in (0,1)$, where $\mathbb{P}(\tau_\infty \leq T) > 2\varepsilon$. Then, there exists $k_1 \geq k_0$, where $\mathbb{P}(\tau_k \leq T) \geq \varepsilon$. There is an integer $k \geq k_1$, denote $\Omega_k = \{\tau_k \leq T\}$, then

$$\mathbb{P}(\Omega_k) \geq \varepsilon. \tag{5}$$

Define the function $V \in \mathcal{C}^2(\mathbb{R}^2_+;\mathbb{R}_+)$ such that

$$\begin{aligned} V(x_1,x_2) &= [2\sqrt{x_1} - \ln(x_1)] \\ &\quad + [2\sqrt{x_2} - \ln(x_2)]. \end{aligned} \tag{6}$$

It is important to note that for each $\omega \in \Omega_k$, there exists an i such that $x_i(\tau_k,\omega)$ takes the value of either k or $\frac{1}{k}$. Moreover, the function $V(x_1(\tau_k,\omega),x_2(\tau_k,\omega))$ is guaranteed to be greater than or equal to either

$$\left[\sqrt{k} - 1 - 0.5\ln(k)\right] \ or \ \left[\frac{1}{\sqrt{k}} - 1 - 0.5\ln\left(\frac{1}{k}\right)\right].$$

Hence, we can deduce

$$V\left(x_1(\tau_k,\omega),x_2(\tau_k,\omega)\right) \geq \check{V}(k), \tag{7}$$

where

$$\begin{aligned} \check{V}(k) &= \left[\sqrt{k} - 1 - 0.5\ln(k)\right] \\ &\quad \wedge \left[\frac{1}{\sqrt{k}} - 1 - 0.5\ln\left(\frac{1}{k}\right)\right]. \end{aligned}$$

Applying Itô formula, we obtain

$$\begin{aligned} dV &= \left(x_1^{-0.5} - x_1^{-1}\right)\Bigg[\Bigg(x_1\left(r_1 - l_1c_0(t) - k_1x_1^{\theta_1}\right) \\ &\quad +\varepsilon_{12}(x_2-x_1)\Bigg)dt + \sum_{i=1}^{n}\Bigg(\alpha_{1i}x_1 + \beta_{1i}x_1^{1+\theta_1} \\ &\quad +\gamma_{1i}x_1c_0(t)\Bigg)dB_i\Bigg)\Bigg] + \left(x_2^{-0.5} - x_2^{-1}\right) \\ &\quad \times\Bigg[\Bigg(x_2(r_2 - l_2c_0(t) - k_2x_2^{\theta_2}) \\ &\quad +\varepsilon_{21}(x_1-x_2)\Bigg)dt + \sum_{i=1}^{n}\Bigg(\alpha_{2i}x_2 + \beta_{2i}x_2^{1+\theta_2} \\ &\quad +\gamma_{2i}x_1c_0(t)\Bigg)dB_i\Bigg] + \left(-0.25x_1^{-1.5} + 0.5x_1^{-2}\right) \\ &\quad \times\sum_{i=1}^{n} x_1^2\Bigg(\alpha_{1i} + \beta_{1i}x_1^{\theta_1} + \gamma_{1i}c_0(t)\Bigg)^2 dt \\ &\quad + \left(-0.25x_2^{-1.5} + 0.5x_2^{-2}\right)\sum_{i=1}^{n} x_2^2\Bigg(\alpha_{2i} \\ &\quad +\beta_{2i}x_2^{\theta_2} + \gamma_{2i}c_0(t)\Bigg)^2 dt. \end{aligned}$$

For $i = 1, 2$, there exists $N_i > 0$, where

$$\left(x_i^{-0.5} - x_i^{-1}\right) \quad < \quad N_i, \quad for \quad t > 0.$$

Letting $N = \max\{N_1, N_2\}$, we get

$$\begin{aligned} dV \leq N\bigg[&- k_1 x_1^{1+\theta_1} + (r_1 - l_1 c_0(t) + \varepsilon_{21} \\ &-\varepsilon_{12})x_1\bigg] dt + \left(-0.25 x_1^{0.5} + 0.5\right) \sum_{i=1}^{n} \bigg(\alpha_{1i} \\ &+\beta_{1i} x_1^{\theta_1} + \gamma_{1i} c_0\bigg)^2 dt + N\bigg(- k_2 x_2^{1+\theta_2} \\ &+(r_2 - l_2 c_0(t) + \varepsilon_{12} - \varepsilon_{21}) x_2\bigg) dt \\ &+ \left(-0.25 x_2^{0.5} + 0.5\right) \sum_{i=1}^{n} \bigg(\alpha_{2i} + \beta_{2i} x_2^{\theta_2} \\ &+\gamma_{2i} c_0\bigg)^2 dt + (x_1^{0.5} - 1) \sum_{i=1}^{n} \bigg(\alpha_{1i} + \beta_{1i} x_1^{\theta_1} \\ &+\gamma_{1i} c_0(t)\bigg) dB_i + \left(x_2^{0.5} - 1\right) \sum_{i=1}^{n} \bigg(\alpha_{2i} \\ &+\beta_{2i} x_2^{\theta_2} + \gamma_{2i} c_0(t)\bigg) dB_i. \end{aligned} \tag{8}$$

Denote

$$\begin{aligned} g(x_1) = N\bigg[&- k_1 x_1^{1+\theta_1} + (r_1 - l_1 c_0(t) + \varepsilon_{21} \\ &-\varepsilon_{12})x_1\bigg] dt + \left(-0.25 x_1^{0.5} + 0.5\right) \\ &\times \sum_{i=1}^{n} \left(\alpha_{1i} + \beta_{1i} x_1^{\theta_1} + \gamma_{1i} c_0(t)\right)^2 dt, \end{aligned}$$

and

$$\begin{aligned} h(x_2) = N\bigg[&- k_2 x_2^{1+\theta_2} + (r_2 - l_2 c_0(t) + \varepsilon_{12} \\ &-\varepsilon_{21})x_2\bigg] dt + (- 0.25 x_2^{0.5} + 0.5) \\ &\times \sum_{i=1}^{n} \left(\alpha_{2i} + \beta_{2i} x_2^{\theta_2} + \gamma_{2i} c_0(t)\right)^2 dt. \end{aligned}$$

One can easily verify that

$$\lim_{x_1 \to +\infty} g(x_1) = \lim_{x_2 \to +\infty} h(x_2) = -\infty,$$

and then there is $M > 0$ such that

$$g(x_1) + h(x_2) < M. \tag{9}$$

From (8) and (9), we obtain

$$\begin{aligned} dV \leq M dt + \left(x_1^{0.5} - 1\right) \sum_{i=1}^{n} \bigg(&\alpha_{1i} + \beta_{1i} x_1^{\theta_1} \\ &+\gamma_{1i} c_0(t)\bigg) dB_i(t) + \left(x_2^{0.5} - 1\right) \\ &\times \sum_{i=1}^{n} \left(\alpha_{2i} + \beta_{2i} x_2^{\theta_2} + \gamma_{2i} c_0(t)\right) dB_i(t). \end{aligned}$$

By integrating both sides from 0 to $\tau_k \wedge T$, one has

$$\begin{aligned} \int_0^{\tau_k \wedge T} dV \leq \int_0^{\tau_k \wedge T} M dt + \int_0^{\tau_k \wedge T} &\bigg[\left(x_1^{0.5} - 1\right) \\ &\times \sum_{i=1}^{n} \left(\alpha_{1i} + \beta_{1i} x_1^{\theta_1} + \gamma_{1i} c_0(t)\right) \bigg] dB_i \\ &+ \int_0^{\tau_k \wedge T} \bigg[\left(x_2^{0.5} - 1\right) \sum_{i=1}^{n} \bigg(\alpha_{2i} \\ &+\beta_{2i} x_2^{\theta_2} + \gamma_{2i} c_0(t)\bigg)\bigg] dB_i(t). \end{aligned}$$

Consequently

$$\begin{aligned} &\mathbb{E}\left[V\left(x_1(\tau_k \wedge T), x_2(\tau_k \wedge T)\right)\right] \\ \leq \ & V\left(x_1(0), x_2(0)\right) + M\mathbb{E}\left[(\tau_k \wedge T)\right], \\ \leq \ & V\left(x_1(0), x_2(0)\right) + MT, \end{aligned}$$

which yields

$$\begin{aligned} &V\left(x_1(0), x_2(0)\right) + MT \\ \geq \ & \mathbb{E}\left[1_{\Omega_k} V\left(x_1(\tau_k, \omega), x_2(\tau_k, \omega)\right)\right]. \end{aligned} \tag{10}$$

By (5) and (10), we obtain

$$\begin{aligned} &V\left(x_1(0), x_2(0)\right) + MT \\ \geq \ & \varepsilon V\left(x_1(\tau_k, \omega), x_2(\tau_k, \omega)\right). \end{aligned} \tag{11}$$

From (7) and (11), one obtains

$$V\left(x_1(0), x_2(0)\right) + MT \quad \geq \quad \varepsilon \check{V}(k).$$

Letting $k \longrightarrow \infty$, we get the following contradiction

$$V\left(x_1(0), x_2(0)\right) + MT \quad = \quad \infty.$$

Therefore, we need to have $\tau_\infty = \infty \quad a.s.$.

3 Extinction

To comprehensively explore the intricate topic of species extinction, [24], utilizing a crucial lemma termed the exponential martingale inequality is paramount (refer to [25], Theorem 7.4, page 44).

Lemma 3.1 *Consider a local martingale $(M_t)_{t\geq 0}$, which vanishes at time zero. Thus, for each positive constants a, b, and c, we have*

$$\mathbb{P}\left[\sup_{0\leq t\leq a}\left(M_t - \frac{b}{2}[M_t, M_t]\right) > c\right] \quad \leq \quad \exp(-bc),$$

where $[M_t, M_t]$ is the quadratic variation of M_t.

Theorem 3.2 *For all $(x_1(0), x_2(0)) \in \mathbb{R}^2_+$, the solution of the SDE (4) obeys*

$$\limsup_{t\to\infty}\frac{1}{t}\log\left(\frac{x_1(t)}{\varepsilon_{12}} + \frac{x_2(t)}{\varepsilon_{21}}\right) \leq M - \frac{m^2}{2} \quad a.s.,$$

where

$$M = \max\left\{r_1 - l_1\inf_{t\geq 0}\{c_0(t)\}; r_2 - l_2\inf_{t\geq 0}\{c_0(t)\}\right\}, \tag{12}$$

and

$$m = \min_{1\leq i\leq n}\left\{\alpha_{1i} + \gamma_{1i}\inf_{t\geq 0}\{c_0(t)\}; \alpha_{2i} + \gamma_{2i}\inf_{t\geq 0}\{c_0(t)\}\right\}.$$

Moreover, if $\left(M - \frac{m^2}{2}\right) < 0$, then the species in (4) is extinct.

Proof 2 *We define*

$$X(t) = \frac{x_1(t)}{\varepsilon_{12}} + \frac{x_2(t)}{\varepsilon_{21}}.$$

Using the Itô formula, we obtain

$$\begin{aligned} d\log(X) = \; & \frac{1}{X(t)}\left[\frac{x_1(t)}{\varepsilon_{12}}(r_1 - l_1c_0(t)) - k_1x_1^{\theta_1}(t) + \frac{x_2(t)}{\varepsilon_{21}}(r_2 - l_2c_0(t)) - k_2x_2^{\theta_2}(t)\right]dt - \frac{1}{2(X(t))^2} \\ & \times\sum_{i=1}^{n}\left[\frac{x_1(t)}{\varepsilon_{12}}\left(\alpha_{1i} + \beta_{1i}x_1^{\theta_1}(t) + \gamma_{1i}c_0(t)\right) + \frac{x_2(t)}{\varepsilon_{21}}\left(\alpha_{2i} + \beta_{2i}x_2^{\theta_2}(t) + \gamma_{2i}c_0(t)\right)\right]^2 dt + \frac{1}{X(t)}\sum_{i=1}^{n}\left[\frac{x_1(t)}{\varepsilon_{12}}\right. \\ & \times\left(\alpha_{1i} + \beta_{1i}x_1^{\theta_1}(t) + \gamma_{1i}c_0(t)\right) + \frac{x_2(t)}{\varepsilon_{21}}\left(\alpha_{2i} + \beta_{2i}x_2^{\theta_2}(t) \left. + \gamma_{2i}c_0(t)\right)\right]dB_i. \end{aligned} \tag{13}$$

By integrating, we obtain

$$\begin{aligned} \log(X(t)) = & \int_0^t \frac{1}{X(s)}\left[\frac{x_1(s)}{\varepsilon_{12}}(r_1 - l_1c_0(s)) - k_1x_1^{\theta_1}(s) + \frac{x_2(s)}{\varepsilon_{21}}(r_2 - l_2c_0(s)) - k_2x_2^{\theta_2}(s)\right]ds - \int_0^t \frac{1}{2(X(s))^2} \\ & \times\sum_{i=1}^{n}\left[\frac{x_1(s)}{\varepsilon_{12}}\left(\alpha_{1i} + \beta_{1i}x_1^{\theta_1}(s) + \gamma_{1i}c_0(s)\right) + \frac{x_2(s)}{\varepsilon_{21}}\left(\alpha_{2i} + \beta_{2i}x_2^{\theta_2}(s) + \gamma_{2i}c_0(s)\right)\right]^2 ds + M_t + \log(X(0)), \end{aligned} \tag{14}$$

where

$$M_t = \int_0^t \frac{1}{X(s)}\sum_{i=1}^{n}\left[\frac{x_1(s)}{\varepsilon_{12}}\left(\alpha_{1i} + \beta_{1i}x_1^{\theta_1}(s) + \gamma_{1i}c_0(s)\right) + \frac{x_2(s)}{\varepsilon_{21}}\left(\alpha_{2i} + \beta_{2i}x_2^{\theta_2}(s) + \gamma_{2i}c_0(s)\right)\right]dB_i(s),$$

is a real-valued continuous local martingale vanishing at $t = 0$ with the quadratic variation

$$[M_t, M_t] = \int_0^t X^{-2}(s)\sum_{i=1}^{n}\left[\frac{x_1(s)}{\varepsilon_{12}}\left(\alpha_{1i} + \beta_{1i}x_1^{\theta_1}(s) + \gamma_{1i}c_0(s)\right) + \frac{x_2(s)}{\varepsilon_{21}}\left(\alpha_{2i} + \beta_{2i}x_2^{\theta_2}(s) + \gamma_{2i}c_0(s)\right)\right]^2 ds. \tag{15}$$

By Lemma 3.1, for any integer $k \geq 1$ and ϵ sufficiently small, we get

$$\mathbb{P}\left[\sup_{0\leq t\leq k}\left(M_t - \frac{\epsilon}{2}[M_t, M_t]\right) > \frac{2}{\epsilon}\log(k)\right] \leq \frac{1}{k^2}.$$

By the Borel-Cantelli Lemma, there is $\Omega_1 \subset \Omega$ with $\mathbb{P}(\Omega_1) = 1$, and for each $\omega \in \Omega_1$, there is $k_1(\omega)$ such that

$$M_t \quad \leq \quad \frac{2}{\epsilon}\log(k) + \frac{\epsilon}{2}[M_t, M_t], \tag{16}$$

where, $0 \leq t \leq k$ and $k \geq k_1(\omega)$.

Hence, from (14) and (16), we get for $\omega \in \Omega_1$, $k \geq k_1(\omega)$ *and* $0 \leq t \leq k$

$$\begin{aligned}
&\log(X(t)) \\
\leqslant\ & \int_0^t \frac{1}{X(s)}\Bigg[\frac{x_1(s)}{\varepsilon_{12}}\Big(r_1 - l_1 c_0(s) \\
&-k_1 x_1^{\theta_1}(s)\Big) + \frac{x_2(s)}{\varepsilon_{21}}\Big(r_2 - l_2 c_0(s) \\
&-k_2 x_2^{\theta_2}(s)\Big)\Bigg]ds - \int_0^t \frac{1-\epsilon}{2\,(X(s))^2} \\
&\times \sum_{i=1}^n \Bigg[\frac{x_1(s)}{\varepsilon_{12}}(\alpha_{1i} + \beta_{1i}x_1^{\theta_1}(s) + \gamma_{1i}c_0(s)) \\
&+\frac{x_2(s)}{\varepsilon_{21}}\Big(\alpha_{2i} + \beta_{2i}x_2^{\theta_2}(s) + \gamma_{2i}c_0(s)\Big)\Bigg]^2 ds \\
&+\log(X(0)) + \frac{2}{\epsilon}\log(k), \qquad (17)
\end{aligned}$$

which implies

$$\begin{aligned}
&\log(X(t)) \\
\leq & \int_0^t \frac{1}{X(s)}\Bigg[\frac{x_1(s)}{\varepsilon_{12}}(r_1 - l_1 c_0(s)) \\
&+\frac{x_2(s)}{\varepsilon_{21}}(r_2 - l_2 c_0(s))\Bigg]ds - \int_0^t \frac{1-\epsilon}{2}\Big(\frac{x_1(s)}{\varepsilon_{12}} \\
&+\frac{x_2(s)}{\varepsilon_{21}}t\Big)^{-2} \sum_{i=1}^n \Bigg[\frac{x_1(s)}{\varepsilon_{12}}(\alpha_{1i} + \gamma_{1i}c_0(s)) \\
&+\frac{x_2(s)}{\varepsilon_{21}}(\alpha_{2i} + \gamma_{2i}c_0(s))\Bigg]^2 ds + \log(X(0)) \\
&+\frac{2}{\epsilon}\log(k). \qquad (18)
\end{aligned}$$

Consequently, it is evident from (18) that

$$\begin{aligned}
\log(X(t)) \leq\ & \Big(M - \frac{1-\epsilon}{2}m^2\Big)t + \log(X(0)) \\
&+\frac{2}{\epsilon}\log(k). \qquad (19)
\end{aligned}$$

Take into account $\omega \in \Omega_1$ *and consider a value of* t *that is sufficiently large such that the maximum integer smaller than t fulfills the condition* $[t] \geq k_1(\omega)$. *Based on equation (19), we get*

$$\begin{aligned}
\frac{1}{t}\log(X(t)) \leq\ & M - \frac{1-\epsilon}{2}m^2 + \frac{1}{[t]}\Bigg[\log(X(0)) \\
&+\frac{1}{\epsilon}(2\log([t]+1))\Bigg], \qquad (20)
\end{aligned}$$

which gives

$$\limsup_{t\to\infty} \frac{1}{t}\log(X(t)) \quad \leq \quad M - \frac{1-\epsilon}{2}m^2.$$

Letting $\epsilon \longrightarrow 0$, *one obtains*

$$\limsup_{t\to\infty} \frac{1}{t}\log(X(t)) \quad \leq \quad M - \frac{1}{2}m^2.$$

This proves the theorem.

Theorem 3.3 *If* $\beta_{1i} = \beta_{2i} = 0$, *then*

$$\begin{aligned}
(i)\ \liminf_{t\to\infty} \frac{1}{t}\int_0^t x_1^{\theta_1}(s)ds \geq\ & \frac{1}{k_1}\Bigg[r_1 - l_1 - \varepsilon_{12} \\
&-\frac{1}{2}\|\alpha_1 + \gamma_1\|^2\Bigg],
\end{aligned}$$

$$\begin{aligned}
(ii)\ \liminf_{t\to\infty} \frac{1}{t}\int_0^t x_2^{\theta_2}(s)ds \geq\ & \frac{1}{k_2}\Bigg[r_2 - l_2 - \varepsilon_{21} \\
&-\frac{1}{2}\|\alpha_2 + \gamma_2\|^2\Bigg].
\end{aligned}$$

Proof 3 (i) *From (2),* $\beta_{1i} = \beta_{2i} = 0$ *and Itô formula, we get*

$$\begin{aligned}
d\log(x_1(t)) \geq & \Bigg[\Big(r_1 - l_1 - k_1 x_1^{\theta_1}(t)\Big) + \frac{\varepsilon_{12}}{x_1(t)} \\
&\times(x_2(t) - x_1(t)) - \frac{1}{2}\|\alpha_1 + \gamma_1\|^2\Bigg]dt \\
&+\sum_{i=1}^n (\alpha_{1i} + \gamma_{1i}c_0(t))\,dB_i, \qquad (21)
\end{aligned}$$

which implies

$$\begin{aligned}
&d\log(x_1(t)) \\
\geq\ & \Bigg[r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1 + \gamma_1\|^2 \qquad (22) \\
&-k_1 x_1^{\theta_1}(t)\Bigg]dt + \sum_{i=1}^n (\alpha_{1i} + \gamma_{1i}c_0(t))\,dB.
\end{aligned}$$

Integrating, we get

$$\begin{aligned}
\log(x_1(t)) \geq & \Bigg[r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1 + \gamma_1\|^2\Bigg]t \\
&-k_1\int_0^t x_1^{\theta_1}(s)ds + \int_0^t \sum_{i=1}^n (\alpha_{1i} \\
&+\gamma_{1i}c_0(t))dB_i + \log(x_1(0)).
\end{aligned}$$

Hence

$$\log\left(x_1^{\theta_1}\right) \leq \theta_1\left[r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1 + \gamma_1\|^2\right] t - \theta_1 k_1 \int_0^t x_1^{\theta_1}(s)ds + \theta_1 \int_0^t \sum_{i=1}^n (\alpha_{1i} + \gamma_{1i} c_0(t)) dB_i + \log(x_1^{\theta_1}(0)). \tag{23}$$

Thus

$$x_1^{\theta_1}(t)\exp\left(\theta_1 k_1 \int_0^t x_1^{\theta_1}(s)ds\right) \geq \exp\left[\theta_1\left(r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1 + \gamma_1\|^2\right) t + \theta_1 N_t + \log\left(x_1^{\theta_1}(0)\right)\right], \tag{24}$$

where

$$N_t = \int_0^t \sum_{i=1}^n \left(\alpha_{1i} + \gamma_{1i} c_0(t)\right) dB_i,$$

represents a continuous martingale with real-valued properties characterized by its quadratic variation.

$$[N_t, N_t] = \int_0^t \sum_{i=1}^n (\alpha_{1i} + \gamma_{1i} c_0(t))^2 ds.$$

According to the law of large numbers for martingales (refer to [25], for further information), one obtains

$$\lim_{t\to\infty} \frac{1}{t}\left[\theta_1\left(r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1 + \gamma_1\|^2\right) t + \theta_1 N_t + \log\left(x_1^{\theta_1}(0)\right)\right] = \theta_1\left[r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1 + \gamma_1\|^2\right] \quad a.s.. \tag{25}$$

From (25), one can easily show that there exists $\Omega_1' \subset \Omega$ such that $\mathbb{P}(\Omega_1') = 1$ and for each $\omega \in \Omega_1'$ and $\epsilon > 0$ sufficiently small, there is $T(\omega, \epsilon)$ such that for any $t \geq T$, we have

$$\left|\frac{1}{t}\left[\theta_1\left(r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1 + \gamma_1\|^2\right) t + \theta_1 N_t + \log\left(x_1^{\theta_1}(0)\right)\right] - \theta_1\left(r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1 + \gamma_1\|^2\right)\right| \leq \epsilon.$$

So

$$\theta_1\left[r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1 + \gamma_1\|^2\right] t + \theta_1 N_t + \log\left(x_1^{\theta_1}(0)\right) \geq \theta_1\left(r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1 + \gamma_1\|^2 - \epsilon\right) t,$$

which gives with (24) that for any $t \geq T$

$$x_1^{\theta_1}(t)\exp\left(\theta_1 k_1 \int_0^t x_1^{\theta_1}(s)ds\right) \geq \exp\left[\theta_1\left(r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1 + \gamma_1\|^2 - \epsilon\right) t\right], \tag{26}$$

or

$$x_1^{\theta_1}(t)\exp\left(\theta_1 k_1 \int_0^t x_1^{\theta_1}(s)ds\right) = \frac{1}{\theta_1 k_1}\frac{d}{dt}\left[\exp\left(\theta_1 k_1 \int_0^t x_1^{\theta_1}(s)ds\right)\right]. \tag{27}$$

By (26) and (27), we have

$$\frac{1}{\theta_1 k_1}\frac{d}{dt}\left[\exp\left(\theta_1 k_1 \int_0^t x_1^{\theta_1}(s)ds\right)\right] \geq \exp\left[\theta_1\left(r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1 + \gamma_1\|^2 - \epsilon\right) t\right]. \tag{28}$$

Hence, by integrating (28) from T to t yields that

$$\frac{1}{\theta_1 k_1}\left[\exp\left(\theta_1 k_1 \int_0^t x_1^{\theta_1}(s)ds\right) - \exp\left(\theta_1 k_1 \int_0^T x_1^{\theta_1}(s)ds\right)\right] \geq \frac{1}{\theta_1}\left(r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1 + \gamma_1\|^2 - \epsilon\right)^{-1} \times\left\{\exp\left[\theta_1\left(r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1 + \gamma_1\|^2 - \epsilon\right) t\right] - \exp\left[\theta_1\left(r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1 + \gamma_1\|^2 - \epsilon\right) T\right]\right\}.$$

Then, we have

$$\frac{1}{t}\int_0^t x_1^{\theta_1}(s)ds \geq \frac{1}{\theta_1 k_1 t}\Lambda(t), \tag{29}$$

where

$$\Lambda(t)=\log\Bigg\{\exp\left(\theta_1 k_1\int_0^T x_1^{\theta_1}(s)ds\right) + k_1\left(r_1-l_1-\varepsilon_{12}-\frac{1}{2}\|\alpha_1+\gamma_1\|^2-\epsilon\right)^{-1} \times\Bigg[\exp\left(\theta_1\left(r_1-l_1-\varepsilon_{12}-\frac{1}{2}\|\alpha_1+\gamma_1\|^2-\epsilon\right)t\right)-\exp\left(\theta_1\left(r_1-l_1-\varepsilon_{12}-\frac{1}{2}\|\alpha_1+\gamma_1\|^2-\epsilon\right)T\right)\Bigg]\Bigg\}.$$

If

$$\left[r_1-l_1-\varepsilon_{12}-\frac{1}{2}\|\alpha_1+\gamma_1\|^2-\epsilon\right] \le 0,$$

then

$$\lim_{t\to\infty}\frac{1}{t}\Lambda(t)=0.$$

Hence the assertion (i) holds trivially. Else if

$$\left(r_1-l_1-\varepsilon_{12}-\frac{1}{2}\|\alpha_1+\gamma_1\|^2-\epsilon\right)>0,$$

we easily have

$$\lim_{t\to\infty}\frac{\Lambda(t)}{t} = \theta_1\left[r_1-l_1-\varepsilon_{12}-\frac{1}{2}\|\alpha_1+\gamma_1\|^2-\epsilon\right].$$

Using (29) and (30), we get

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x_1^{\theta_1}(s)ds \ge \frac{1}{k_1}\left(r_1-l_1-\varepsilon_{12}-\frac{1}{2}\|\alpha_1+\gamma_1\|^2-\epsilon\right).$$

By letting $\epsilon\longrightarrow 0$, we get the required estimation (i). Similarly, we get the assertion (ii).

Based on Theorem 3.3, we can deduce the subsequent corollary

Corollary 3.1 *Suppose that $\beta_{1i}=\beta_{2i}=0$, thus*
(i) If $\left(r_1-l_1-\varepsilon_{12}-\frac{1}{2}\|\alpha_1+\gamma_1\|^2\right)>0$, then $x_1^{\theta_1}$ is strongly persistent in mean,
(ii) If $\left(r_2-l_2-\varepsilon_{21}-\frac{1}{2}\|\alpha_1+\gamma_2\|^2\right)>0$, then $x_2^{\theta_2}$ is strongly persistent in mean.

4 Stationary distribution

In this section, our investigation determines whether a solution to the SDE represented by equation (4) exhibits an asymptotically invariant distribution, indicating stability in a stochastic context. To shed light on this matter, we turn to the insightful analysis conducted by [26], whose theorem provides a comprehensive answer to this question. Let us consider a homogeneous Markov process denoted as $X(t)$, which is characterized by the following stochastic differential equation

$$dX(t)=b(X)dt+\sum_{r=1}^{n}\sigma_r(X)dB_r(t), \qquad (30)$$

$$X(t)=\begin{pmatrix} x_1(t)\\ x_2(t)\end{pmatrix},\quad b(X)=\begin{pmatrix} b_1(x_1,x_2)\\ b_2(x_1,x_2)\end{pmatrix},$$

$$b(X)=\begin{pmatrix} x_1(t)\left(r_1-l_1c_0(t)-k_1x_1^{\theta_1}(t)\right)\\ +\varepsilon_{12}\left(x_2(t)-x_1(t)\right)\\ x_2(t)\left(r_2-l_2c_0(t)-k_2x_2^{\theta_2}(t)\right)\\ +\varepsilon_{21}\left(x_1(t)-x_2(t)\right)\end{pmatrix},$$

$$\sigma_r(X)=\begin{pmatrix}\sigma_r^1(x)\\ \sigma_r^2(x)\end{pmatrix},$$

$$\sigma_r(X)=\begin{pmatrix}\alpha_{1r}x_1(t)+\beta_{1r}x_1^{1+\theta_1}(t)\\ +\gamma_{1r}x_1c_0(t)\\ \alpha_{2r}x_2(t)+\beta_{2r}x_2^{1+\theta_2}(t)\\ +\gamma_{2r}x_2c_0(t)\end{pmatrix}.$$

Let $V(x)\in\mathcal{C}^2(\mathbb{R}^2)$ be a function that is twice continuously differentiable. The differential operator $\mathcal{L}$ mentioned in equation (30) can be defined as follows

$$\mathcal{L}V(x)=\nabla V(x)b(x)+\frac{1}{2}Tr\left(A(x)\nabla^2V(x)\right),$$

where the gradient of the function $V(x)$, denoted as $\nabla V(x)$, and the hessian of $V(x)$, represented as $\nabla^2V(x)$, play crucial roles in this context. Additionally, the diffusion matrix A is defined to be associated with these quantities.

$$A(x)=(a_{ij}(x))_{1\leqslant i,j\leqslant 2}\ ,\ a_{ij}=\sum_{r=1}^{n}\sigma_r^i(x)\sigma_r^j(x).$$

So, the diffusion matrix associated with the system (30) can be expressed as follows

$$A(x_1,x_2)=\begin{pmatrix}\|Y\|^2 & <Y,Z>\\ <Y,Z> & \|Z\|^2\end{pmatrix}, \qquad (31)$$

where

$$Y = \Big[\alpha_{1r}x_1(t) + \beta_{1r}x_1^{1+\theta_1}(t) + \gamma_{1r}x_1c_0(t)\Big]_{1\le i\le n},$$

$$Z = \Big[\alpha_{2r}x_2(t) + \beta_{2r}x_2^{1+\theta_2}(t) + \gamma_{2r}x_2c_0(t)\Big]_{1\le i\le n}.$$

We can readily ascertain the positive definiteness of matrix A by verifying the strict inequality of the Cauchy-Schwarz inequality, expressed as

$$|< Y, Z >| \quad \le \quad \|Y\|.\|Z\|.$$

In cases where a system (30) lacks equilibrium states, exploring the potential existence of an asymptotically invariant distribution remains feasible. The authors [26] propose that it is sufficient to establish the positive recurrence of system solutions (30) concerning a bounded open set to pursue this investigation. We consider the $\mathbb{R}^2$-valued process $X(t, x_0)$ is recurrent for the bounded set $U \subset \mathbb{R}^2$ if $\mathbb{P}(\tau^{x_0} < \infty) = 1$, for each $x_0 \in U^c$. Here, τ^{x_0} represents the stopping time of U for the process $X(t, x_0)$, defined as

$$\tau^{x_0} \quad = \quad \inf\{t > 0,\ X(t, x_0) \in U\}.$$

The process $X(t, x_0)$ is said to be positive recurrent for the set U if it satisfies two conditions: first, it is recurrent for U, meaning that it revisits the set U infinitely often, and second, for any $x_0 \notin U$, $\mathbb{E}(\tau^{x_0}) < \infty$. A theorem exists that provides a criterion for positive recurrence based on the Lyapunov function (see, e.g., [26], and the references cited therein).

Theorem 4.1 *The system represented by (2) is said to be positively recurrent if there exists a bounded open subset $D \subset \mathbb{R}^2$ with a smooth boundary that satisfies the following conditions*
(i) *There is some $\kappa \in (0, 1]$ such that for all $(x_1, x_2) \in D$, $\xi \in \mathbb{R}^2$*

$$\kappa\|\xi\|^2 \le \xi^T A(x_1, x_2)\xi \le \kappa^{-1}\|\xi\|^2,$$

(ii) *There exists $V \in \mathcal{C}(D^c; \mathbb{R}^+)$ a nonnegative function that is twice continuously differentiable and for some $\varrho > 0$*

$$\mathcal{L}V(x_1, x_2) \le -\varrho \quad \textit{for all} \ (x_1, x_2) \in D^c.$$

Furthermore, the system (2) possesses a distinctive ergodic stationary distribution denoted as π, and the solution $(x_1(t), x_2(t))$ exhibits uniqueness concerning this distribution. If the function f is integrable with respect to the measure π, then

$$\mathbb{P}\left[\lim_{t\to\infty} \frac{1}{t}\int_0^t f(X(s))ds = \int_{\mathbb{R}^2} f(x)\pi(dx)\right] = 1.$$

The subsequent theorem provides a condition sufficient for a stationary distribution in our model (2).

Theorem 4.2 *Consider the stochastic system (2) with an initial condition in $\mathbb{R}^2_+$, where $n \ge 4$. Let us assume that α_1, α_2, β_1, β_2, γ_1, and γ_2 are linearly independent and satisfy the following conditions*

$$\left(r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1 + \gamma_1\|^2\right) > 0, \tag{32}$$

and

$$\left(r_2 - l_2 - \varepsilon_{21} - \frac{1}{2}\|\alpha_2 + \gamma_2\|^2\right) > 0.$$

Then, the solution $(x_1(t), x_2(t))$ of the SDE (2) admits a unique stationary distribution and is ergodic.

Proof 4 *Given a bounded open subset as described below*

$$D = \left(\frac{1}{\mu}, \mu\right) \times \left(\frac{1}{\mu}, \mu\right) \subset \mathbb{R}^2_+, \tag{33}$$

where μ represents a sufficiently large number. So, $\overline{D} \subset \mathbb{R}^2_+$.
(i) *If $\alpha_1, \alpha_2, \beta_1$, and β_2 are linearly independent, it follows that Y and Z are linearly independent. Consequently, based on the abovementioned observation, the matrix A has the potential to be positive and definite. This leads us to the following expression:*

$$\lambda_{min} = \lambda_{min}(A(x_1, x_2)) > 0, \tag{34}$$

and

$$\lambda_{max} = \lambda_{max}(A(x_1, x_2)) > 0,$$

where the quantities $\lambda_{min}(A(x_1, x_2))$ and $\lambda_{max}(A(x_1, x_2))$ denote the smallest and largest eigenvalues of the matrix $A(x_1, x_2)$, respectively. In addition, we can establish the following for all $\xi \in \mathbb{R}^2$

$$\lambda_{min}\|\xi\|^2 \leqslant \xi^T A(x_1, x_2)\xi \leqslant \lambda_{max}\|\xi\|^2. \tag{35}$$

It is evident that the functions $\lambda_{min}(A(.,.))$ and $\lambda_{max}(A(.,.))$ are continuous with respect to the variables (x_1, x_2). Consequently, using equation (34), we can deduce that

$$\lambda_1 = \min_{(x_1,x_2)\in\overline{D}} \lambda_{min}(A(x_1, x_2)) > 0, \tag{36}$$

and

$$\lambda_2 = \max_{(x_1,x_2)\in\overline{D}} \lambda_{max}(A(x_1, x_2)) > 0.$$

Furthermore, by (35), we get, for all $\xi \in \mathbb{R}^2$

$$\kappa\|\xi\|^2 \leq \xi^T A(x_1, x_2)\xi \leq \frac{1}{\kappa}\|\xi\|^2,$$

where $\kappa = \min\{\lambda_1, \lambda_2, 1\}$. *Consequently, we can confirm that the condition* (i) *specified in Theorem 4.1 holds for SDE (2).*

(ii) *Consider the following positive functions*

$$\psi_1(x_1) = \frac{1}{2}\log^2(x_1), \quad \psi_2(x_2) = \frac{1}{2}\log^2(x_2),$$

and

$$\psi_3(x_1, x_2) = \varepsilon_{21}x_1 + \varepsilon_{12}x_2,$$

and

$$\psi(x_1, x_2) = \psi_1(x_1) + \psi_2(x_2) + \psi_3(x_1, x_2).$$

Using Itô formula on the function ψ_1*, we get*

$$\begin{aligned}\mathcal{L}\psi_1(x_1) = \log(x_1)\Big[& r_1 - l_1c_0 - k_1x_1^{\theta_1} \\ &+\varepsilon_{12}\left(\frac{x_2}{x_1} - 1\right)\Big] + \frac{1}{2}(1 - \log(x_1)) \\ &\times \sum_{r=1}^{n}\left(\alpha_{1r} + \beta_{1r}x_1^{\theta_1} + \gamma_{1r}c_0\right)^2.\end{aligned}$$

Using $\log(x_1) \leq x_1$ *and rearranging yields*

$$\begin{aligned}\mathcal{L}\psi_1(x_1) \leq & \left[r_1 - l_1c_0 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1 + \gamma_1c_0\|^2\right] \\ &\times \log(x_1) + \varepsilon_{12}x_2 - k_1x_1^{\theta_1}\log(x_1) \\ &+\frac{1}{2}\|\alpha_1 + \gamma_1c_0\|^2 \\ &+\frac{1}{2}\Big[2 < (\alpha_1 + \gamma_1c_0), \beta_1 > \\ &+\|\beta_1\|^2x_1^{\theta_1}\Big]x_1^{\theta_1}(1 - \log(x_1)).\end{aligned} \tag{37}$$

Similarly, we have

$$\begin{aligned}\mathcal{L}\psi_2(x_2) \leq & \left(r_2 - l_2c_0 - \varepsilon_{21} - \frac{1}{2}\|\alpha_2 + \gamma_2c_0\|^2\right) \\ &\times \log(x_2) + \varepsilon_{21}x_1 - k_2x_2^{\theta_2}\log(x_2) \\ &+\frac{1}{2}\|\alpha_2 + \gamma_2c_0\|^2 \\ &+\frac{1}{2}\Big[2 < (\alpha_2 + \gamma_2c_0), \beta_2 > \\ &+\|\beta_2\|^2x_2^{\theta_2}\Big]x_2^{\theta_2}\left(1 - \log(x_2)\right),\end{aligned} \tag{38}$$

and

$$\begin{aligned}\mathcal{L}\psi_3(x_1, x_2) = & \ \varepsilon_{21}\left(r_1x_1 - l_1c_0x_1 - k_1x_1^{1+\theta_1}\right) \\ &+\varepsilon_{12}\left(r_2x_2 - l_2c_0x_2 - k_2x_2^{1+\theta_2}\right).\end{aligned} \tag{39}$$

From (37), (38) and (39), we have

$$\mathcal{L}\psi(x_1, x_2) \leq \phi_1(x_1) + \phi_2(x_2), \tag{40}$$

where

$$\begin{aligned}\phi_1(x_1) = & \left(r_1 - l_1c_0 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1 + \gamma_1c_0\|^2\right) \\ &\times \log(x_1) + \frac{1}{2}\Big[2 < (\alpha_1 + \gamma_1c_0), \beta_1 > \\ &+\|\beta_1\|^2x_1^{\theta_1}\Big]x_1^{\theta_1}(1 - \log(x_1)) \\ &-k_1x_1^{\theta_1}\log(x_1) + (\varepsilon_{21}r_1 + \varepsilon_{21} \\ &-\varepsilon_{21}l_2c_0)x_1 - k_1\varepsilon_{21}x_1^{1+\theta_1} \\ &+\frac{1}{2}\|\alpha_1 + \gamma_1c_0\|^2,\end{aligned}$$

and

$$\begin{aligned}\phi_2(x_2) = & \left(r_2 - l_2c_0 - \varepsilon_{21} - \frac{1}{2}\|\alpha_2 + \gamma_2c_0\|^2\right) \\ &\times \log(x_2) + \frac{1}{2}\Big[2 < (\alpha_2 + \gamma_2c_0), \beta_2 > \\ &+\|\beta_2\|^2x_2^{\theta_2}\Big]x_2^{\theta_2}(1 - \log(x_2)) \\ &-k_2x_2^{\theta_2}\log(x_2) + (\varepsilon_{12}r_2 + \varepsilon_{12} \\ &-\varepsilon_{12}l_1c_0)x_2 - k_2\varepsilon_{12}x_2^{1+\theta_2} \\ &+\frac{1}{2}\|\alpha_2 + \gamma_2c_0\|^2.\end{aligned}$$

Hence, if $x_1 \sim \infty$*, then*

$$\phi_1(x_1) \simeq \frac{1}{2}\|\beta_1\|^2x_1^{2\theta_1}\log(x_1) - k_1\varepsilon_{21}x_1^{1+\theta_1},$$

and if $x_1 \sim 0$*, then*

$$\phi_1(x_1) \simeq \left[r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1 + \gamma_1\|^2\right]\log(x_1),$$

and if $x_2 \sim \infty$*, then*

$$\phi_2(x_2) \simeq -\frac{1}{2}\|\beta_2\|^2x_2^{2\theta_2}\log(x_2) - k_2\varepsilon_{12}x_2^{1+\theta_2},$$

and if $x_2 \longrightarrow 0$*, then*

$$\phi_2(x_2) \simeq \left[r_2 - l_2 - \varepsilon_{21} - \frac{1}{2}\|\alpha_1 + \gamma_2\|^2\right]\log(x_2).$$

Since

$$\left(r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1 + \gamma_1\|^2\right) > 0,$$

and

$$\left(r_2 - l_2 - \varepsilon_{21} - \frac{1}{2}\|\alpha_1 + \gamma_2\|^2\right) > 0,$$

then

$$\lim_{x_1 \to 0} \phi_1(x_1) = \lim_{x_1 \to +\infty} \phi_1(x_1) = -\infty,$$

and

$$\lim_{x_2 \to 0} \phi_2(x_2) = \lim_{x_2 \to +\infty} \phi_2(x_2) = -\infty.$$

This gives together with (33) and (40) that for a sufficiently large μ*,*

$$\mathcal{L}\psi(x_1, x_2) \quad \leq \quad -1 \quad \textit{for each} \quad (x_1, x_2) \in D^c.$$

As a result, the stochastic system described by equation (2) possesses an invariant distribution, which is characterized by a density of zero in $\mathbb{R}^2_+$*.*

5 Computer simulations

To demonstrate our findings, we will utilize the widely-known Euler scheme (refer to, [27], for further details). We shall analyze the discretized system presented below

$$\begin{cases}
x_1(k+1) = x_1(k) + \Bigg[x_1(k)\bigg(r_1 - k_1 x_1^{\theta_1}(k) \\
\qquad -l_1 c_0(k)\bigg) + \varepsilon_{12}\bigg(x_2(k) \\
\qquad -x_1(k)\bigg)\Bigg]h + \sum_{i=1}^{n} \alpha_{1i} x_1(k)\sqrt{h}\eta_i \\
\qquad + \sum_{i=1}^{n} \beta_{1i} x_1^{1+\theta_1}(k)\sqrt{h}\eta_i \\
\qquad + \sum_{i=1}^{n} \gamma_{1i} x_1(k) c_0(k)\sqrt{h}\eta_i, \\
x_2(k+1) = x_2(k) + \Bigg[x_2(k)\bigg(r_2 - k_2 x_2^{\theta_2}(k) \\
\qquad -l_2 c_0(k)\bigg) + \varepsilon_{21}\bigg(x_1(k) \\
\qquad -x_2(k)\bigg)\Bigg]h + \sum_{i=1}^{n} \alpha_{2i} x_2(k)\sqrt{h}\eta_i \\
\qquad + \sum_{i=1}^{n} \alpha_{2i} x_2^{1+\theta_2}(k)\sqrt{h}\eta_i \\
\qquad + \sum_{i=1}^{n} \gamma_{2i} x_2(k) c_0(k)\sqrt{h}\eta_i,
\end{cases}$$

where η_i represents independent Gaussian random variables with a standard normal distribution, denoted as $\mathcal{N}(0, 1)$. In the following figures, we choose

$$c_0(t) = 0.1 + 0.05\sin(t).$$

5.1 Extinction

Example 5.1 *We choose* $x_1(0) = 0.7$, $r_1 = 0.06$, $l_1 = 1$, $k_1 = 0.7$, $\alpha_1 = 0.5$, $\beta_1 = 0.95$, $\theta_1 = 0.5$, $\varepsilon_{12} = 0.9$, $\gamma_1 = 0.05$, $x_2(0) = 0.3$, $r_2 = 0.05$, $l_2 = 1$, $k_2 = 0.8$, $\alpha_2 = 0.51$, $\beta_2 = 0.85$, $\theta_2 = 0.6$, $\varepsilon_{21} = 0.8$ *and* $\gamma_2 = 0.1$. *This gives*

$$M - \frac{1}{2}m^2 = -0.09125 < 0.$$

Therefore, the extinction condition stated in Theorem 3.2 is fulfilled, as confirmed by the computer simulations depicted in Figure 1.

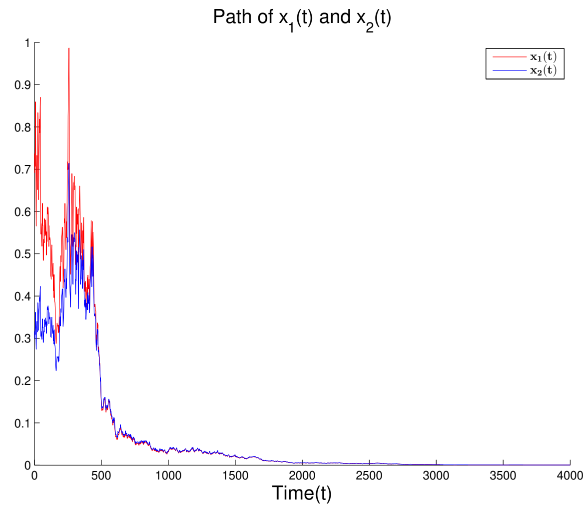

Fig. 1. *Trajectories of* x_1 *and* x_2 *for SDE (2).*

5.2 Persistence and Stationary distribution

Example 5.2 *Set* $x_1(0) = 0.7$, $r_1 = 0.7$, $l_1 = 0.2$, $k_1 = 0.4$, $\alpha_1 = 0.15$, $\gamma_1 = 0.1$, $\beta_1 = 0$, $\theta_1 = 1$, $\varepsilon_{12} = 0.35$, $x_2(0) = 0.8$, $r_2 = 0.7$, $l_2 = 0.1$, $k_2 = 0.3$, $\alpha_2 = 0.2$ $\gamma_2 = 0.1$, $\beta_2 = 0$, $\theta_2 = 1$ *and* $\varepsilon_{21} = 0.4$. *This gives*

$$\left(r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1 + \gamma_1\|^2\right) \approx 0.2969 > 0,$$

and

$$\left(r_2 - l_2 - \varepsilon_{21} - \frac{1}{2}\|\alpha_2 + \gamma_2\|^2\right) \approx 0.5167 > 0.$$

Hence, the persistence condition of Corollary 3.1 is verified. The computer simulations in Figure 2 *and Figure* 3 *support this outcome.*

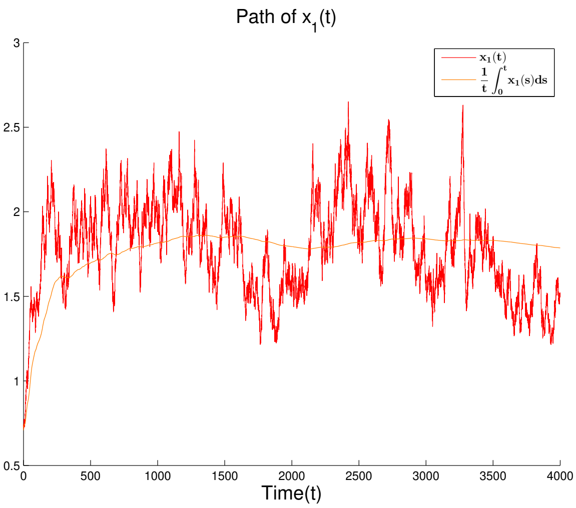

Fig. 2. *Trajectories of x_1 and x_2 for SDE (2).*

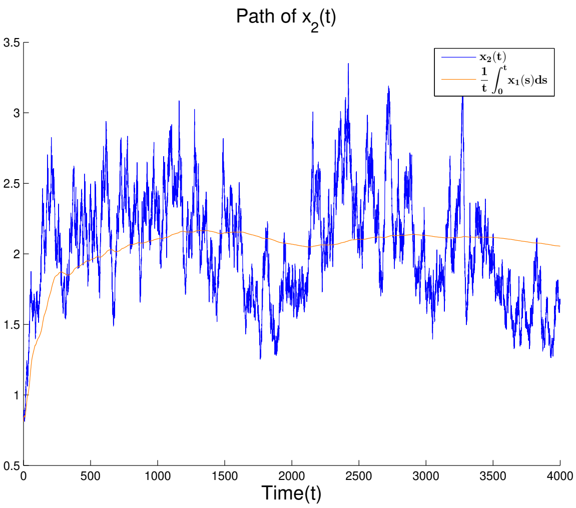

Fig. 3. *Trajectories of x_1 and x_2 for SDE (2).*

Example 5.3 *Set $x_1(0) = 0.7$, $r_1 = 0.4$, $l_1 = 0.1$, $k_1 = 0.4$, $\alpha_1 = 0.15$, $\gamma_1 = 0.05$, $\beta_1 = 0.15$, $\theta_1 = 0.85$, $\varepsilon_{12} = 0.1$, $x_2(0) = 0.8$, $r_2 = 0.5$, $l_2 = 0.15$, $k_2 = 0.3$, $\alpha_2 = 0.2$, $\gamma_2 = 0.1$, $\beta_2 = 0.3$, $\theta_2 = 0.95$ and $\varepsilon_{21} = 0.15$. This gives*

$$\left(r_1 - l_1 - \varepsilon_{12} - \frac{1}{2}\|\alpha_1 + \gamma_1\|^2\right) = 0.18 > 0,$$

and

$$\left(r_2 - l_2 - \varepsilon_{21} - \frac{1}{2}\|\alpha_2 + \gamma_2\|^2\right) = 0.155 > 0.$$

Therefore, the condition for a stationary distribution as stipulated in Theorem 4.2 is validated. This assertion is substantiated by the computational simulations depicted in Figure 4.

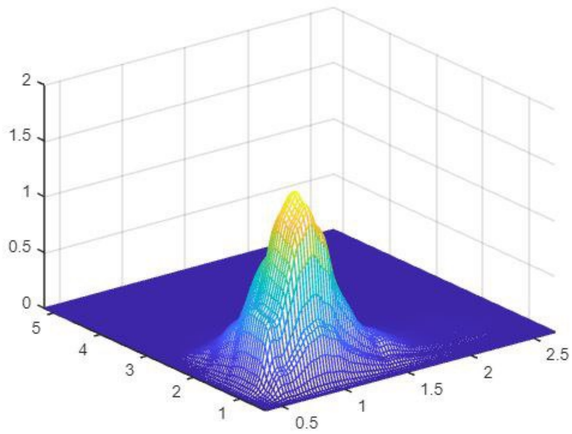

Fig. 4. *Kernel density functions of (x_1, x_2).*

6 Discussion

Investigating stochastic dynamics using the Gilpin-Ayala model, particularly within dispersed polluted environments, is essential in modern ecological research. This investigation delves into the intricate interplay between environmental processes and stochastic fluctuations in scenarios where pollutants are disseminated throughout ecosystems. The Gilpin-Ayala model is a fundamental tool for understanding population dynamics, offering insights into species interactions and coexistence dynamics. However, extending this model to encompass stochastic elements in polluted environments introduces a more realistic depiction of ecological systems, where inherent randomness and external perturbations play pivotal roles. One of the primary implications of incorporating stochasticity in the Gilpin-Ayala model within polluted environments is its capacity to capture the variability and uncertainty inherent in real-world ecological systems. As an exogenous factor, pollution introduces fluctuations in vital parameters such as growth rates, mortality rates, and species interactions. Consequently, the model's stochastic version enables exploring how these uncertainties influence species survival, coexistence, and potential extinction. By studying the stochastic dynamics of the Gilpin-Ayala model in dispersed polluted environments, researchers can attain more profound insights into the resilience of ecosystems against pollution-induced disturbances. Examining critical factors influencing extinction and persistence in such scenarios provides a comprehensive understanding of how ecosystems respond to environmental challenges. Moreover, this investigation advances theoretical ecology and applied environmental science. The development of mathematical frameworks to analyze stochastic ecological models underlines the interdisciplinary nature of this research, fostering collaboration between ecologists, mathematicians, and statisticians. These models can aid in devising strategies for mitigating the adverse effects of pollution on ecosystems, informing conservation efforts, and facilitating sustainable resource management. In conclusion, delving into the stochastic dynamics of the Gilpin-Ayala model in dispersed, polluted environments opens up new avenues for understanding the intricate dynamics of ecological systems in the face of uncertainty and external stressors. This endeavor enhances our theoretical understanding of ecology and offers practical implications for managing and conserving biodiversity in polluted environments.

7 Conclusion

In conclusion, investigating the stochastic dynamics of the Gilpin-Ayala model in dispersed polluted environments has provided valuable insights into the complex behavior of species populations and pollutant concentrations. The incorporation of stochastic elements has allowed for a more realistic representation of the inherent variability and unpredictability in these systems. The findings have underscored the importance of considering stochastic dynamics when studying and managing dispersed polluted environments, contributing to a more comprehensive understanding of their ecological dynamics and developing effective environmental conservation and pollution control strategies.

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Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

A. Nait Brahim, B. Harchaoui, S. Boutouil: were responsible for the conceptualization, validation, formal analysis, writing - original draft, methodology, writing - review & editing. **M. El Idrissi, S. Aznague:** have implemented the software, formal analysis, writing - original draft, writing - review & editing. **A. Settati, A. Lahrouz, M. El Jarroudi:** carried out the validation, investigation, conceptualization, writing - review & editing.

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