# Conformal Self Mappings of the Fundamental Domains of Analytic Functions and Computer Experimentation

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*Abstract:* - Conformal self mappings of a given domain of the complex plane can be obtained by using the Riemann Mapping Theorem in the following way. Two different conformal mappings $\varphi$ and $\psi$ of that domain onto one of the standard domains: the unit disc, the complex plane or the Riemann sphere are taken and then $\psi^{-1} \circ \varphi$ is what we are looking for. Yet, this is just a theoretical construction, since the Riemann Mapping Theorem does not offer any concrete expression of those functions. The Möbius transformations are concrete, but they can be used only for particular circular domains. We are proving in this paper that conformal self mappings of any fundamental domain of an arbitrary analytic function can be obtained via Möbius transformations as long as we allow that domain to have slits. Moreover, those mappings enjoy group properties. This is a totally new topic. Although fundamental domains of some elementary functions are well known, the existence of such domains for arbitrary analytic functions has been proved only in our previous publications mentioned in the References section. No other publication exists on this topic and the reference list is complete. We deal here with conformal self mappings of fundamental domains in its whole generality and present sustaining illustrations. Those related to the case of Dirichlet functions represent a real achievement. Computer experimentation with these mappings are made for the most familiar analytic functions.

*Key-Words:* - Conformal mappings, Fundamental domains, Möbius transformations, Euler Gamma function, Riemann Zeta function, Dirichlet functions, Computer experimentation

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Dedicated to the memory of Professor Gabriela Kohr

## 1 Introduction

Let $f(z)$ be a holomorphic function in $\overline{\mathbb{C}}$ with the exception of isolated singular points, which can be poles or essential singular points. It is known, [1], that $\overline{\mathbb{C}} = \cup_{k=1}^{n\leq\infty}\overline{\Omega}_k$, where $\Omega_k$ are open connected sets, $\Omega_k \cap \Omega_j = \varnothing$, when $j \neq k$ and every $\Omega_k$ is conformal (hence bijectively) mapped by $f$ onto $\overline{\mathbb{C}}\setminus L_k$, where $L_k$ is a *slit,* or a *cut, i.e., a Jordan arc* or a Jordan infinite curve. We will treat slits mostly as point sets. If $E \subset \overline{\mathbb{C}}$ is any point set we will denote by $f(E)$ the *image* of $E$ by $f$, i.e.

$$f(E) = \{f(z)|z \in E\}$$

and by $f^{-1}(E)$ the *pre-image* of $E$ by $f$ i.e.,

$$f^{-1}(E) = \{z|f(z) \in E\}.$$

This convention cannot produce any confusion. The uniqueness theorem of analytic functions guarantees that for a non constant analytic function $f(z)$ the pre-image by $f(z)$ of a point $w$ is a discrete set of points $\{z_n\}$ such that $f(z_n) = w$.

A slit exhibits two distinct *edges*, [2], and a point of the slit can be n on one edge or on the other. The inverse function $f^{-1}_{|\Omega_k}$, which exists for every $k$, in view of bijectiveness of $f$ in $\Omega_k$, fails to have a continuous extension to $L_k$, since for sequences of points tending to the same point on $L_k$ from the sides of different edges the function has different limits. However, the function $f_{|\Omega_k}$ can be extended by continuity to the boundary $\partial\Omega_k$ of $\Omega_k$ and it maps $\partial\Omega_k$ onto $L_k$. This fact is granted by the Riemann-Caratheodory Theorem of boundary correspondence in the conformal mapping [3] and [4]. The same theorem allows the extension of $f^{-1}_{|\Omega_k}$ not to $L_k$, but to the two edges of $L_k$ which then are mapped one to one by the extended function onto $\partial\Omega_k$.

Ahlfors, [2], called the domains $\Omega_k$ fundamental regions of $f$. When $f$ is a rational function of degree $m$, then $n = m$, [5].

Fundamental domains are known also for analytic functions having non isolated singular points, as for example the modular function or the infinite Blaschke products. For the modular function every point of the real axis is a singular point (obviously, non isolated), while for the infinite Blaschke products such points are the cluster points of the poles.

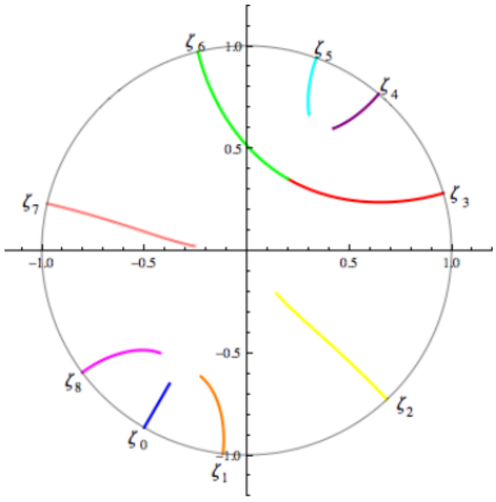
Examples of fundamental domains for different classes of analytic functions were given in [6], [7]. We have n there that they are not uniquely determined, yet some points of their boundaries must be the same for any partition of the complex plane into fundamental domains and these are the *branch points* of the function $f$. The branch points of $f$ are the zeros of derivative of $f$, the multiple poles and the essential singular points of this function. Since in any neighborhood of such a point the function fails to be injective, the branch points cannot belong to any fundamental domain and therefore they should be located on the boundaries of these domains. This simple remark, as well as the simultaneous continuation technique, [5], [6], [7], [8], [9], allowed us to implement procedures of finding fundamental domains for different classes of analytic functions.

It is known that for a Blaschke product $w = B(z)$ of degree $n$ the equation $B(z) = 1$ has exactly $n$ distinct solutions $\zeta_k$ all located on the unit circle and for every $k$ the image by $B(z)$ of the arc of the unit circle between $\zeta_k$ and $\zeta_{k+1}$ is the whole unit circle. The equation $B'(z) = 0$ has no solution on the unit circle, [5], [7], [8]. Then the pre-image by $B(z)$ of the point 1 is the set of points $\{\zeta_k\}$ and when $w$ moves from 1 towards 0 on the real axis, $n$ points $z_k$ will move each one from the corresponding $\zeta_k$ inside the unit disc describing some Jordan arcs. These arcs can only meet each other at the branch points of $B(z)$. When two adjacent arcs starting from consecutive points $\zeta_k$ and $\zeta_{k+1}$ on the unit circle meet each other at $a_k$, they bound together with the arc of the unit circle between $\zeta_k$ and $\zeta_{k+1}$ a domain which is conformal mapped by $B(z)$ onto the unit disc with a slit alongside the real axis from $B(a_k)$ to 1. The symmetric of this domain with respect to the unit circle is conformal mapped by $B(z)$ onto the exterior of the unit circle with a slit alongside the real axis from 1 to $1/\overline{B(a_k)}$ and therefore the union $\Omega_k$ of the two domains and of the common part of the boundary is conformal mapped by $B(z)$ onto the whole complex plane with a slit alongside the real axis from $B(a_k)$ to $1/\overline{B(a_k)}$. The domain $\Omega_k$ is a fundamental domain of $B(z)$ and we obtain the remaining $n - 1$ fundamental domains of $B(z)$ in a similar way.

Although we use the words *move* and *describe* this is a static situation, where arcs are mapped bijectively onto some intervals of the real axis.

Let us notice that the topological facts listed above have as corollary a surprising algebraic result, namely, that the image by $B(z)$ of the roots of $B'(z)$ are all real.

Fig. 1, Fig. 2, Fig. 3 and Fig. 4 illustrate this affirmation for $B(z)$ having the triple zeros $a$, $-a$ with $|a| < 1$ and 0. Different stages of the construction of the fundamental domains are exhibited in these figures.



$(a)$

Figure 1: Building fundamental domains of a Blaschke product of degree 9

This result has been generalized in [7] to infinite Blaschke products and we have found that in every neighborhood of a cluster point of poles (which is necessarily on the unit circle) of an infinite Blaschke product there are infinitely many fundamental domains of that function. This is a completion of the Big Picard Theorem in the sense that such a point is not an isolated singular point (being limit of poles) and instead of infinitely many points having the same image it states that infinitely many domains have the same image.

We notice also that every neighborhood of an isolated essential singular point of any analytic function $f$ intersects infinitely many fundamental domains. Indeed, let $a$ be such a point and let $V$ be a neighborhood of $a$. If $w$ is a non omitted value of $f$, then there are infinitely many points $z_k \in V$ such that $f(z_k) = w$. Each one of the points $z_k$ is either an interior point of a fundamental domain $\Omega_k$ or a finite number $k_m$ of fundamental domains meet in $z_k$. As the set $(k_m)$ is infinite, infinitely many fundamental domains $f$ intersect $V$.

The exponential function $f(z) = e^z$ has as fundamental domains horizontal strips of width $2\pi$, as for example

$$\Omega_k = \{z|2k\pi < \Im z < 2(k+1)\pi\}$$

where $k \in \mathbb{Z}$.

Each one of these domains is conformal mapped by $f(z)$ onto the complex plane with a slit alongside the positive real half axis.

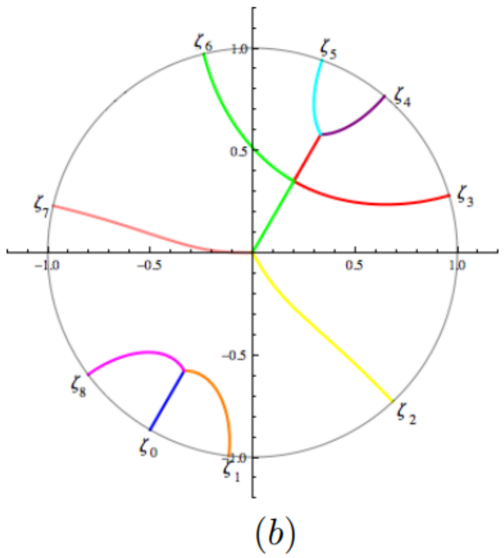



Figure 2: Building fundamental domains of a Blaschke product of degree 9

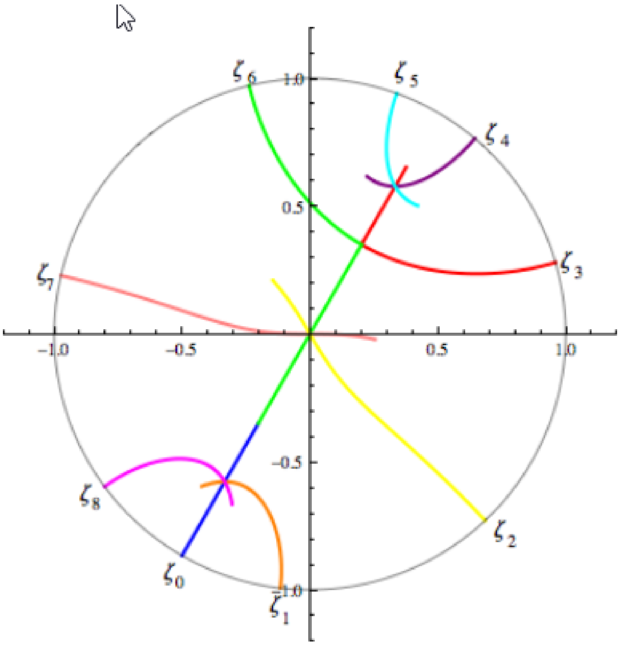


Figure 3: Building fundamental domains of a Blaschke product of degree 9

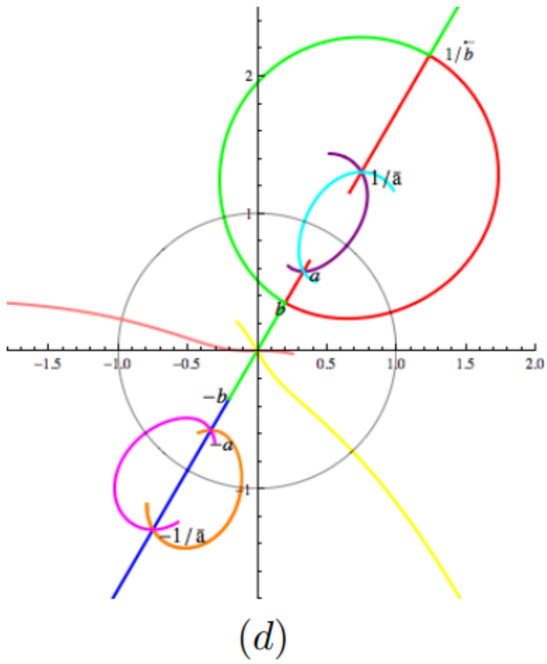



Figure 4: Fundamental domains of a Blaschke product of degree 9

Let us notice that $f(z) = e^z$ maps one to one the interval

$$\{z = iy | 0 \leq y < 2\pi\} \subset \Omega_0$$

onto the unit circle. Moreover, since

$$e^{x_0 + iy_0} = e^{x_0}(\cos y + i \sin y)$$

is the equation of a circle centered at the origin and of radius $e^{x_0}$, when $0 \leq y < 2\pi$, $f(z)$ is a conformal mapping of the half of $\Omega_0$ on the left side of the imaginary axis (corresponding to $x_0 < 0$) onto the unit disc (since $e^{x_0} < 1$) with a slit alongside the real axis from 0 to 1. It is also a conformal mapping of the half of $\Omega_0$ at the right side of the imaginary axis onto the exterior of the unit disc (since $e^{x_0} > 1$) with a slit alongside the real axis from 1 to $\infty$. This remark gives a good idea of the geometry of the conformal mapping of $\Omega_0$ by $f(z) = e^z$. We have a similar situation for every $\Omega_k$. Fundamental domains of the exponential function are illustrated in Fig. 5.

The fundamental domains of $f(z) = \cos z$ are bounded by vertical lines $\Re z = k\pi$ and $\Re z = (k + 1)\pi$, $k \in \mathbb{Z}$. This function realizes a conformal mapping of each one of these strips onto the complex plane with a slit alongside the real axis complemen-

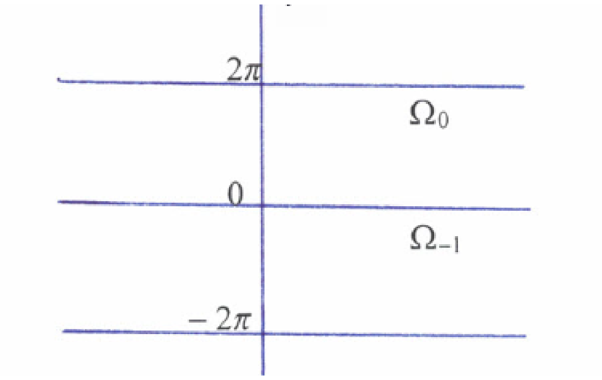



Figure 5: Fundamental domains of $e^z$

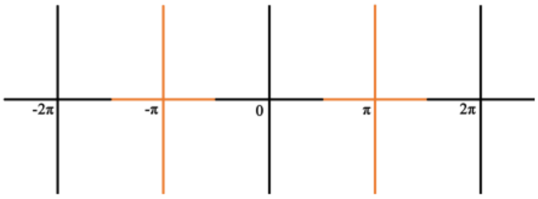



Figure 6: Fundamental domains of $\cos z$

tary to the interval $(-1, 1)$. Indeed, we have

$$\begin{aligned}\cos z &= \frac{1}{2}(e^{iz} + e^{-iz}) = \frac{1}{2}(e^{ix-y} + e^{-ix+y}) \\ &= \frac{e^{-y}}{2}(\cos x + i \sin x) + \frac{e^{y}}{2}(\cos x - i \sin x) \\ &= \cos x \left[\frac{1}{2}(e^{y} + e^{-y})\right] - i \sin x \left[\frac{1}{2}(e^{y} - e^{-y})\right] \\ &= \cos x \cosh y - i \sin x \sinh y.\end{aligned}$$

Thus,

$$\cos(k\pi + iy) = (-1)^k \cosh y,$$

which shows that every vertical line $z = k\pi + iy$, $k \in \mathbb{Z}$ is mapped two to one by $f(z) = \cos z$ onto the interval of the real axis from $-\infty$ to $-1$ when $k$ is odd and from $1$ to $+\infty$ when $k$ is even. Therefore every vertical strip bounded by two consecutive such lines is a fundamental domain for $f(z) = \cos z$, which is mapped by $f(z)$ onto the complex plane with a slit alongside the real axis complementary to the interval $(-1, 1)$. Fundamental domains of the cosine function are given in Fig. 6. All the other trigonometric functions have similar fundamental domains.

The Euler Gamma function is an extension to the whole complex plane of the arithmetic function $\Gamma(n) = (n-1)!$. For $\Re z > 1$ the extension is given by

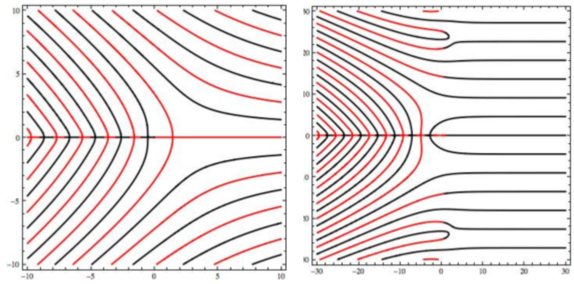


Figure 7: Fundamental domains of the Euler Gamma and the Riemann Zeta function

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt \tag{1}$$

Integrating by parts, we find that $\Gamma(z)$ satisfies the functional equation $z\Gamma(z) = \Gamma(z+1)$ which allows its extension to the half-plane $\Re z \le 1$. The pre-image by $\Gamma(z)$ of the real axis displays fundamental domains of this function as shown in Fig. 7. The red curves are the pre-image of the positive half axis and the black ones that of the negative half axis. The domains bounded by consecutive curves of the same color are fundamental domains of $\Gamma(z)$. They are conformal mapped by the function onto the complex plane with a slit alongside some intervals of the real axis. On the right side of Fig. 7, the preimage of the real axis by the Riemann Zeta function is illustrated. It is similar to that of the Gamma function, yet not all the domains bounded by its components are fundamental domains.

The modular function $\lambda(\tau)$ has been built, [2], starting with a domain $\Omega_1$ bounded by the half-lines $\Re\tau = \pm 1$, $\Im\tau \ge 0$ and the half-circles $|\tau \pm 1/2| = 1/2$, $\Im\tau \ge 0$. The Riemann Mapping Theorem states that there is a unique conformal mapping $\lambda(\tau)$ of the domain $\Omega_1$ onto the complex plane with a slit alongside the real axis from $-\infty$ to $-1$ and from $1$ to $+\infty$ such that $\tau = 0, 1, \infty$ corresponds to $\lambda = 1, \infty, 0$, [2]. Using the Schwartz Symmetry Principle this mapping can be extended analytically to the whole upper half plane. The domains obtained by iterated symmetries with respect to the half-lines and half-circles are all fundamental domains of $\lambda(\tau)$, which are illustrated in Fig. 8. In every neighborhood of a point of the real axis there are infinitely many such domains and therefore the real axis is a singular line of $\lambda(\tau)$.

The fundamental domains mentioned above are obvious, yet for most of the classes of analytic functions they have to be found. On the other hand, there is no hope to compute the values of $\lambda(\tau)$ since the Riemann Mapping Theorem states only the fact that

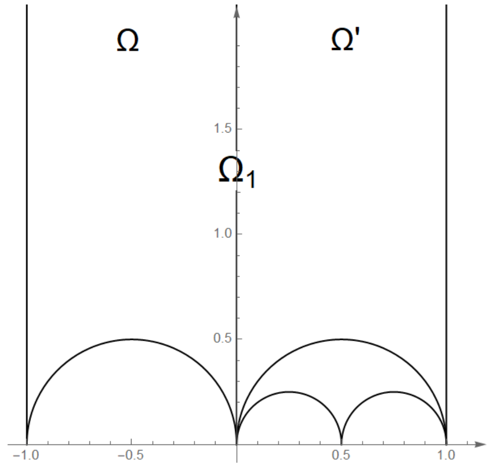



Figure 8: The fundamental domains of the modular function

such a function exists and is unique. However, it has this special feature that the conformal mappings of the fundamental domains one of each other are known, namely those generated by $\tau \to \tau+1$ and $\tau \to -1/\tau$. We will show later that for any analytic function such mappings can be found as long as we allow fundamental domains to have slits.

The Weierstrass $\wp$ function has been defined, [2], by the formula

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left[ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right] \tag{2}$$

where $\omega = m\omega_1 + n\omega_2$, $(m,n) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}$ with $\omega_1$ and $\omega_2$ arbitrary complex numbers having non real ratio $\omega_1/\omega_2$. It is a doubly periodic function with the periods $\omega_1$ and $\omega_2$. The parallelogram determined by $\omega_1$ and $\omega_2$ is divided by the diagonal from $\omega_1$ to $\omega_2$ into two triangles which are fundamental domains for $\wp$, [8]. The conformal mapping of these triangles made by $\wp$ onto the complex plane with three slits is illustrated in Fig. 9.

## 2 Fundamental Domains of Dirichlet Functions

The conformal mappings by *Dirichlet functions* have been studied in [10], [11], [12], [13] and the way the fundamental domains of these functions can be revealed has been explained. However, for a better understanding of this topic we will repeat some of the findings in those papers.

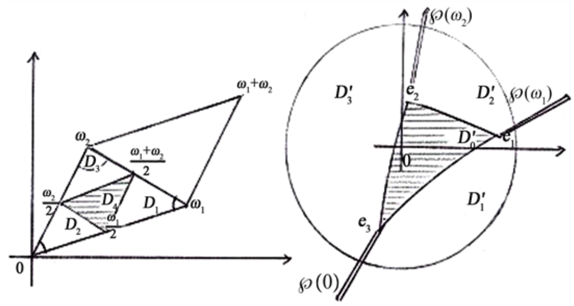



Figure 9: The fundamental domains of $\wp$ and their conformal mapping onto the whole complex plane with three slits

Since the *Dirichlet L-functions* have been implemented in different packages of software and for illustration purposes we only make use of these functions, it is necessary to present them separately.

A *Dirichlet character* modulo $q$ is an arithmetic function $\chi(n)$ which is periodic of period $q$ and such that if $q$ and $n$ are not relatively prime, then $\chi(n) = 0$ and if they are relatively prime then $\chi(n)$ is a root of order $\varphi(q)$ of the unity, where $\varphi$ is the *Euler totient function*. For every $q$ we have $\chi(1) = 1$.

Let us take $q = 7$. Then $\varphi(q) = 6$ and the six Dirichlet characters modulo seven are given in Table 1, where $\omega = e^{\pi i/3}$.

| $\chi\backslash n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $\chi_1(n)$ | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_2(n)$ | 0 | 1 | $\omega^2$ | $\omega$ | $-\omega$ | $-\omega^2$ | -1 |
| $\chi_3(n)$ | 0 | 1 | $-\omega$ | $\omega^2$ | $\omega^2$ | $-\omega$ | 1 |
| $\chi_4(n)$ | 0 | 1 | 1 | -1 | 1 | -1 | -1 |
| $\chi_5(n)$ | 0 | 1 | $\omega^2$ | $-\omega$ | $-\omega$ | $\omega^2$ | 1 |
| $\chi_6(n)$ | 0 | 1 | $-\omega$ | $-\omega^2$ | $\omega^2$ | $\omega$ | -1 |

Table 1: Dirichlet characters modulo 7

A *Dirichlet function* is obtained by performing analytic continuation to the whole complex plane of the sum of a *Dirichlet series*

$$\zeta_{A,\Lambda}(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \tag{3}$$

where $\Lambda = (\lambda_n)$ is a non decreasing sequence of positive numbers, $A = (a_n)$ is an arbitrary sequence of complex numbers and $s = \sigma + it$ is a complex variable. It make sense to deal only with *normalized* series (3) in which $a_1 = 1$ and $\lambda_1 = 0$. For every normalized Dirichlet series $\zeta_{A,\Lambda}(s)$ we have

$$\lim_{\sigma \to +\infty} \zeta_{A,\Lambda}(\sigma + it) = 1. \tag{4}$$

This apparently trivial property has surprising consequences regarding the geometry of the mappings by Dirichlet series. We list here a few of them.

The pre-image of the real axis by $\zeta_{A,\Lambda}(s)$ has infinitely many components which are of the three types:

a) $\Gamma'_k$ , $k \in \mathbb{Z}$ extending for $\sigma$ from $-\infty$ to $+\infty$ and which are mapped by $\zeta_{A,\Lambda}(s)$ bijectively onto the interval $(1, +\infty)$ of the real axis. These curves do not intersect each other and $\Gamma'_k$ and $\Gamma'_{k+1}$ form infinite open strips $S_k$ which are mapped by $\zeta_{A,\Lambda}(s)$ not necessarily one to one onto the whole complex plane with a slit alongside the interval $(1, +\infty)$ of the real axis. We count them from $-\infty$ to $+\infty$ such that $S_{k+1}$ is above $S_k$ and $0 \in S_0$.

b) Every strip $S_k$, $k \neq 0$ contains a unique component $\Gamma_{k,0}$ of the pre-image of the real axis which is mapped by $\zeta_{A,\Lambda}(s)$ bijectively onto the interval $(-\infty, 1)$ of the real axis. This component also extends for $\sigma$ from $-\infty$ to $+\infty$.

c) Every strip $S_k$, $k \neq 0$ contains a finite number of components $\Gamma_{k,j}$, $j \neq 0$ of the pre-image of the real axis which are mapped bijectively by $\zeta_{A,\Lambda}(s)$ onto the whole real axis. These components extend for $\sigma$ from $-\infty$ to some finite values. They are parabola shaped curves with the branches extending to $-\infty$. The strip $S_0$ has infinitely many components $\Gamma_{0,j}$.

The strip $S_k$, $k \neq 0$ having $m$ components $\Gamma_{k,j}$ of the pre-image of the real axis can be partitioned into $m$ sub-strips, the interiors of which are fundamental domains of $\zeta_{A,\Lambda}(s)$. The strip $S_0$ contains infinitely many fundamental domains.

For $k \neq 0$ the fundamental domains of every function $\zeta_{A,\Lambda}(s)$ are bounded by components of the pre-image by $\zeta_{A,\Lambda}(s)$ of the interval $(1, +\infty)$ of the real axis to which components of the pre-image of the segment between 1 and $\zeta_{A,\Lambda}(v_{k,j})$ is added, where $\zeta'_{A,\Lambda}(v_{k,j}) = 0$. We have shown in [10] that such a construction is always possible.

There is number $\sigma_c \leq \infty$ such that the series (3) converges locally uniformly for $\Re s > \sigma_c$ and diverges for $\Re s < \sigma_c$. The number $\sigma_c$ is called the *abscissa of convergence* of the series (3).

When $a_n = \chi(n)$ are Dirichlet characters of some module $q$ and $\lambda_n = \log n$ then we have a *Dirichlet L-series:*

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \tag{5}$$

The character whose values are only 0 and 1 is called *principal character*. Every Dirichlet L-series defined by a non principal character has the abscissa of convergence 0, while the Dirichlet series defined by principal characters have the abscissa of convergence 1.

When $q = 1$ the only character is the principal one and the corresponding series is the Riemann series, whose abscissa of convergence is known to be 1. The analytic continuation of this series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{6}$$

is the famous *Riemann Zeta Function.*

The Riemann Zeta Function is one of the most studied analytic functions, in view of its many applications in number theory, algebra, complex analysis, statistics, as well as in physics.

The pre-image of the real axis by $\zeta(s)$ shows infinite strips, Fig. 7, which can be divided into sub-strips representing fundamental domains of this function. The way it can be done is described in [10].

We have found later, [11], [12], [13], that this property is common to the whole class of *Dirichlet functions* to which the Riemann Zeta function belongs. The Dirichlet functions are analytic continuations to the whole complex plane of Dirichlet series (3). We will keep the notation $\zeta_{A,\Lambda}(s)$ for such a function.

The components of the pre-image by $\zeta_{A,\Lambda}(s)$ of circles centered at the origin and of radius $r$ are of three types depending on the values $r$.

When $r < 1$, these components are all closed curves around some zeros of the function. For $r$ small enough each one of these curves contains just one zero.

When $r = 1$, (the unit circle) every strip $S_k, k \neq 0$ contains exactly one unbounded component of the pre-image of the unit circle and some bounded ones. The unbounded component has the branches tending asymptotically to the boundary $\partial S_k$ of $S_k$.

When $r$ increases over one, the unbounded components fuse all into one unbounded curve expanding from $t = -\infty$ to $t = +\infty$. The bounded components expand also and two of them can fuse into one containing the zeros of both of them.

These components and the pre-image of the real axis form an orthogonal net of quadrilaterals which are conformal mapped by $\zeta_{A,\Lambda}(s)$ on half-rings centered at the origin with two opposite sides on the real axis and included respectively in the upper and the lower half-planes.

When $a_n$ are all real, then the real axis is included in the pre-image by the respective Dirichlet function of the real axis. Otherwise, such an inclusion does not take place. This is obvious for the Riemann Zeta

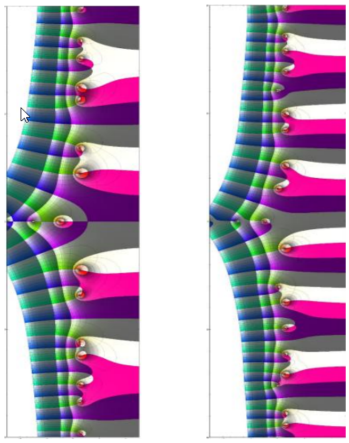


Figure 10: The pre-images by two Dirichlet L-functions of an orthogonal net formed with circles centered at the origin and the real axis

function, Fig. 7, but also for the Dirichlet L-functions illustrated in Fig. 10 and Fig. 11.

## 3 Fundamental Domains with Slits

The list of the classes of analytic functions we have visited is not exhaustible, yet it gives us an idea of what happens in general. When dealing with any analytic function we can focus on a particular fundamental domain $\Omega$. The function performs a conformal mapping of $\Omega$ onto the complex plane with a slit $L$. If we do also a Möbius transformation $M$ of the image plane, the function $M\circ f$ is a conformal mapping of the domain $\Omega$ onto the complex plane with two slits: $L$ and $M(L)$. One of them is the image by $M\circ f$ of a slit in $\Omega$, thus we need to deal with fundamental domains with slits. This is even more obvious when we study conformal self mappings of $\Omega$ of the form $f_{|\Omega}^{-1} \circ M \circ f$.

More exactly, there is a slit $L'$ in the complex plane carried by $M$ onto $L$ and which is the image by $f$ of a slit $L_M$ of $\Omega$. Therefore, $f$ carries $L_M$ into $L'$, $M$ carries $L'$ into $L$, $f_{|\Omega}^{-1}$ carries the two edges of $L$ into $\partial\Omega$, thus

$$\chi_M = f_{|\Omega}^{-1} \circ M \circ f \tag{7}$$

carries $L_M$ into $\partial\Omega$. The inverse function of $\chi_M$, where defined, is

$$\chi_M^{-1} = f_{|\Omega}^{-1} \circ M^{-1} \circ f.$$

There is a slit $L''$ of the complex plane, the image of $L$ by $M^{-1}$ and which is also the image by $f$ of a slit

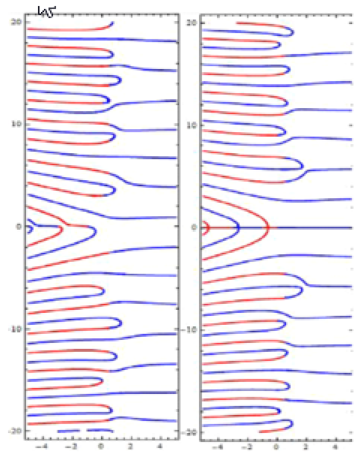


Figure 11: The pre-image of the real axis by two Dirichlet L-functions defined by $\chi_2(n)$ and $\chi_4(n)$ modulo 7

$L_{M^{-1}}$ of $\Omega$, thus $f$ carries $L_{M^{-1}}$ into $L''$, which is carried by $M$ into $L$ and $L$ is carried by $f_{|\Omega}^{-1}$ into $\partial\Omega$.

**Theorem 1**. The function $\chi_M$ is a conformal mapping of $\Omega \setminus L_{M^{-1}}$ onto $\Omega \setminus L_M$. The boundaries $\partial\Omega \cup L_{M^{-1}}$ and $\partial\Omega \cup L_M$ of the two double connected domains correspond one to each other through $\chi_M$ in the following way: $\partial\Omega$ is carried into $L_M$ and $L_{M^{-1}}$ is carried into $\partial\Omega$.

*Proof:* The functions $f$, $M$ and $f_{|\Omega}^{-1}$ are analytic functions in their domains, except for the simple pole of $M$, if there is one. The function $M^{-1} \circ f$ is a conformal mapping of $\Omega$ and carries $\partial\Omega$ into $L''$, which is carried by $f_{|\Omega}^{-1}$ into $L_{M^{-1}}$, hence $\chi_M$ is a conformal mapping of $\Omega \setminus L_{M^{-1}}$ which carries $L_{M^{-1}}$ into $\partial\Omega$. On the other hand, the function $\chi_M$ does not take any value belonging to $L_M$ since it is injective and $\chi_M^{-1}$ carries $L_M$ into $\partial\Omega$. By the boundary correspondence theorem $L_M$ must be carried by $\chi_M$ into $L_{M^{-1}}$. This completely proves the theorem. $\square$

It is known that for two arbitrary Möbius transformations $M_1$ and $M_2$ the function $M_1 \circ M_2$ is also a Möbius transformation. On the other hand, where all the functions are defined, we have:

$$\begin{aligned}\chi_{M_1\circ M_2}(s) &= f_{|\Omega}^{-1} \circ (M_1 \circ M_2) \circ f(s) \\ &= (f_{|\Omega}^{-1} \circ M_1 \circ f) \circ (f_{|\Omega}^{-1} \circ M_2 \circ f(s)) \\ &= \chi_{M_1} \circ \chi_{M_2}(s).\end{aligned} \tag{8}$$

Thus, the composition law in the family $\{\chi_M\}$ of transformations of $\Omega$ with slits defined by all Möbius transformations $M$ through the formula (7) is an internal operation. It is obvious that if $M_0$ is the identity mapping, them $\chi_{M_0}$ is the identity transformation of $\Omega$. Also, for every $M$ the transformation $\chi_{M^{-1}}$ is the inverse mapping of $\chi_M$. The associativity of the compositions of mappings $\chi_M$ is a direct corollary of the same property of the mappings $M$. Thus, we have:

**Theorem 2.** Any subgroup of Möbius transformations defines through the formula (7) a group of transformations of any fundamental domain $\Omega$ with slits of an analytic function $f$.

*Proof:* Let $G$ be a subgroup of Möbius transformations. We can think at any subgroup well studied in the literature, as for example the subgroup of Möbius transformations which represent the unit disc onto itself, or those with real coefficients etc.. Let $G_\Omega = \{\chi_M \mid M \in G\}$, where $\chi_M$ is defined by the formula (7). It doesn't harm to use the same sign for the composition in $G$ and in $G_\Omega$. We need to prove that $G_\Omega$ with the assumed composition law is a group of transformations of $\Omega$. If $\chi_{M_1}, \chi_{M_2} \in G_\Omega$, where $M_1, M_2 \in G$, then by (8) we have: $\chi_{M_1} \circ \chi_{M_2}(s) = \chi_{M_1 \circ M_2}(s)$, thus the composition in $G_\Omega$ is an internal operation. If $M_0$ is the unit element of $G$, i.e. for every $M \in G$ we have $M \circ M_0 = M_0 \circ M = M$, then $\chi_M \circ \chi_{M_0} = \chi_{M \circ M_0} = \chi_M$ and $\chi_{M_0} \circ \chi_M = \chi_{M_0 \circ M} = \chi_M$, therefore $\chi_{M_0}$ is the unit element of $G_\Omega$. Finally, $\chi_M \circ \chi_{M^{-1}} = \chi_{M \circ M^{-1}} = \chi_{M_0}$ and the conclusion is that $G_\Omega$ is indeed a group of transformations of $\Omega$ with slits. □

# 4 Fixed Points of Self-Mappings of the Fundamental Domains

Suppose that for $s_0 \in \Omega$ we have that $f(s_0)$ is a fixed point of the Möbius transformation $M$, i.e. $M(f(s_0)) = f(s_0)$. We notice that $f(s_0)$ does not belong to the slit $L$ corresponding to $\Omega$. We have

$$\chi_M(s_0) = f_{|\Omega}^{-1}(M(f(s_0))) = f_{|\Omega}^{-1}(f(s_0)) = s_0,$$

therefore $s_0$ is a fixed point of $\chi_M$. Reciprocally, if $s_0$ is a fixed point of $\chi_M$ this means

$$f_{|\Omega}^{-1}(M(f(s_0))) = s_0,$$

so

$$M(f(s_0)) = f(s_0),$$

therefore $f(s_0)$ is a fixed point of $M$, which obviously does not belong to $L$. Moreover:

**Theorem 3.** The fixed points of the transformation $\chi_M$ of $\Omega$ are those points $s$ for which $z = f(s)$ are fixed points of $M$.

When $M$ has a fixed point $z$ located on $L$, then to this point correspond two fixed points $s_1$ and $s_2$ of the extended $\chi_M$ such that $\chi_M(s_1) = \chi_M(s_2) = z$ ( Fig. 13). If no fixed point of $M$ is located on $L$, then there is a one to one correspondence between the fixed points of $M(z)$ and those of $\chi(s)$.

It is known, [15], that any Möbius transformation $M$ has either just one fixed point (the parabolic case), or two fixed points in which case the configuration of the mapping by $M$ can be of three kinds: elliptic, hyperbolic or loxodromic. All depends on the so-called multiplier of $M$, which is a number $\mu$ such that

$$\frac{M(z) - \xi_1}{M(z) - \xi_2} = \mu \frac{z - \xi_1}{z - \xi_2} \tag{9}$$

where $\xi_1$ and $\xi_2$ are the fixed points of $M$. The number $\mu$ is associated to $\xi_1$.

The multiplier associated to $\xi_2$ is $1/\mu$, [16].

If $s_1$ and $s_2$ are the fixed points of $\chi_M(s)$ corresponding to the fixed points $\xi_1$ and $\xi_2$ of $M(z)$, then the formula (9) translates into a similar formula for $\chi_M(s)$, namely:

$$\frac{f(\chi_M(s)) - f(s_1)}{f(\chi_M(s)) - f(s_2)} = \mu \frac{f(s) - f(s_1)}{f(s) - f(s_2)}.$$

An important property of Möbius transformations is that of transforming circles into circles, where by circle we understand a proper circle or a straight line (circle of infinite radius). We will call $\Omega$-*circle* the pre-image by $f_{|\Omega}$ of a circle from the complex plane. We notice that if for a circle $C$ we have $C \cap L = \varnothing$, then the pre-image $C_\Omega$ of $C$ is a closed curve in $\overline{\mathbb{C}}$. Otherwise, if $C$ traverses $L$ then $C_\Omega$ ends up at two different points on the boundary $\partial\Omega$ of $\Omega$ and if $C$ is tangent to $L$ then $C_\Omega$ is a closed curve tangent to $\partial\Omega$. These are obvious topological properties of the conformal mapping $f_{|\Omega}$.

It is known that to every non-parabolic Möbius transformation with fixed points $\xi_1$ and $\xi_2$ we can associate two families of circles, namely all circles passing through $\xi_1$ and $\xi_2$ and all circles orthogonal to them centered on the line between $\xi_1$ and $\xi_2$ and containing just one of the points $\xi_1$ or $\xi_2$. These last circles are called the *Apollonius circles.* Together the two families of circles form the so called *Steiner net.* The pre-image by $f_{|\Omega}$ of a Steiner net will be called $\Omega$-*Steiner net,* which is a net of orthogonal curves in $\Omega$ since $f_{|\Omega}^{-1}$ is a conformal mapping.

The Steiner net of $M$ describes the way $M$ moves the points of the complex plane and this depends on the value of $\mu$. If $\mu = e^{i\theta}$, $\theta \in \mathbb{R}$ (the elliptic case) the points are moved alongside the Apollonius circles counterclockwise around $\xi_1$ and clockwise around $\xi_2$ and the orthogonal circles move one into the other. If $\mu \in \mathbb{R}$, $\mu \neq 1$ (the hyperbolic case) the points are moved such that the Apollonius circles expand around $\xi_1$ and they shrink around $\xi_2$. When $\mu = \rho e^{i\theta}, \theta \in \mathbb{R}$, $\rho \neq 1$ (the loxodromic case) the motion of the points by $M$ is a combination of the previous two, resulting in trajectories of double spirals issuing from $\xi_1$ and entering in $\xi_2$. When the transformation is parabolic, i. e., it has a unique fixed point $\xi$ then we have two families of orthogonal circles passing through $\xi$ and a combination of elliptic and hyperbolic motions.

**Theorem 4.** The pre-image by $f_{|\Omega}$ of the Steiner net generated by a Möbius transformation $M$ is a net of orthogonal $\Omega$-circles, the $\Omega$-*Steiner net*. This net describes the motion of the points $s$ by the self mapping $\chi_M$ of $\Omega$ in the same way the Steiner net of $M$ does with the points $f(s)$.

*Proof:* Suppose that $M$ is non parabolic, therefore it has two distinct fixed points $\xi_1$ and $\xi_2$. If none of them belong to the slit associated to $\Omega$, then they are the image by $f$ of two distinct points $s_1, s_2 \in \Omega$, which are fixed points of $\chi_M$. If one of them belongs to the slit, then it is the image by the extended $f$ of two points on $\partial\Omega$.

The pre-image by $f_{|\Omega}$ of every Apollonius circle around $\xi_k$ is a $\Omega$-circle around $s_k$, $k = 1, 2$, the *Apollonius* $\Omega$-*circle.* The pre-image by $f_{|\Omega}$ of every circle orthogonal to the Apollonius circles is a $\Omega$-circle orthogonal to all the Apollonius $\Omega$-circles. Since $f_{|\Omega}^{-1}$ is a conformal mapping, the sense of the motion of the points on the $\Omega$-circles by $\chi_M$ is the same as that of their $f_{|\Omega}$-images by $M$. So, in the elliptic case the points $s$ move on the Apollonius $\Omega$-circles around $s_k$ in counterclockwise around $s_1$ and clockwise around $s_2$. In the hyperbolic case, the Apollonius $\Omega$-circles around $s_1$ expand while those around $s_2$ shrink. Finally, in the loxodromic case the motion of the points $s$ is on a double spiral issuing from $s_1$ and entering in $s_2$. $\square$

In the parabolic case, where $M$ has a unique fixed point $\xi$ there are four families of orthogonal $\Omega$-circles passing through $s_0 = f_{|\Omega}^{-1}(\xi)$. The transformation $\chi_M$ of $\Omega$ will move the points $s$ in the same way as $M$ moves the points $z$ of the complex plane in the corresponding Steiner net.

The computer experimentation with Dirichlet L-functions revealed some surprising facts. Since the formula (4) suggests that the components of the pre-image of circles passing through $z = 1$ tend to $\infty$ as $\sigma \to \infty$ one could conclude that the function $\chi_M$ corresponding to $M(z) = (2z - 1)/(2 - z)$ has only one fixed point, which is attractive, while $\infty$ is repelling. However, in Fig. 19 appeared some other repelling points. We realized that these are points on the boundaries of the fundamental domains where the respective Dirichlet L-function takes the value $z = 1$. Fig. 19 portrays the conformal self mapping of several fundamental domains whose boundaries cannot be n in the image. What we know for sure is the fact that these boundaries are necessarily between components of the pre-image of Apollonius circles asymptotically tangent at infinity.

# 5 Some Subgroups of the Group of Transformations of $\Omega$

The subgroups of the group of transformations of $\Omega$ correspond to subgroups of the group of Möbius transformations. The most familiar subgroup of Möbius transformations is

$$G = \left\{ M_{a,\theta} \middle| M_{a,\theta}(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z}, |a| \neq 1, \theta \in \mathbb{R} \right\} \tag{10}$$

which leaves invariant the unit circle and transforms the unit disc into itself when $|a| < 1$ and onto the exterior of the unit disc when $|a| > 1$. Also the exterior of the unit disc is transformed into itself when $|a| < 1$ and onto the unit disc when $|a| > 1$.

It can be easily checked that $M_{a,\theta} \circ M_{b,\eta} = M_{c,\phi}$, where

$$c = \frac{a + e^{-i\theta} b}{1 + e^{i\theta} a\bar{b}} \text{ and } \phi = \theta + \eta,$$

and thus the composition in $G$ is an internal law. The unit element of $G$ is obtained for $a = 0$ and $\theta = 0$. The inverse element $M_{b,\eta}$ of $M_{a,\theta}$ is obtained for $b = -e^{i\theta} a$ and $\eta = -\theta$. It is obvious that the elements of $G$ with $|a| < 1$ form a subgroup of $G$. This is not true for the elements of $G$ with $|a| > 1$ since the unit element of $G$ is not among them.

The interval $(1, +\infty)$ of the real axis is transformed by $M_{a,\theta}$ into an arc of a circle and if $a \in \mathbb{R}$ and $\theta = 0$ it is transformed into an interval of the real axis. This remark will be useful for the examples which follow.

Let us deal first with the Möbius transformation

$$M(z) = \frac{2z - 1}{2 - z} \tag{11}$$

which is an element of $G$ above with $a = 1/2$ and $\theta = 0$. This is a non-parabolic Möbius transformation with the fixed points 1 and $-1$. We will classify the circles of the corresponding Steiner net taking into account their position with respect to the slit

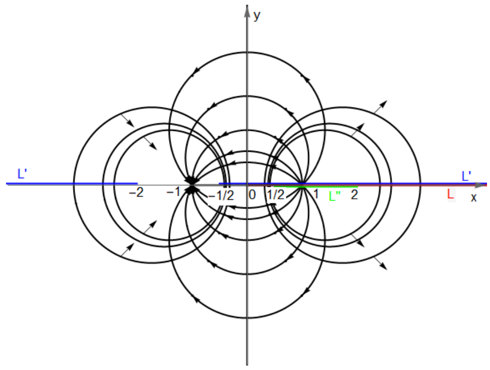



Figure 12: The Steiner net of the Möbius transformation (11) with the slits corresponding to the exponential function

$L = (1, +\infty)$ associated to a fundamental domain $\Omega$ of a Dirichlet function $\zeta_{A,\Lambda}(s)$. We notice that the Apollonius circles corresponding to this transformation are of two kinds: those around 1, which intersect the slit in one point and therefore their pre-images by $\zeta_{A,\Lambda}(s)$ are arcs having the ends on each one of the components of $\partial\Omega$ and those around $-1$ which do not intersect the slit and therefore their pre-images are closed $\Omega$-circles. All these $\Omega$-circles fill the domain bounded by the pre-image of the imaginary axis, which is a parabola-like curve with the branches tending asymptotically towards $\partial\Omega$ at $\sigma = -\infty$. The remaining part of $\Omega$ is filled by the pre-images of circles intersecting the slit. Each point of $\partial\Omega$ is the end point of such an $\Omega$-circle.

Let us notice that since $M$ has real coefficients, it maps the real axis onto itself and so does $M^{-1}$. Thus, for $L = (1, +\infty)$ we have that $L'$ and $L''$ are parts of the real axis. Since $M(1) = 1$, $M(2) = \infty$ and $M(\infty) = -2$, we have $L' = (-\infty, -2) \cup (1, +\infty)$. Also, since $M^{-1}(1) = 1$ and $M^{-1}(\infty) = 2$, we have that $L'' = (1, 2)$. Since $L'' \subset L$, for these functions we have $L_{M^{-1}} \subset \partial\Omega$, thus $\chi_M : \Omega \to \Omega \setminus L_M$. Since $M$ has the fixed points 1 and $-1$, the function $\chi_M$ should have the fixed points $s_1$ and $s_{-1}$ for which $\zeta_{A,\Lambda}(s_1) = 1$ and $\zeta_{A,\Lambda}(s_{-1}) = -1$. Yet, there is no such $s_1 \in \Omega$ such that $\zeta_{A,\Lambda}(s_1) = 1$. We only have $\lim_{\sigma\to+\infty} \zeta_{A,\Lambda}(\sigma + it) = 1$. We cannot say that $\infty$ is a fixed point for $\chi_M$ since $\chi_M$ is not defined at $\infty$. However, $\infty$ is repelling for $\chi_M$ and $\chi_M$ has the attracting fixed point $s_{-1}$.

This is a hyperbolic Möbius transformation with the multiplier $\mu = 3$. Let us notice that if a circle $C$ of the net passing through $-1$ and 1 has the center at $ih$, $h \in \mathbb{R}$, then its radius is $r = (h^2 + 1)^{1/2}$. For any Apollonius circle orthogonal to it of center $x \in \mathbb{R}$ and radius $\rho$ we have

$$r^2 + \rho^2 = x^2 + h^2 = x^2 + r^2 - 1,$$

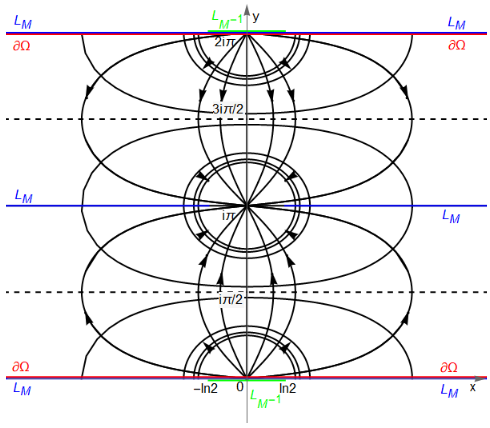



Figure 13: The conformal self-mapping by $\chi_M$ of the fundamental domain of the exponential function and the corresponding $\Omega$-Steiner net when M is given by the relation (11)

hence $x^2 = \rho^2 + 1$, which means that $x$ and $\rho$ depend on each other, but not on $h$ and $r$, confirming the known fact that the Apollonius circles are all orthogonal to every circle $C$ passing through the fixed points of the Möbius transformation. These relations help us to illustrate the corresponding Steiner net by choosing conveniently the parameters of the two families of circles. In the picture above we have chosen for $h$ the values: $\pm 1/2$, $\pm 1$ and $\pm 3/2$. Correspondingly, we have for $r$ the values: $\sqrt{5}/2$, $\sqrt{2}$ and $\sqrt{13}/2$. If we let $\rho = r$, then we have for $x$ respectively: $\pm 3/2$, $\pm\sqrt{3}$ and $\pm\sqrt{17}/2$. Now we can write the equations of the 12 circles chosen to represent the Steiner net and use a software to draw the respective net and its pre-image by any analytic function into any fundamental domain of that analytic function.

Fig. 12 is so simple since the Möbius transformation $M(z)$ we have taken has real coefficients and therefore the real axis is mapped by $M(z)$ onto itself and $L_M$ and $L_{M^{-1}}$ are located on the preimage of the real axis, Fig. 13, Fig. 15 and Fig. 17 . In particular, when the slit is $L = (1, +\infty)$, which is the case of Dirichlet functions and the fundamental domains $R_1$ and $R_{-1}$, then $L'$ and $L''$ are also on the real axis. The situation is a little more complicated when the coefficients of $M(z)$ are complex. Then the pre-images of $L$ by $M$ and $M^{-1}$ are arcs of a circle, Fig. 14, which are mapped by $f_\Omega^{-1}$ into some curves on the $\Omega$-Steiner net, Fig. 14 and Fig. 15. Let us deal with the Möbius transformation:

$$M(z) = \frac{z - a}{1 - \bar{a}z}, a \notin \mathbb{R}, |a| \neq 1 \tag{12}$$

This is a non parabolic Möbius transformation with

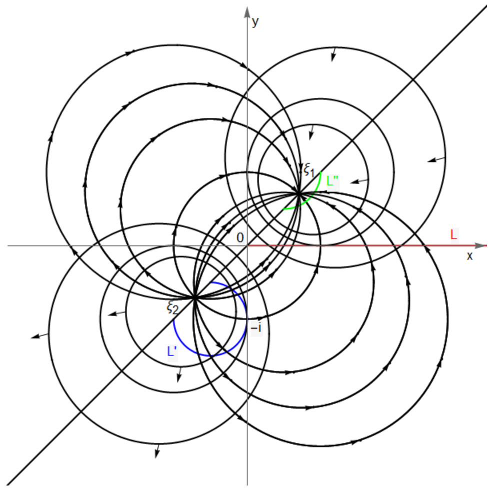



Figure 14: The Steiner net of to the Möbius transformation (12) with the slits corresponding to the exponential function

the fixed points $\xi_{1,2} = \pm e^{i\theta}$, where $\theta = \arg a$. It can be easily checked that if $a = re^{i\theta}$ then the multiplier of the transformation (12) is $\mu = \frac{1+r}{1-r}$, therefore this transformation is hyperbolic.

The function $M(z)$ transforms the interval $(1, +\infty)$ into an arc of circle with the ends at $\frac{1-a}{1-\overline{a}} = e^{2i\phi}$, where $\phi = \arg(1-a)$ and $-1/\overline{a}$ and this arc of circle is mapped by $f_{|\Omega}^{-1}$ into $L_M$. Also, $M^{-1}$ transforms $(1, +\infty)$ into an arc of circle with the ends at $\frac{1+a}{1+\overline{a}} = e^{2i\psi}$, where $\psi = \arg(1+a)$ and at $1/\overline{a}$ and this arc of circle is transformed by $f_{|\Omega}^{-1}$ into $L_{M^{-1}}$.

When the slit corresponding to $f_{|\Omega}$ is $(0, \infty)$, as in the case of the exponential function, we have $M(0) = -a$, $M(\infty) = -1/\overline{a}$, $M^{-1}(0) = a$ and $M^{-1}(\infty) = 1/\overline{a}$.

Let us take $a = \frac{1}{2}(1+i)$, hence $\arg a = \pi/4$. We get the non parabolic Möbius transformation with the fixed points $\pm(1+i)/\sqrt{2}$. It can be easily checked that the image by $M$ of the interval $(1, +\infty)$ is an arc of circle with the ends in $-i$ and $-(1+i)$ passing through $(-1/2)(1+i)$ and the image by $M^{-1}$ of the same interval is an arc of circle with the ends in $(4+3i)/5$ and $1+i$ passing through $(9+7i)/10$. When the slit corresponding to $f_{|\Omega}$ is $(0, \infty)$, then $L'$ is an arc of circle ending at $-(1+i)$ and $(-1/2)(1+i)$ and $L''$ is an arc of circle ending at $(1/2)(1+i)$ and $1+i$, Fig. 15.

Additional Steiner nets for different functions are illustrated in Fig. 16 and Fig. 18.

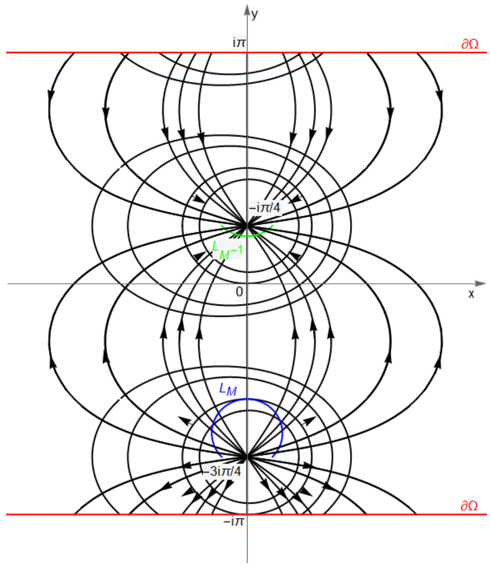



Figure 15: The conformal self-mapping by $\chi_M$ of the fundamental domain of the exponential function and the corresponding $\Omega-$Steiner net when M is given by the relation (12)

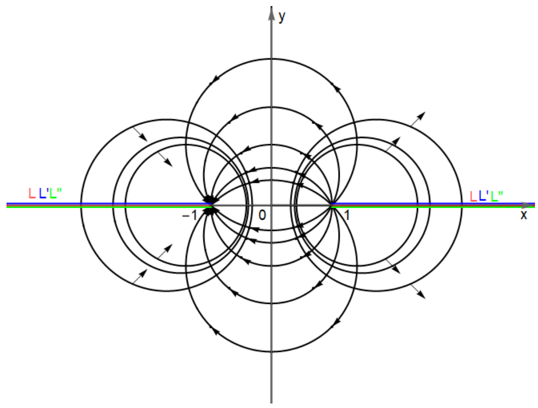



Figure 16: The Steiner net of the Möbius transformation (11) with the slits corresponding to the cosine function

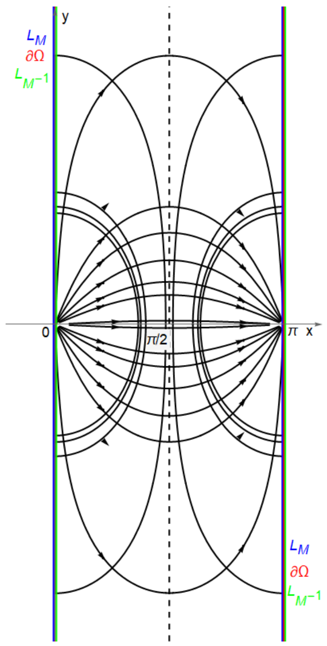



Figure 17: The conformal self-mapping by $\chi_M$ of the fundamental domain of the cosine function and the corresponding Ω-Steiner net when M is given by the relation (11)

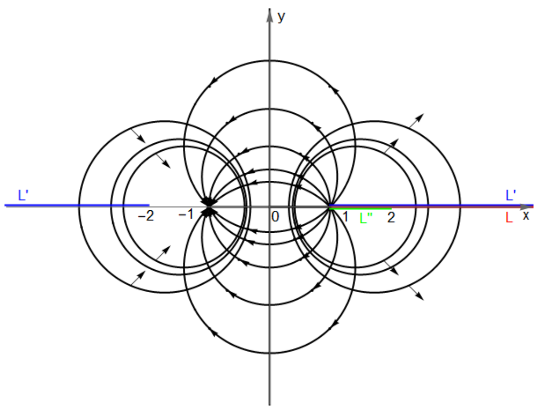



Figure 18: The Steiner net of the Möbius transformation (11) with the slits corresponding to the Dirichlet L-function L(7,2,s)

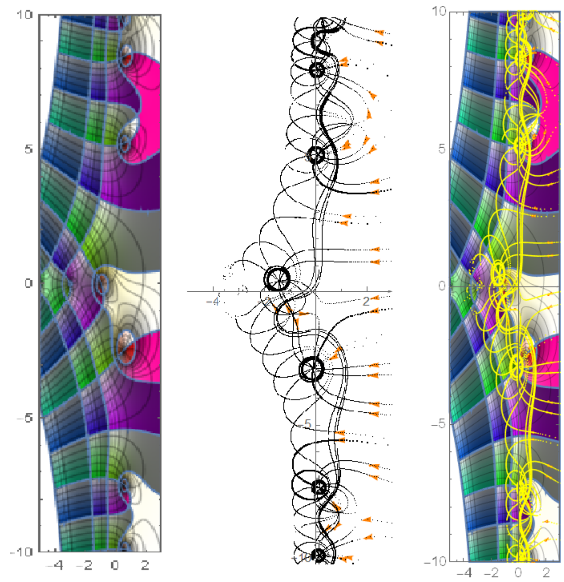



Figure 19: The conformal self-mapping by $\chi_M$ of the fundamental domain of the Dirichlet L(7,2,s) function and the corresponding $\Omega-$Steiner net when M is given by the relation (11)

# 6 Conformal Mappings of the Fundamental Domains with Slits One Onto Each Other

Let $\Omega_k$ and $\Omega_j$ be two fundamental domains of the analytic function $f$ and let $f_{|\Omega_k}$ and $f_{|\Omega_j}$ be conformal maps of these domains onto the complex plane with the slits $L_k$ and respectively $L_j$. There is a slit $L_{k,j}$ of $\Omega_k$ which is mapped by $f_{|\Omega_k}$ onto $L_j$ and there is a slit $L_{j,k}$ of $\Omega_j$ which is mapped by $f_{|\Omega_j}$ onto $L_k$ , in other words $L_{k,j} = f^{-1}_{|\Omega_k}(L_j)$ and $L_{j,k} = f^{-1}_{|\Omega_j}(L_k)$. The function $\chi_{k,j}(z) = f^{-1}_{|\Omega_j} \circ f_{|\Omega_k}(z)$ is a conformal mapping of $\Omega_k \backslash L_{k,j}$ onto $\Omega_j \backslash L_{j,k}$. These two double connected domains have the boundaries $\partial\Omega_k \cup L_{k,j}$ and respectively $\partial\Omega_j \cup L_{j,k}$ and $\chi_{k,j}$ carries $\partial\Omega_k$ into $L_{j,k}$ and $L_{k,j}$ into $\partial\Omega_j$. when $k = j$ then $L_{k,j} = \varnothing$ and $\chi_{k,k}$ is the identity mapping of $\Omega_k$. It is also obvious that $\chi^{-1}_{k,j} = \chi_{j,k}$.

We can only compose functions $\chi_{k,j}$ with $\chi_{m,k}$ for different $j$ and $m$ and we obtain $\chi_{m,j}$.

Each one of the domains $\Omega_k$, $\Omega_j$ and $\Omega_m$ have now two slits , namely $L_{k,j}$ and $L_{k,m}$, respectively $L_{j,k}$ and $L_{j,m}$ and finally $L_{m,k}$ and $L_{m,j}$.

The mappings can be extended to the boundaries.

We notice that

$$(\chi_{k,j} \circ \chi_{m,k}) \circ \chi_{l,m} = \chi_{m,j} \circ \chi_{l,m} = \chi_{l,j}$$

and

$$\chi_{k,j} \circ (\chi_{m,k} \circ \chi_{l,m}) = \chi_{k,j} \circ \chi_{l,k} = \chi_{l,j},$$

thus the composition law is associative. However, the set of these mappings fails to form a group since no two arbitrary mappings can be composed.

Let us notice that when $L_k = L_j$ then $L_{k,j} = L_{j,k} = \varnothing$ and $\chi_{k,j}$ is a conformal mapping of $\Omega_k$ onto $\Omega_j$. This mapping can be extended by continuity to $\partial\Omega_k$ and the extended function maps one to one $\partial\Omega_k$ onto $\partial\Omega_j$. A lot of classes of analytic functions fulfill this condition, as for example the exponential, the modular function, the trigonometric and hyperbolic functions.

Finally, if $f$ has a finite number $m$ of fundamental domains, as is the case of the rational functions of degree $m$,then the number of functions $\chi_{k,j}$ is $m^2$.

# 7 Conclusion

The study of the fundamental domains of analytic functions initiated by Ahlfors, proved to be useful in revealing global mapping properties of these functions and in particular in the theory of distribution of zeros of Dirichlet functions. In our opinion the famous Riemann Hypothesis related to the non trivial zeros of the Riemann Zeta function is an aspect of a deeper inside of these global mapping properties of Dirichlet functions. This topic has been treated in the literature only by number theory techniques. Using the powerful tool of conformal mappings has the advantage of a global view. Moreover, this tool enabled a strategy of divide and conquer, in which it was enough to deal with a specific strip containing a finite number of fundamental domains in order to draw conclusions valid for the whole complex plane. The graphics we included have shown that the behavior of those mappings is like that of the Möbius transformations. They do to the fundamental domains the same thing that the Möbius transformations are doing to the whole complex plane. In this paper we have focused on a particular property of fundamental domains of any analytic function, namely that of allowing conformal mappings onto itself. These mappings are obtained by combining the function with Möbius transformations and the inverse function. We have also studied the conformal mappings between two fundamental domains.

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