On the properties of Generalized Jacobsthal and Generalized Jacobsthal-Lucas sequences

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Abstract: In this paper, we study the Generalized Jacobsthal and Generalized Jacobsthal-Lucas sequences. We also present some properties on relationship between the Generalized Jacobsthal sequence and Generalized Jacobsthal-Lucas sequence by using Binet's formula for derivation. In addition, we showed colollaries and exmples of Jacobsthal and Jacobsthal-Lucas sequence.

Key-Words: Generalized Jacobsthal sequence, Generalized Jacobsthal-Lucas sequence , Binet's formula.

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1 Introduction

In the last years, many researchers have been interested in studying the sequence of numbers and recurrence relation, for example, Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas, etc. It can also be applied in applied mathematics, number theory, computers, etc. Also, many applications in diverse fields such as finance, architectural science, art, petals on flower, arrangement of seeds on flower etc., [1]. The Jacobsthal and Jacobsthal-Lucas sequences, [2], is defined by the recurrence relation, respectively,

$$J_n = J_{n-1}+2J_{n-2} \text{ and } j_n = j_{n-1}+2j_{n-2} \quad (1)$$

for $n \geq 2$ where $J_0 = 0,\ J_1 = 1$ and $j_0 = 2,\ j_1 = 1$. The Binet's formulas, respectively, for Jacobsthal and Jacobsthal-Lucas sequences is given by

$$J_n = \frac{r_2^n - r_1^n}{r_2 - r_1} \quad \text{and} \quad j_n = r_1^2 + r_2^2$$

where $r_1 = 2$ and $r_2 = -1$ are the root of the characteristic equation

$$x^2 - x - 2 = 0.$$

For equation (1), the first Jacobsthal and Jacobsthal–Lucas sequences are written as, respectively,

$$J_n = \{0, 1, 1, 3, 5, 11, 21, 43, 85, 171, ...\}$$

and

$$j_n = \{2, 1, 5, 7, 17, 31, 65, 127, 257, 511, ...\}.$$

In 2014, [3], studied the product and sum of the Jacobsthal and Jacobsthal–Lucas sequences by using the explicit Binet's formulas to show their properties. In 2008, [4], presented some properties of the Jacobsthal-Lucas E-matrix and R-matrix by the Matrix Method. In 2014, [5], derive various formula for sum of k-Jacobsthal numbers with indexes in an arithmetic sequence. In 2016, [6], presented some properties of k-Jacobsthal and k-Jacobsthal–Lucas sequences and some relationship of these sequence by using the Binet's formulas. Moreover, many researchers are also interested in some properties of Jacobsthal-like, (s,t)-Jacobthal, (s,t)-Jacobthal-Lucas etc. by using Binet's formula or Matrix Method for derivation see more details in [7], [8]. Based on extensive research on the properties of Jacobsthal and Jacobsthal-Lucas numbers, it was discovered that additional properties could be derived, and the results of the study could be extended in the form of generalized Jacobsthal and generalized Jacobsthal-Lucas sequences. The proof will use the Binet's formula.

Next, in sections 2, 3 and 4, we discuss some basic facts, definitions, proven theorems, corollaries, providing examples and conclusions.

2 Preliminaries

In section 2, we recall some definitions of the Generalized Jacobsthal sequence and Generalized Jacobsthal-Lucas sequences.

Definition 2.1, [10]. Let $J(k,n)$ be the Generalized Jacobsthal sequence defined recursively as follows :

$$J(k,n) = (k-1)\,J(k,n-1) + kJ(k,n-2),$$

for $n \geq 2$ with $J(k,0) = 0$ and $J(k,1) = 1$. That is, from relationship of the Generalized Jacobsthal sequence, we can write the first Generalized Jacobsthal sequence for case $0 \leq n \leq 7$ as follows :

$J(k,0) = 0$, $J(k,1) = 1$, $J(k,2) = k-1$,
$J(k,3) = k^2 - k + 1$, $J(k,4) = k^3 - k^2 + k - 1$,
$J(k,5) = k^4 - k^3 + k^2 - k + 1$,
$J(k,6) = k^5 - k^4 + k^3 - k^2 + k - 1$,
$J(k,7) = k^6 - k^5 + k^4 - k^3 + k^2 - k + 1$.

Next, Table 1 shows the first few terms of the sequences $J(k,n)$ for case $2 \leq k \leq 10$ and $2 \leq n \leq 8$,

Table 1: Generalized Jacobsthal sequence.

$J(k,n)$	2	3	4	5	6	7	8
2	1	3	5	11	21	43	85
3	2	7	20	61	182	547	1640
4	3	13	51	205	819	3277	13107
5	4	21	104	521	2604	13021	65104
6	5	31	185	1111	6665	39991	239945
7	6	43	300	2101	14706	102943	720600
8	7	57	455	3641	29127	233017	1864135
9	8	73	656	5905	53144	478297	4304672
10	9	91	909	9091	90909	909091	9090909

Definition 2.2, [10]. Let $j(k,n)$ be the Generalized Jacobsthal-Lucas sequence defined recursively as follows :

$$j(k,n) = (k-1)\, j(k,n-1) + kj(k,n-2),$$

for $n \geq 2$ with $j(k,0) = 2$ and $j(k,1) = 1$. That is, from relationship of the Generalized Jacobsthal-Lucas sequence, we can write the first Generalized Jacobsthal-Lucas sequence for case $0 \leq n \leq 7$ as follows :

$j(k,0) = 2$, $j(k,1) = 1$, $j(k,2) = 3k - 1$,
$j(k,3) = 3k^2 - 3k + 1$, $j(k,4) = 3k^3 - 3k^2 + 3k - 1$,
$j(k,5) = 3k^4 - 3k^3 + 3k^2 - 3k + 1$,
$j(k,6) = 3k^5 - 3k^4 + 3k^3 - 3k^2 + 3k - 1$,
$j(k,7) = 3k^6 - 3k^5 + 3k^4 - 3k^3 + 3k^2 - 3k + 1$.

Next, Table 2 shows the first few terms of the sequences $j(k,n)$ for case $2 \leq k \leq 10$ and $2 \leq n \leq 8$,

Table 2: Generalized Jacobsthal-Lucas sequence.

$j(k,n)$	2	3	4	5	6	7	8
2	5	7	17	31	65	127	257
3	8	19	62	181	548	1639	4922
4	11	37	155	613	2459	9829	39323
5	14	61	314	1561	7814	39061	195314
6	17	91	557	3331	19997	119971	719837
7	20	127	902	6301	44120	308827	2161802
8	23	169	1367	10921	87383	699049	5592407
9	26	217	1970	17713	159434	1434889	12914018
10	29	271	2729	27271	272729	2727271	27272719

Let r_1 and r_2 be the roots of the characteristic equation

$$r^2 - (k-1)r - k = 0.$$

Since $\triangle = (k+1)^2 > 0$ for $k \geq 2$, we have

$$r_1 = -1 \text{ and } r_2 = k.$$

They satisfy the following equations,

$$r_1 + r_2 = k - 1,\ r_2 - r_1 = k + 1 \text{ and } r_1 r_2 = -k.$$

For $n \geq 0$, $k \geq 2$. The Binet's formula, respectively, for Generalized Jacobsthal and Generalized Jacobsthal-Lucas sequence is given by

$$J(k,n) = \frac{r_2^n - r_1^n}{r_2 - r_1} \text{ and } j(k,n) = c_1 r_1^2 + c_2 r_2^2$$

where $c_1 = \dfrac{2k-1}{k+1}$ and $c_2 = \dfrac{3}{k+1}$. Particular cases of the previous definition are :

- If $k = 2$ and $J_0 = 0$, $J_1 = 1$, the Jacobsthal sequence is obtained.
- If $k = 2$ and $j_0 = 2$, $j_1 = 1$, the Jacobsthal-Lucas sequence is obtained.

Researchers have been talking about the properties of Generalized Jacobsthal and Generalized Jacobsthal-Lucas sequences. For example, in 2021, [9], introduced the Generalization of Jacobsthal and Jacobsthal-Lucas numbers and gave the properties of related matrix and the sum of terms of the sequences. In 2022, [10], studied the Generalization of Jacobsthal and Jacobsthal-Lucas numbers and give generating functions, Binet's formulas for these numbers. They show properties generalize the well-known results for classical Jacobstal numbers and Jacobsthal-Lucas numbers. Moreover, many authors have studied and presented some important relationships between Generalized Jacobsthal and Generalized Jacobsthal–Lucas sequences. For example, in 2005, [11], definined and proved theorems and corollaries of the incomplete generalized Jacobsthal and incomplete generalized Jacobsthal-Lucas number. In addition, in 2015, [12], presented new families and generating functions for generalized Jacobsthal and the Jacobsthal-Lucas sequences.

The purpose of this paper is to establish the relationship between generalized Jacobsthal and generalized Jacobsthal-Lucas sequences in terms of the product and the sum of both numbers, expressing them in real number form and the generalized Jacobsthal number format using the Binet's formula. From the results, it is found that the properties we have proven also hold true for Jacobsthal and Jacobsthal-Lucas sequences when considering the case where $k = 2$.

3 Main Theorem

In this section, we gave some theorem of the Generalized Jacobsthal and Generalized Jacobsthal-Lucas sequences. Moreover, we presented corollaries and examples as follows.

Theorem 3.1. Let k, n, s be integers. For $k \geq 2$ and $n \geq s \geq 1$. Then

$$J(k,2n)\,J(k,2s) - J^2(k,n+s) = -k^{2s} J^2(k,n-s). \tag{2}$$

Proof. By using Binet's formula, we have

$$\begin{aligned}
& J(k,2n)\,J(k,2s) - J^2(k,n+s) \\
&= \frac{\left(r_2^{2n} - r_1^{2n}\right)\left(r_2^{2s} - r_1^{2s}\right)}{(r_2 - r_1)^2} - \frac{\left(r_2^{n+s} - r_1^{n+s}\right)^2}{(r_2 - r_1)^2} \\
&= \frac{r_2^{2n+2s} - r_1^{2s} r_2^{2n} - r_1^{2n} r_2^{2s} + r_1^{2n+2s}}{(r_2 - r_1)^2} \\
&\quad - \frac{\left(r_2^{2n+2s} - 2(r_1 r_2)^{n+s} + r_1^{2n+2s}\right)}{(r_2 - r_1)^2} \\
&= \frac{2(r_1 r_2)^{n+s} - r_1^{2s} r_2^{2n} - r_1^{2n} r_2^{2s}}{(r_2 - r_1)^2} \\
&= \frac{(r_1 r_2)^{n+s}}{(r_2 - r_1)^2} \left(2 - \left(\frac{r_1}{r_2}\right)^{n-s} - \left(\frac{r_2}{r_1}\right)^{n-s}\right) \\
&= \frac{(r_1 r_2)^{n+s}}{-(r_1 r_2)^{n-s}(r_2 - r_1)^2} \\
&\quad \times \left(r_1^{2(n-s)} - 2(r_1 r_2)^{n-s} + r_2^{2(n-s)}\right) \\
&= -(r_1 r_2)^{2s} \left(\frac{r_2^{n-s} - r_1^{n-s}}{r_2 - r_1}\right)^2 \\
&= (-1)\,k^{2s} J^2(k,n-s) \\
&= -k^{2s} J^2(k,n-s).
\end{aligned}$$

Thus, the proof is complete.

From Theorem 3.1, we considered the equation (2) by choosing $n = 3$ and $s = 1$ as an example. It was found that on the left side of equation (2), it could be calculated as $-k^4 + 2k^3 - k^2$. And on the right, it could also be calculated as $-k^4 + 2k^3 - k^2$, where both sides were equal. Based on the calculated values from Table 1, when we considered the case where $k = 3$, we found that both sides resulted in the same value, which is -36.

Theorem 3.2. Let k, n, s be integers and $k \geq 2,\ n \geq s \geq 1$. Then, we have

$$j(k,2n)\,j(k,2s) - j^2(k,n+s) = (6k-3)k^{2s} J^2(k,n-s). \tag{3}$$

Proof. By using Binet's formula, we have

$$\begin{aligned}
& j(k,2n)\,j(k,2s) - j^2(k,n+s) \\
&= \left(c_1 r_1^{2n} + c_2 r_2^{2n}\right)\left(c_1 r_1^{2s} + c_2 r_2^{2s}\right) \\
&\quad - \left(c_1 r_1^{n+s} + c_2 r_2^{n+s}\right)^2 \\
&= c_1^2 r_1^{2n+2s} + c_1 c_2 r_1^{2n} r_2^{2s} + c_1 c_2 r_1^{2s} r_2^{2n} + c_2^2 r_2^{2n+2s} \\
&\quad - c_1^2 r_1^{2n+2s} - 2c_1 c_2 r_1^{n+s} r_2^{n+s} - c_2^2 r_2^{2n+2s} \\
&= c_1 c_2 r_1^{2n} r_2^{2s} + c_1 c_2 r_1^{2s} r_2^{2n} - 2c_1 c_2 r_1^{n+s} r_2^{n+s} \\
&= c_1 c_2 (r_1 r_2)^{n+s} \left(\left(\frac{r_1}{r_2}\right)^{n-s} - 2 + \left(\frac{r_2}{r_1}\right)^{n-s}\right) \\
&= \frac{(c_1 c_2)(r_1 r_2)^{n+s}}{(r_1 r_2)^{n-s}} \left(r_1^{2(n-s)} - 2(r_1 r_2)^{n-s} + r_2^{2(n-s)}\right) \\
&= (c_1 c_2)(r_1 r_2)^{2s} \left(r_2^{n-s} - r_1^{n-s}\right)^2 \\
&= (c_1 c_2)(r_1 r_2)^{2s} (r_2 - r_1)^2 \left(\frac{r_2^{n-s} - r_1^{n-s}}{r_2 - r_1}\right)^2 \\
&= \left(\frac{2k-1}{1+k}\right)\left(\frac{3}{1+k}\right)(-k)^{2s}(k+1)^2 J^2(k,n-s) \\
&= (6k-3)k^{2s} J^2(k,n-s).
\end{aligned}$$

Thus, the proof is complete.

From Theorem 3.2, we considered the equation (3) by choosing $n = 3$ and $s = 1$ as an example. It was found that on the left side of equation (3), it could be calculated as $6k^5 - 15k^4 + 12k^3 - 3k$. And on the right, it could also be calculated as $6k^5 - 15k^4 + 12k^3 - 3k$, where both sides were equal. Based on the calculated values from Table 1 and Table 2, when we considered the case where $k = 3$, we found that both sides resulted in the same value, which is 540.

Next, from the above theorem, we obtained the well-known identities for Generalized Jacobsthal and Generalized Jacobsthal-Lucas sequence. For $k = 2$, i.e. $J(2,n) = J_n$ and $j(2,n) = j_n$, Theorem 3.1 and Theorem 3.2 reduces to new identities in the following corollary.

Corollary 3.3. Let n, s be integer. For $n \geq s$, we have

$$J_{2n} J_{2s} - J_{n+s}^2 = -2^{2s} J_{n-s}^2 \tag{4}$$

Moreover,

$$j_{2n} j_{2s} - j_{n+s}^2 = 9\,(2)^{2s}\, j_{n-s}^2. \tag{5}$$

From Corollary 3.3, we considered the Table 1 and Table 2, by choosing $n = 3$ and $s = 2$ as an example. It was found that on the left side of equation (4), it could be calculated as -16. On the right side, it could be calculated as -16. It was found that both were equal. Similarly, for $n = 3$ and $s = 2$ it was found that on the left and right side of equation (5), it could be calculated as 144. It was found that both were equal.

Theorem 3.4. Let k, n be integers and $k \geq 2,\ n \geq 1$. Then, we have

$$kJ^2(k,n-1) + (k-1)J(k,n-1)J(k,n) - J^2(k,n) = (-1)^n k^{n-1}. \tag{6}$$

Proof. Since,

$$J(k,n) = (k-1)J(k,n-1) + kJ(k,n-2),$$

we have

$$J^2(k,n) = (k-1)J(k,n-1)J(k,n) + kJ(k,n-2)J(k,n)$$

i.e.,

$$(k-1)J(k,n-1)J(k,n) - J^2(k,n) = -kJ(k,n-2)J(k,n).$$

By using Binet's formula, we have

$$\begin{aligned}
&kJ^2(k,n-1) + (k-1)J(k,n-1)J(k,n) - J^2(k,n)\\
&= kJ^2(k,n-1) - kJ(k,n-2)J(k,n)\\
&= k\left[\frac{\left(r_2^{n-1} - r_1^{n-1}\right)^2}{(r_2-r_1)^2} - \frac{\left(r_2^{n-2} - r_1^{n-2}\right)\left(r_2^n - r_1^n\right)}{(r_2-r_1)^2}\right]\\
&= \frac{k}{(r_2-r_1)^2}\left(r_2^{2(n-1)} - 2r_1^{n-1}r_2^{n-1} + r_1^{2(n-1)}\right)\\
&- \frac{k}{(r_2-r_1)^2}\left(r_2^{2(n-1)} - r_1^n r_2^{n-2} - r_1^{n-2}r_2^n + r_1^{2(n-1)}\right)\\
&= \frac{k}{(r_2-r_1)^2}\left(-2(r_1r_2)^{n-1} + r_1^n r_2^{n-2} + r_1^{n-2}r_2^n\right)\\
&= \frac{k(r_1r_2)^{n-1}}{(r_2-r_1)^2}\left(-2 + \frac{r_1}{r_2} + \frac{r_2}{r_1}\right)\\
&= \frac{k(-k)^{n-1}}{(r_2-r_1)^2}\left(\frac{-2r_1r_2 + r_1^2 + r_2^2}{r_1r_2}\right)\\
&= \frac{k(-k)^{n-1}}{(r_2-r_1)^2}\left(\frac{(r_2-r_1)^2}{-k}\right)\\
&= -(-k)^{n-1}\\
&= (-1)^n k^{n-1}.
\end{aligned}$$

Thus, the proof is complete.

From Theorem 3.4, we considered the equation (6) by choosing $n = 3$ as an example. It was found that on the left side of equation (6), it could be calculated as $-k^2$. And on the right, it could also be calculated as $-k^2$, where both sides were equal. Based on the calculated values from Table 1, when we considered the case where $k = 3$, we found that both sides resulted in the same value, which is -9.

Theorem 3.5. Let k, n be integers and $k \geq 2, n \geq 1$. Then, we have

$$kj^2(k,n-1) + (k-1)j(k,n-1)j(k,n) - j^2(k,n) = (-k)^{n-1}(6k-3). \tag{7}$$

Proof. Since,

$$j(k,n) = (k-1)j(k,n-1) + kj(k,n-2),$$

we have

$$j^2(k,n) = (k-1)j(k,n-1)j(k,n) + kj(k,n-2)j(k,n)$$

i.e.,

$$(k-1)j(k,n-1)j(k,n) - j^2(k,n) = -kj(k,n-2)j(k,n).$$

By using Binet's formula, we have

$$\begin{aligned}
&j^2(k,n-1) + (k-1)j(k,n-1)J(k,n)\\
&-j^2(k,n)\\
&= j^2(k,n-1) - kj(k,n-2)j(k,n)\\
&= k\left(c_1r_1^{n-1} + c_2r_2^{n-1}\right)^2\\
&\quad -k\left(c_1r_1^{n-2} + c_2r_2^{n-2}\right)\left(c_1r_1^n + c_2r_2^n\right)\\
&= k\left(c_1^2r_1^{2(n-1)} + 2c_1c_2(r_1r_2)^{n-1} + c_2^2r_2^{2(n-1)}\right)\\
&\quad -k\left(c_1^2r_1^{2(n-1)} + c_1c_2r_1^{n-2}r_2^n\right)\\
&\quad -k\left(c_1c_2r_1^nr_2^{n-2} + c_2^2r_2^{2(n-1)}\right)\\
&= k\left(2c_1c_2(r_1r_2)^{n-1} - c_1c_2r_1^{n-2}r_2^n - c_1c_2r_1^nr_2^{n-2}\right)\\
&= k(c_1c_2)(r_1r_2)^{n-1}\left[2 - \left(\frac{r_2}{r_1}\right) - \left(\frac{r_1}{r_2}\right)\right]\\
&= k(c_1c_2)(r_1r_2)^{n-1}\left[\frac{2r_1r_2 - r_2^2 - r_1^2}{r_1r_2}\right]\\
&= k(c_1c_2)(r_1r_2)^{n-2}\left(2r_1r_2 - r_2^2 - r_1^2\right)\\
&= -k(c_1c_2)(r_1r_2)^{n-2}(r_2-r_1)^2
\end{aligned}$$

$$= -k\left(c_1c_2\right)(-k)^{n-2}(k+1)^2$$
$$= (-k)^{n-1}\left(\frac{2k-1}{k+1}\cdot\frac{3}{k+1}\right)(k+1)^2$$
$$= (-k)^{n-1}(6k-3).$$

Thus, the proof is complete.

From Theorem 3.5, we considered the equation (7) by choosing $n=3$ as an example. It was found that on the left side of equation (7), it could be calculated as $6k^3-3k^2$. And on the right, it could also be calculated as $6k^3-3k^2$, where both sides were equal. Based on the calculated values from Table 2, when we considered the case where $k=3$, we found that both sides resulted in the same value, which is 135.

Next, from the above theorem, we obtained the well-known identities for Generalized Jacobsthal and Generalized Jacobsthal-Lucas sequences. For $k=2$, i.e. $J(2,n)=J_n$ and $j(2,n)=j_n$, Theorem 3.4 and Theorem 3.5 reduce to new identities in the following corollary.

Corollary 3.6. For n be integer, we have

$$2J_{n-1}^2+J_{n-1}J_n-J_n^2=(-1)^n\,2^{n-1} \tag{8}$$

Moreover,

$$2j_{n-1}^2+j_{n-1}j_n-j_n^2=9\,(-2)^{n-1}. \tag{9}$$

From Corollary 3.6, we considered the Table 1 and Table 2, by choosing $n=3$ as an example. It was found that on the left side of equation (8), it could be calculated as -4. On the right side, it could be calculated as -4. It was found that both were equal. Similarly, for $n=3$, it was found that on the left and right side of equation (9), it could be calculated as 36. It was found that both were equal.

Theorem 3.7. Let k,n,s be integers and $k\geq 2, n\geq s\geq 1$. Then, we have

$$J(k,n)\,j(k,s+1)-j(k,n+1)\,J(k,s) = (-k)^s\,J(k,n-s). \tag{10}$$

Proof. By using Binet's formula, we have

$$J(k,n)\,j(k,s+1)-j(k,n+1)\,J(k,s)$$
$$= \left(\frac{r_2^n-r_1^n}{r_2-r_1}\right)\left(c_1r_1^{s+1}+c_2r_2^{s+1}\right) - \left(c_1r_1^{n+1}+c_2r_2^{n+1}\right)\left(\frac{r_2^s-r_1^s}{r_2-r_1}\right)$$
$$= \left(\frac{c_1r_1^{s+1}r_2^n+c_2r_2^{n+s+1}-c_1r_1^{n+s+1}-c_2r_1^nr_2^{s+1}}{r_2-r_1}\right) - \left(\frac{c_1r_1^{n+1}r_2^s-c_1r_1^{n+s+1}+c_2r_2^{n+s+1}-c_2r_1^sr_2^{n+1}}{r_2-r_1}\right)$$
$$= \frac{\left(r_1^sr_2^n\right)\left(c_1r_1+c_2r_2\right)}{r_2-r_1} - \frac{\left(r_1^nr_2^s\right)\left(c_1r_1+c_2r_2\right)}{r_2-r_1}$$
$$= \left(\frac{c_1r_1+c_2r_2}{r_2-r_1}\right)\left(r_1^sr_2^n-r_1^nr_2^s\right)$$
$$= \left(c_1r_1+c_2r_2\right)\left(r_1r_2\right)^s\left(\frac{r_2^{n-s}-r_1^{n-s}}{r_2-r_1}\right)$$
$$= j(k,1)\,(-k)^s\,J(k,n-s)$$
$$= (-k)^s\,J(k,n-s).$$

Thus, the proof is complete.

From Theorem 3.7, we considered the equation (10) by choosing $n=3$ and $s=2$ as an example. It was found that on the left side of equation (10), it could be calculated as $-k^2$. And on the right, it could also be calculated as $-k^2$, where both sides were equal. Based on the calculated values from Table 1 and Table 2, when we considered the case where $k=3$, we found that both sides resulted in the same value, which is -9.

Theorem 3.8. Let k,n,s be integers and $k\geq 2, n\geq s\geq 1$. Then, we have

$$j(k,n)\,J(k,s+1)-J(k,n+1)\,j(k,s) = (-1)^{s+1}(2k-3)\,k^sJ(k,n-s). \tag{11}$$

Proof. By using Binet's formula, we have

$$j(k,n)\,J(k,s+1)-J(k,n+1)\,j(k,s)$$
$$= \left(c_1r_1^n+c_2r_2^n\right)\left(\frac{r_2^{s+1}-r_1^{s+1}}{r_2-r_1}\right) - \left(\frac{r_2^{n+1}-r_1^{n+1}}{r_2-r_1}\right)\left(c_1r_1^s+c_2r_2^s\right)$$
$$= \left(\frac{c_1r_1^nr_2^{s+1}+c_2r_2^{n+s+1}-c_1r_1^{n+s+1}-c_2r_1^{s+1}r_2^n}{r_2-r_1}\right) - \left(\frac{c_1r_1^sr_2^{n+1}-c_1r_1^{n+s+1}+c_2r_2^{n+s+1}-c_2r_1^{n+1}r_2^s}{r_2-r_1}\right)$$
$$= \frac{c_1r_2\left(r_1^nr_2^s\right)-c_2r_1\left(r_1^sr_2^n\right)}{r_2-r_1} - \frac{c_1r_2\left(r_1^sr_2^n\right)-c_2r_1\left(r_1^nr_2^s\right)}{r_2-r_1}$$
$$= \frac{1}{r_2-r_1}\left(c_1r_2+c_2r_1\right)\left(r_1^nr_2^s-r_1^sr_2^n\right)$$

$$= \frac{1}{r_2 - r_1}\left(c_1 r_2 + c_2 r_1\right)\left(r_1 r_2\right)^s \left(\frac{r_1^n}{r_1^s} - \frac{r_2^n}{r_2^s}\right)$$
$$= -\frac{1}{r_2 - r_1}\left(c_1 r_2 + c_2 r_1\right)\left(r_1 r_2\right)^s \left(r_2^{n-s} - r_1^{n-s}\right)$$
$$= -\left(c_1 r_2 + c_2 r_1\right)(-k)^s J\left(k, n-s\right)$$
$$= (-1)^{s+1}\left(2k-3\right) k^s J\left(k, n-s\right).$$

Thus, the proof is complete.

From Theorem 3.8, we considered the equation (11) by choosing $n = 3$ and $s = 2$ as an example. It was found that on the left side of equation (11), it could be calculated as $-2k^3 + 3k^2$. And on the right, it could also be calculated as $-2k^3 + 3k^2$, where both sides were equal. Based on the calculated values from Table 1 and Table 2, when we considered the case where $k = 3$, we found that both sides resulted in the same value, which is -27.

Corollary 3.9. Let $n \geq s$ be integers. Then, we have

$$J_n j_{s+1} - j_{n+1} J_s = (-2)^s J_{n-s} \tag{12}$$

Moreover,

$$j_n J_{s+1} - J_{n+1} j_s = (-1)^{s+1} (2)^s J_{n-s}. \tag{13}$$

From Corollary 3.9, we considered the Table 1 and Table 2, by choosing $n = 3$ and $s = 2$ as an example. It was found that on the left side of equation (12), it could be calculated as 4 . On the right side, it could be calculated as 4. It was found that they both were equal. Similarly, by choosing $n = 3$ and $s = 2$ as an example. It was found that on the left side of equation (13), it could be calculated as -4. On the right side, it could be calculated as -4. It was found that both were equal.

4 Conclusion

We have studied the definitions of the Generalized Jacobsthal and Generalized Jacobsthal-Lucas sequences. From the study, we have discovered some important properties of those relationships. In the proof, we will use the Binet form. Moreover, for the properties that we have discovered, if we consider the case where $k = 2$, we will obtain the properties that cover the relationships of the Jacobsthal and Jacobsthal-Lucas sequences. Furthermore, we have also presented corollaries and demonstrated some examples to support the properties that we have studied. In the future, we are interested in studying the relationships of the Generalized Jacobsthal and Generalized Jacobsthal-Lucas sequences in the form of matrices or defining new relationships of these numbers. In addition, we aim to find some specific properties of these numbers as well.

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