Distributed optimization algorithms for heterogeneous linear multi-agent systems with inequality constraints

ZHENGQUAN YANG[1], WENJIE YU[2], ZHIYUN GAO[1]
[1]College of transportation science and Engineering, Civil Aviation University of China, Tianjin 300300, CHINA
[2]College of Science, Civil Aviation University of China, Tianjin 300300, CHINA

Abstract: - In this paper, for heterogeneous linear multi-agent systems, a distributed constrained optimization problem about digraphs is studied. Every agent only utilizes local interaction and information such that all agents can achieve the global objective function. The state of each agent is limited to a local inequality constraint set. First, this paper proposes a distributed continuous-time optimization algorithm by designing a left eigenvector corresponding to the zero eigenvalue of the Laplacian matrix, which removes the imbalance of the communication graph. Next, the asymptotical convergence about the algorithm is demonstrated using Lyapunov stability. Finally, two numerical examples are given to illustrate the effectiveness of the algorithm.

Key-Words: - Asymptotical convergence, Weight-unbalanced digraphs, Multi-agent systems, Distributed optimization,

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1 Introduction

Distributed optimization has broad application prospects in the fields of economic dispatch in smart grid, [1], wireless sensor networks, [2], UAV formation, [3], etc., so it has become a research hotspot at present. Several types of distributed algorithms based on the framework of multi-agent systems have been proposed successively and are used to solve various optimization problems.

Up to now, in most of the literature (see, [4], [5]), many works have been derived based on the setting of discrete-time. Owing to the fact that continuous-time dynamic systems are more convenient to study convergence and their applications in physical systems are more flexible, distributed continuous-time algorithms have also received widespread attention from scholars. In [6], the authors put forward the continuous-time distributed adaptive coordination protocols to address the given optimization problems. Aiming at the distributed optimization problem with general constraints, a proportional-integral (PI) consensus protocol was proposed by [7]. However, in light of these distributed strategies depend on the differentiability of local objective functions, they may not be able to solve the problems that local objective functions are non-smooth. The dual decomposition technique was given to decouple the equality constraints and a new initialization-free distributed continuous-time algorithm was developed in [8]. In [9], the authors adopted the Laplacian-gradient method and designed an adaptive control to resolve the resource allocation problem, and constructed a distance-based exact penalty function to handle local convex sets. Different from the above distributed optimization problems based on single integrator multi-agent systems, the Euler-Lagrange systems were considered in [10], and the system dynamics were second-order dynamical systems in [11].

These above researches are all based on undirected networks, which require the information transmission between agents to be symmetrical and bidirectional. However, in the actual systems, the information exchange between agents is often one-way and easy to be interfered with external factors, such as data packet loss and signal attenuation. Therefore, in recent years, distributed optimization on digraphs has attracted more and more attention. In [12], the authors proposed a subgradient optimization algorithm of distributed average consistency and in [13], the authors established an improved subgradient push algorithm by introducing push-sum protocol and distributed subgradient algorithm for the deterministic directed switching network. In [14], the authors solved the constrained optimization problems without any initialization conditions and in [15], the authors researched the resource allocation problems with the help of a projection operator over weighted balanced digraphs. In [16], [17], [18], [19], the directed network is generalized to the situation of weight-unbalanced. To avoid knowing certain information of both in-neighbors and out-neighbors, the effort was made in [16], to design a continuous-time coordination algorithm. In [17], the authors designed an adaptive algorithm by virtue of dynamic coupling gains and the requirements of the Lipschitz constants, meanwhile, the network connectivity was removed. Resorting to the gradient-tracking and PI strategies, in [18], the authors established a distributed optimization algorithm with a fixed step-size, which ensured that the convergence rate was not affected by the decreasing step-size. In [19], the authors designed an adaptive distributed continuous-time algorithm, in which the out-Laplacian matrix was adopted to overcome the difficulties brought about by a weight-unbalanced communication graph.

As an important carrier of distributed information processing, linear multi-agent systems have achieved many research results. For homogeneous linear multi-agent systems on general digraphs, some relevant studies have been conducted in [20]. Actually, different multi-agents have different system states and even diverse dimensions of state space. Many works in [21], [22], [23], have been derived for heterogeneous linear multi-agent systems. By means of the KKT condition and primal-dual control scheme, in [22], the authors proposed the state-based and output-based adaptive control algorithms without initialization. Based on output feedback, the PI control technique was used to handle distributed optimal output consensus problem in [23]. This did not require the convexity of the local cost functions, but the global control parameters were adopted. For more related works on distributed optimization algorithms, readers can refer to some literature reviews, [24], [25], [26], [27], [28], [29].

Most of theses above-mentioned researches are limited to distributed optimization problems under undirected graphs or unconstrained digraphs. The design of constrained distributed algorithms under weight-unbalanced digraphs still has some limitations. When the balance of communication topology is damaged, the original distributed algorithms suitable for undirected and balanced directed graphs may become invalid. The Laplacian matrix for the unbalanced communication graph is asymmetric, so the equilibrium point of the general distributed algorithm is not equal to the optimal solution. Hence, it is quite necessary to extend the constrained distributed optimization algorithms to the weight-unbalanced digraphs. Besides, most systems are generally heterogeneous in practice, in the meantime, notice that more and more scholars focus on the heterogeneity of multi-agent systems. Generally speaking, on account of the more complex dynamics involved in systems, dealing with the consistency of heterogeneous systems is undoubtedly a challenge. thus, the study of distributed constrained optimization problems over weight-unbalanced digraphs considering the heterogeneous structure of multi-agent systems can be further discussed.

To handle the distributed optimization problem with inequality constraints, in which the local objective functions are strongly convex, this paper aims to design a novel continuous-time optimization algorithm for heterogeneous linear multi-agent systems. Simultaneously, each agent interacts information with other agents through a communication network modeled by the weight-unbalanced digraph. The main contributions of this paper are given as follows.

(1) Compared with [6], [7], [8], [22], which require the graphs to be undirected, our algorithm is designed over weight-unbalanced digraphs. In addition, some of the methods used to prove convergence cannot be applied to unbalanced graphs such as the symmetry of undirected graphs in [6]. The distributed optimization problems on unbalanced digraphs have also been studied in [16], [17], [23]. However, in contrast to [16], [17], [23], which can only deal with unconstrained optimization problems, these approaches they used do not enable agents to reach consensus on the intersection of feasible sets subject to set constraints.

(2) We devote ourselves to researching the heterogeneous linear multi-agent systems in this study while the works of [15], [20] are restricted to homogeneous linear multi-agent systems. The subsystems of this paper have different dynamics, so the method is also suitable for agents with identical dynamics.

We arrange the remaining parts of this paper in the following order. Section 2 introduces some useful preliminaries. Section 3 formulates the constrained optimization problem and gives some indispensable assumptions and lemmas. The main result of this article presented in Section 4 is to seek the optimal output of a given problem by designing a distributed continuous time optimization algorithm, and to analyze its asymptotic convergence.Then in Section 5, the effectiveness of the proposed algorithm is illustrated via two numerical examples. Finally, Section 6 discusses our conclusions and future work.

2 Preliminaries

2.1 Notations

Let $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$ stand for the sets of real numbers, n-dimensional real vectors, and $n \times m$-dimensional real matrices, respectively. Let $I_n \in \mathbb{R}^{n\times n}$, $\mathbf{1}_n \in \mathbb{R}^n$, $\mathbf{0}_n \in \mathbb{R}^n$ represent the identity matrix, the vector with entries equal to one and zero, respectively. $A^\top$ and $\|\cdot\|$ are respectively the transpose of a matrix A and the Euclidean norm. $col(x_1,\dots,x_n) = (x_1^\top,\dots,x_n^\top)^\top$ is a column vector sequentially stacked by vectors $x_1,\dots,x_n$. $(p)^+ = p$, if $p > 0$, and $(p)^+ = 0$ otherwise. $\otimes$ represents the Kronecker product.

2.2 Convex Analysis and Projection Operation

For a set $\Omega \subseteq \mathbb{R}^d$, if $\tau x_1 + (1+\tau)x_2 \in \Omega$ for any $x_1, x_2 \in \mathbb{R}^d$, $x_1 \neq x_2$ and $\tau \in (0,1)$, then Ω is a convex set. $f : \mathbb{R}^d \to \mathbb{R}$ is a continuously-differentiable function, if there exists $m > 0$ such that $(x_1 - x_2)^\top(\nabla f(x_1) - \nabla f(x_2)) \geq m\|x_1 - x_2\|^2$, for any $x_1, x_2 \in \mathbb{R}^d$ holds, the function f is said to be strongly convex over a convex set $\Omega \subseteq \mathbb{R}^d$. And moreover, if there exists $\epsilon_x, M > 0$ such that for any $x_1, x_2 \in \mathcal{B}(x, \epsilon_x)$, there is $|f(x_1) - f(x_2)| \leq M\|x_1 - x_2\|$, then we say f is locally Lipschitz at $x \in \mathbb{R}^d$.

We signify $\mathcal{P}_\Omega(x) = argmin_{y\in\Omega}\|x - y\|$ as the projection of a vector x, where Ω denotes a closed convex set. A useful lemma on the based on the definition of $\mathcal{P}_\Omega(x)$ is following:

Lemma 1 ([7]) Define $\Omega \subseteq \mathbb{R}^d$ to be a nonempty closed convex set. Then $x = \mathcal{P}_\Omega(y)$ if and only if $x \in \Omega$ and $(x - y)^\top(\varrho - x) \geq 0$ holds for any $\varrho \in \Omega$.

2.3 Graph Theory

We describe $\mathcal{G} = (\mathcal{V}, \varepsilon, \mathcal{A})$ as a communication graph consisting of N agents with node set $\mathcal{V} = \{1, \ldots, N\}$ and edge set $\varepsilon \subseteq \mathcal{V} \times \mathcal{V}$, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is a weighted adjacency matrix. Particularly, $a_{ij} > 0$ if $(i,j) \in \varepsilon$, and $a_{ij} = 0$ otherwise. A directed path from agent i to agent j is a sequence of ordered edges in the form of $(i, i_1), (i_1, i_2), \ldots, (i_l, j)$. If there exists a directed path connecting every pair of nodes, then $\mathcal{G}$ is called a strongly connected directed graph. Let $d_i^{\text{in}} = \sum_{j=1}^N a_{ij}$ and $d_i^{\text{out}} = \sum_{j=1}^N a_{ji}$ respectively be the in-degree and out-degree of agent i. The Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{N\times N}$ associated with $\mathcal{G}$ is denoted by $L = D^{in} - A$, where $D^{\text{in}} = \text{diag}(d_1^{\text{in}}, d_2^{\text{in}}, \ldots, d_N^{\text{in}})$. For all $i \in \mathcal{V}$, if $d_i^{\text{in}} = d_i^{\text{out}}$, then we say the directed graph is weight-balanced, otherwise weight-unbalanced.

Lemma 2 ([16]) For a strongly connected graph $\mathcal{G}$, and $L \in \mathbb{R}^{N\times N}$ is its Laplacian matrix. Then, the following properties hold.

(1) $L\mathbf{1}_N = \mathbf{0}_N$;

(2) there is a positive left eigenvector $\xi = (\xi_1, \xi_2, \ldots, \xi_N)^\top$ concerning eigenvalue zero such that $\xi^\top L = \mathbf{0}_N^\top$ and $\sum_{i=1}^N \xi_i = 1$;

(3) $\tilde{L} = \frac{\Xi L + L^\top \Xi}{2}$ is a valid Laplacian matrix for a strongly connected and balanced graph, where $\Xi = diag(\xi_1, \ldots, \xi_N)$, and $\tilde{L}$ is positive semidefinite. Zero is a simple eigenvalue of $\tilde{L}$ and $\lambda_2(\tilde{L}) = \min_{\mathbf{1}_N^\top = \mathbf{0}_N} \frac{x^\top \tilde{L} x}{x^\top x}$ is its second smallest eigenvalue.

3 Problem formulation

We consider that a multi-agent system is composed of N agents, and each agent i is given the following heterogeneous linear dynamics:

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i, \\ y_i &= C_i x_i, \end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}^{n_i}$ is the state variable of agent i, $u_i \in \mathbb{R}^{p_i}$ is the control input of agent i, and $y_i \in \mathbb{R}^d$ is the control output of agent i. $A_i \in \mathbb{R}^{n_i\times n_i}$, $B_i \in \mathbb{R}^{n_i\times p_i}$, $C_i \in \mathbb{R}^{d\times n_i}$ are the state, input, and output matrices, respectively, which are constant matrices.

Our goal is to design a proper controller $u_i(t)$ for each agent i, where each agent only knows its own information and local interaction, such that all agents can reach the optimal output y^* under the local inequality constraints. The optimization problem with inequality constraints is formulated as follows:

$$\begin{aligned} &\min f(y) = \sum_{i=1}^N f_i(y), \\ &\text{s.t.} \quad g_i(y) \leq 0, i = 1, \ldots, N, \end{aligned} \tag{2}$$

where $y \in \mathbb{R}^d$, $f_i : \mathbb{R}^d \to \mathbb{R}$, $g_i : \mathbb{R}^d \to \mathbb{R}^{r_i}$. Here, $f_i(y)$ is known as the local objective function of ith agent, and $g_i(y) = (g_{i1}(y), \ldots g_{ir_i}(y))^\top$ is the local inequality constraints of ith agent, where r_i denotes the number of local constraints. Clearly, the optimization problem can be resolved in a distributed way due to both information of the local objective functions $f_i(y)$ and local constraints $g_i(y)$ are only obtained by agent i.

Before the introduction of our main results, we make a few assumptions about the communication graph, the local objective functions and the local constraint functions.

Assumption 1 The communication network $\mathcal{G}$ is strongly connected.

Assumption 2 The function $f_i(y), i = 1, \ldots, N$ is m_i-strongly convex. Local constraint function $g_i(y), i = 1, \ldots, N$ is convex.

Assumption 3 The gradient function of the local objective function $\nabla f_i, i = 1, \ldots, N$ as well as the local constraint function $\nabla g_i, i = 1, \ldots, N$ is M_i-Lipschitz with $M_i > 0$.

Under the whole output variable $y = col(y_1, y_2, \ldots, y_N) \in \mathbb{R}^{Nd}$, the global objective function incurred by all agents is $f(y) = \sum_{i=1}^N f_i(y_i)$. Under Assumption 1, we can reformulate the problem (2) as

$$\begin{aligned} &\min f(y) = \sum_{i=1}^{N} f_i(y_i), \\ &\text{s.t.} \quad (L \otimes I_d)y = \mathbf{0}, \quad g_i(y_i) \le 0, i = 1, \dots, N. \end{aligned} \tag{3}$$

The equality constraint $(L \otimes I_d)y = \mathbf{0}$ ensures that $y_1 = y_2 = \ldots = y_N$, the problem (2) and problem (3) are equivalent in terms of the optimal solution set. Then to solve the problem (2), we can solve the problem (3) instead.

Remark 1 Define $\mathcal{Y} = \bigcap_{i=1}^N \mathcal{Y}_i$ with $\mathcal{Y}_i = \{y_i \in \mathbb{R}^d | g_i(y_i) \le 0, i = 1, \dots, N\}$ as the global constraint set on the output variable. And the optimal set of the considered optimization problem is denoted by $\mathcal{Y}^*$. Assumption 1 permits the graph to be unbalanced. Assumption 2 ensures that the optimal solution to (3) is unique and the optimal set $\mathcal{Y}^*$ is convex.

Assumption 4 (A_i, B_i) is controllable, and

$$\text{rank}(C_i B_i) = d, i = 1, \dots, N. \tag{4}$$

Lemma 3 ([23]) Under Assumption 4, the following matrix equations

$$\begin{aligned} C_i B_i \Upsilon_i &= C_i A_i, \\ C_i B_i \Psi_i &= I_d. \end{aligned} \tag{5}$$

exist solutions $\Upsilon_i \in \mathbb{R}^{p_i \times n_i}$, $\Psi_i \in \mathbb{R}^{p_i \times d}$.

4 Distributed General Continuous-Time Convex Optimization Algorithm

In this part, we give a novel distributed optimization algorithm to solve the problem (3) and prove its convergence property in detail. The distributed optimization algorithm is given as follows

$$\begin{aligned} u_i &= -\Upsilon_i x_i + \Psi_i \Big(-\frac{\nabla f_i(y_i)}{z_i^i} - (\nabla g_i(y_i))^\top (\mu_i + \dot{\mu}_i) \\ &\quad - \alpha \sum_{j=1}^N a_{ij}(y_i - y_j) - \sum_{j=1}^N a_{ij}(\eta_i - \eta_j)\Big), \\ \dot{\eta}_i &= \alpha \sum_{j=1}^N a_{ij}(y_i - y_j), \\ \dot{z}_i &= -\sum_{j=1}^N a_{ij}(z_i - z_j), \\ \dot{\mu}_i &= (\mu_i + g_i(y_i))^+ - \mu_i, \end{aligned} \tag{6}$$

where $\eta_i \in \mathbb{R}^d$ and $\mu_i \in \mathbb{R}_+^r$ with $r = \sum_{i=1}^N r_i$ are the auxiliary states of agent i, $z_i \in \mathbb{R}^N$ and z_i^i is the ith component of z_i, x_i, u_i, y_i are same defined as (1). α is a positive parameter. Υ_i, Ψ_i are feedback matrices based on Lemma 3. ∇f_i is the gradient of f_i, ∇g_i is the gradient of g_i. The simulation structure diagram of the control algorithm (6) is shown in Figure 1 (Appendix)

Remark 2 In comparison, the related work in [7], [8], [22] only give the results on undirected graphs, while (6) is designed over unbalanced digraphs. We evaluate the left eigenvector associated with the zero eigenvalue of the Laplacian matrix $\tilde{L}$ through (6) by designing the variable z_i, which removes the imbalance of the communication graph, and ensures that the constrained optimization algorithm can converge to the optimal output y^* without knowing the information of the left eigenvector. However, in [20], the information of ξ needs to be obtained in advance. And moreover, the μ_i is applied to handle local inequality constraints. It should be noted that the algorithms in [16], [17], [23], do not take the local state constraints into account. Furthermore, for some other unconstrained optimization problems over weight-balanced directed graphs or undirected graphs, the algorithm (6) we designed is still applicable.

Let $x = col(x_1, \dots, x_N)$, $y = col(y_1, \dots, y_N)$, $z = col(z_1, \dots, z_N)$, $\eta = col(\eta_1, \dots, \eta_N)$, $\mu = col(\mu_1, \dots, \mu_N)$, $\Psi = diag(\Psi_1, \dots, \Psi_N)$,$A = diag(A_1, \Upsilon = diag(\Upsilon_1, \dots, \Upsilon_N), \dots, A_N)$, $B = diag(B_1, \dots, B_N)$, $C = diag(C_1, \dots, C_N)$, $Z_N = diag(z_1^1, \dots, z_N^N)$, $g(y) = col(g_1(y_1), \dots, g_N(y_N)), \nabla f(y) = col(\nabla f_1(y_1), \dots, \nabla f_N(y_N)), \nabla g(y) = col(\nabla g_1(y_1), \dots, \nabla g_N(y_N))$, and $\bar{\mu} = (\mu + g(y))^+$. On the basis of the above definition, the compact form of the algorithm (6) can be written as

$$\begin{aligned} \dot{x} &= (A - B\Upsilon)x + B\Psi(-(Z_N^{-1} \otimes I_d)\nabla f(y) \\ &\quad - (\nabla g(y))^\top \bar{\mu} - \alpha(L \otimes I_d)y - (L \otimes I_d)\eta), \end{aligned} \tag{7a}$$

$$\dot{\eta} = \alpha(L \otimes I_d)y, \tag{7b}$$

$$y = Cx, \tag{7c}$$

$$\dot{z} = -(L \otimes I_N)z, \tag{7d}$$

$$\dot{\mu} = \bar{\mu} - \mu. \tag{7e}$$

Pre-multiplying (7a) by C, substituting Lemma 3 into the above system, then we can get the following result.

$$\begin{aligned}\dot{y} &= -(Z_N^{-1} \otimes I_d)\nabla f(y) - (\nabla g(y))^\top \bar{\mu} \\ &\quad - \alpha(L \otimes I_d)y - (L \otimes I_d)\eta, \\ \dot{\eta} &= \alpha(L \otimes I_d)y, \\ \dot{z} &= -(L \otimes I_N)z, \\ \dot{\mu} &= \bar{\mu} - \mu.\end{aligned} \tag{8}$$

Remark 3 In order to guarantee the existence of Z_N^{-1}, it is essential that the initial value $z(0)$ satisfies $z_i^i = 1$ for $i = 1, \ldots, N$ and $z_i^j = 0$ for all $i \neq j$. With Assumption 1, it can be readily obtained that $e^{-Lt}(t > 0)$ is a nonnegative matrix with positive diagonal entries. This implies that $z_i^i > 0$ for all $t \geq 0$. Therefore, there exists $(z_i^i)^{-1}$, that is, Z_N^{-1} is well-defined.

Next, the optimality condition is given in Lemma 4 which plays a crucial part in the following analysis.

Lemma 4 Let the Lagrangian function of problem (3) be $\mathbb{L}(y, \eta, \mu) = f(y) + \frac{1}{2}\|(\Xi^{\frac{1}{2}} \otimes I_d)\bar{\mu}\|^2 + y^\top(\tilde{L} \otimes I_d)\eta$, where $\eta \in \mathbb{R}^{Nd}$ and $\mu \in \mathbb{R}_+^{Nr}$ are Lagrange multipliers. Then, if Assumptions 1-4 hold, the point y^* is an optimal solution of the optimization problem (3) iff there exist Lagrangian multipliers $(\eta^*, \mu^*) \in \mathbb{R}^{Nd} \times \mathbb{R}_+^{Nr}$ such that the following KKT condition holds:

$$\begin{aligned}&\mathbf{0} = \nabla f(y^*) + (\mu^*)^\top(\Xi \otimes I_d)\nabla g(y^*) + (\tilde{L} \otimes I_d)\eta^*, \\ &\mu^* \geq 0, \ \ g(y^*) \leq 0, \ \ (\mu^*)^\top g(y^*) = 0.\end{aligned} \tag{9}$$

Proof 1 The second formula of (9) can be discussed in the following two cases.

(1) It follows from $\mu^* \geq 0$, $g(y^*) = 0$ that $(\mu^*)^\top g(y^*) = 0$, then we can get that $(\mu^* + g(y^*))^+ = \max(\mu^* + g(y^*), 0) = \mu^* + g(y^*) = \mu^*$.

(2) It follows from $\mu^* = 0$, $g(y^*) \leq 0$ that $(\mu^*)^\top g(y^*) = 0$, by similar discussion, we can also have that $(\mu^* + g(y^*))^+ = 0 = \mu^*$.

And vice versa.

Combined with the definition of $(\cdot)^+$ and premultiplying (9) by $(\Xi^{-1} \otimes I_d)$, (9) is equivalent to

$$\begin{aligned}\mathbf{0} &= (\Xi^{-1} \otimes I_d)\nabla f(y^*) + (\nabla g(y^*))^\top \mu^* \\ &\quad + (L \otimes I_d)\eta^*, \\ \mu^* &= (\mu^* + g(y^*))^+.\end{aligned} \tag{10}$$

The lemma below gives the relation between an optimal solution to the optimization problem (3) and an equilibrium point of the algorithm (8).

Lemma 5 Assume that Assumptions 1-4 hold. Given $z(0)$ in Remark 3 and the parameter $\alpha(\alpha > 0)$, if $(y^*, \eta^*, \mu^*, z^*)$ is the equilibrium point of the algorithm (8), then y^* is an optimal solution of the distributed constrained optimization problem (3).

Proof 2 Let $(y^*, \eta^*, \mu^*, z^*)$ is the equilibrium point of the algorithm (8), so

$$\begin{aligned}\mathbf{0} &= -((Z_N^{-1})^* \otimes I_d)\nabla f(y^*) - (\nabla g(y^*))^\top \bar{\mu}^* \\ &\quad - \alpha(L \otimes I_d)y^* - (L \otimes I_d)\eta^*, \\ \mathbf{0} &= \alpha(L \otimes I_d)y^*, \\ \mathbf{0} &= -(L \otimes I_N)z^*, \\ \mathbf{0} &= \bar{\mu}^* - \mu^*.\end{aligned} \tag{11}$$

It follows from [16], that $\lim_{t\to\infty} z = \lim_{t\to\infty} e^{-(L\otimes I_N)t}z(0) = (1_N\xi^\top \otimes I_N)z(0) = 1_N \otimes \xi$, which implies that $\lim_{t\to\infty} Z_N^{-1} = \Xi^{-1}$. Thus, $z^* = 1_N \otimes \xi$ and $(Z_N^{-1})^* = \Xi^{-1}$. Based on the above discussion and combined with the definition of $\bar{\mu} = (\mu + g(y))^+$, then we have

$$\mathbf{0} = -(\Xi^{-1} \otimes I_d)\nabla f(y^*) - (\nabla g(y^*))^\top \mu^* - (L \otimes I_d)\eta^*, \tag{12a}$$

$$\mathbf{0} = (L \otimes I_d)y^*, \tag{12b}$$

$$\bar{\mu}^* = \mu^* = (\mu^* + g(y^*))^+. \tag{12c}$$

It is obvious that (12b) generates $y^* = \mathbf{1}_N \otimes \tilde{y}$ with $\tilde{y} \in \mathbb{R}^d$. Furthermore, by comparing (12a), (12c) with (10), we can obtain that the equilibrium point of (8) satisfies the condition (10). By Lemma 4, we know that y^* is an optimal solution to the problem (3).

Then, Theorem 1 provides the proof of convergence.

Theorem 1 Under Assumptions 1-4, given $z(0)$ in Remark 3 and the parameter $\alpha(\alpha > 0)$, then the output variable $y(t)$ asymptotically converge to the optimal solution y^* of (3) for any initial value $(y(0), \eta(0), \mu(0)) \in \mathbb{R}^{Nd} \times \mathbb{R}^{Nd} \times \mathbb{R}_+^{Nr}$.

Proof 3 Let $\theta = col(y, \eta, \mu)$, it follows from (8) that θ satisfies

$$\dot{\theta} = f(\theta) + g(t, \theta) + v(t), \tag{13}$$

where

$$f(\theta) = \begin{pmatrix} f_1(\theta) \\ f_2(\theta) \\ f_3(\theta) \end{pmatrix}.$$

$f_1(\theta) = -(\Xi^{-1} \otimes I_d)\nabla f(y) - (\nabla g(y))^\top \bar{\mu} - \alpha(L \otimes I_d)y - (L \otimes I_d)\eta$, $f_2(\theta) = \alpha(L \otimes I_d)y$, $f_3(\theta) = \bar{\mu} - \mu$,

$$g(t,\theta)=\begin{pmatrix}((\Xi^{-1}-Z_N^{-1})\otimes I_d)(\nabla f(y)-\nabla f(y^*))\\ \mathbf{0}\\ \mathbf{0}\end{pmatrix},$$

$$v(t)=\begin{pmatrix}((\Xi^{-1}-Z_N^{-1})\otimes I_d)\nabla f(y^*)\\ \mathbf{0}\\ \mathbf{0}\end{pmatrix}.$$

First, consider the stability of the system:

$$\dot{\theta}=f(\theta). \tag{14}$$

Let $\theta^*=col(y^*,\eta^*,\mu^*)$ be an equilibrium point of system (14) and the Lyapunov candidate is defined as

$$V_1=\frac{1}{2}(y-y^*)^\top(\Xi\otimes I_d)(y-y^*), \tag{15}$$

Then we have

$$\begin{aligned}\dot{V}_1&=(y-y^*)^\top(\Xi\otimes I_d)(-(\Xi^{-1}\otimes I_d)\nabla f(y)\\&\quad-(\nabla g(y))^\top\bar{\mu}-\alpha(L\otimes I_d)y-(L\otimes I_d)\eta)\\&=-(y-y^*)^\top(\nabla f(y)-\nabla f(y^*))\\&\quad-(y-y^*)^\top(\Xi\otimes I_d)((\nabla g(y))^\top\bar{\mu}-(\nabla g(y^*))^\top\mu^*)\\&\quad-\frac{\alpha}{2}(y-y^*)^\top((\Xi L+L^\top\Xi)\otimes I_d)(y-y^*)\\&\quad-\frac{1}{2}(y-y^*)^\top((\Xi L+L^\top\Xi)\otimes I_d)(\eta-\eta^*)\\&=-(y-y^*)^\top(\nabla f(y)-\nabla f(y^*))\\&\quad-(y-y^*)^\top(\Xi\otimes I_d)((\nabla g(y))^\top\bar{\mu}-(\nabla g(y^*))^\top\mu^*)\\&\quad-\alpha(y-y^*)^\top(\tilde{L}\otimes I_d)(y-y^*)\\&\quad-(y-y^*)^\top(\tilde{L}\otimes I_d)(\eta-\eta^*),\end{aligned} \tag{16}$$

where $\tilde{L}=\frac{\Xi L+L^\top\Xi}{2}$ we have used in Lemma 2.

Next, define $V_2=\frac{1}{2}(\mu-\mu^*)^\top(\Xi\otimes I_d)(\mu-\mu^*)$. According to the definition of $(\cdot)^+$ and Lemma 1, we can conclude that

$$(\dot{\mu}+\mu-(\mu+g(y)))^\top(\mu^*-\dot{\mu}-\mu)\geq 0, \tag{17}$$

expand the equation (17), that is,

$$\dot{\mu}^\top(\mu^*-\mu)-\|\dot{\mu}\|^2-g(y)^\top(\mu^*-\bar{\mu})\geq 0, \tag{18}$$

pre-multiplying (7a) by $(\Xi\otimes I_d)$, it can be obtained that

$$\begin{aligned}&\dot{\mu}^\top(\Xi\otimes I_d)(\mu^*-\mu)-\|(\Xi\otimes I_d)^{\frac{1}{2}}\dot{\mu}\|^2\\&-g(y)^\top(\Xi\otimes I_d)(\mu^*-\bar{\mu})\geq 0,\end{aligned} \tag{19}$$

by combining (19) and the definition of V_2, it is simplified as

$$\dot{V}_2\leq-\|(\Xi\otimes I_d)^{\frac{1}{2}}\dot{\mu}\|^2-g(y)^\top(\Xi\otimes I_d)(\mu^*-\bar{\mu}). \tag{20}$$

Take $V_3=\frac{1}{2\alpha}(\eta-\eta^*)^\top(\Xi\otimes I_d)(\eta-\eta^*)$, where $\alpha>0$. Then the derivative of V_3 along with (14) is

$$\begin{aligned}\dot{V}_3&=\frac{1}{2\alpha}\cdot\alpha(y-y^*)^\top((\Xi L+L^\top\Xi)\otimes I_d)(\eta-\eta^*)\\&=(y-y^*)^\top(\tilde{L}\otimes I_d)(\eta-\eta^*).\end{aligned} \tag{21}$$

Let

$$V=V_1+V_2+V_3 \tag{22}$$

be a Lyapunov function candidate, obviously, which is positive semi-definitive. By utilizing (16),(20) and (21), and from (22), we can obtain that

$$\begin{aligned}\dot{V}&=-(y-y^*)^\top(\nabla f(y)-\nabla f(y^*))\\&\quad-\alpha(y-y^*)^\top(\tilde{L}\otimes I_d)(y-y^*)\\&\quad-\|(\Xi\otimes I_d)^{\frac{1}{2}}\dot{\mu}\|^2-g(y)^\top(\Xi\otimes I_d)(\mu^*-\bar{\mu})\\&\quad-(y-y^*)^\top(\Xi\otimes I_d)((\nabla g(y))^\top\bar{\mu}\\&\quad-(\nabla g(y^*))^\top\mu^*).\end{aligned} \tag{23}$$

Since the function f is strongly convex, we have

$$(y-y^*)^\top(\nabla f(y)-\nabla f(y^*))\geq m\|y-y^*\|^2. \tag{24}$$

Utilized the fact that the communication graph is a connected digraph, one can obtain that

$$\alpha(y-y^*)^\top(\tilde{L}\otimes I_d)(y-y^*)\geq\alpha\lambda_2(\tilde{L})\|y-y^*\|^2, \tag{25}$$

where $\lambda_2(\tilde{L})$ is the second smallest eigenvalue of $\tilde{L}$.

Next analyzing $\omega=-g(y)^\top(\Xi\otimes I_d)(\mu^*-\bar{\mu})-(y-y^*)^\top(\Xi\otimes I_d)((\nabla g(y))^\top\bar{\mu}-(\nabla g(y^*))^\top\mu^*)$.

By rearranging these terms, it follows that

$$\begin{aligned}\omega&=+(\mu^*)^\top(\Xi\otimes I_d)((\nabla g(y^*))^\top(y-y^*)-g(y))\\&\quad-\bar{\mu}^\top(\Xi\otimes I_d)((\nabla g(y))^\top(y-y^*)-g(y)).\end{aligned} \tag{26}$$

By adding and subtracting $g(y^*)$ in (26), one has

$$\begin{aligned}\omega&=(\bar{\mu}^*)^\top(\Xi\otimes I_d)((\nabla g(y^*))^\top(y-y^*)\\&\quad+g(y^*)-g(y))-(\bar{\mu}^*)^\top(\Xi\otimes I_d)g(y^*)\\&\quad-\bar{\mu}^\top(\Xi\otimes I_d)((\nabla g(y))^\top(y-y^*)\\&\quad+g(y^*)-g(y))+\bar{\mu}^\top(\Xi\otimes I_d)g(y^*)\end{aligned} \tag{27}$$

where the convex property of $g(y)$ and the nonnegative property of $\bar{\mu}$ and μ^* are used in the first and third term. It follows from Lemma 4 that y^* is an optimal solution to the optimization problem (3). This makes it easy to verify that $g(y^*)\leq 0$, $\bar{\mu}^\top(\Xi\otimes I_d)g(y^*)\leq 0$, and $(\bar{\mu}^*)^\top(\Xi\otimes I_d)g(y^*)=0$. Hence, what we can gain from the above analysis is that

$$\begin{aligned}\dot{V}&\leq-\|(\Xi\otimes I_d)^{\frac{1}{2}}\dot{\mu}\|^2-(m+\alpha\lambda_2)\|y-y^*\|^2\\&\leq 0.\end{aligned} \tag{28}$$

Due to $\dot{V}$ is continuous and negative for any $\alpha > 0$, $V(t)$ is bounded, then we can conclude that $y(t)$, $\mu(t)$ and $\eta(t)$ are bounded. This together with the Lipschitz condition of f_i and g_i, which imply that ∇f_i and ∇g_i are bounded, then $-(Z_N^{-1} \otimes I_d)\nabla f(y) - (\nabla g(y))^\top \bar{\mu} - \alpha(L \otimes I_d)y - (L \otimes I_d)\eta$ is also bounded, and denotes $\sigma, \sigma > 0$ as its upper bound. Furthermore, we have that $\|x(t)\| \le \|x_0\|e^{-t} + (1 - e^{-t})\sigma \le \|x_0\| + \sigma$ according to the expression of $\dot{x}(t)$, so $x(t)$ is also bounded. For the reason that V is radially unbounded, then by using LaSalle Invariance Principle, it can be found that $\lim_{t\to\infty} y_i(t) = y^*$, for $i = 1, \ldots, N$.

Next, we discuss the asymptotical stability of the terms $g(t, \theta)$ and $v(t)$.

From [16], there exist two positive constants ϱ_1 and ι_1 such that $\max|\xi_i^{-1} - (z_i^i)^{-1}| \le \varrho_1 e^{-\iota_1 t}$ holds. Thus, $g(t, \theta)$ satisfies the inequality

$$\begin{aligned} \|g(t,\theta)\| &= \|((\Xi^{-1} - Z_N^{-1}) \otimes I_d)(\nabla f(y) - \nabla f(y^*))\| \\ &\le \max|\xi_i^{-1} - (z_i^i)^{-1}| \cdot \|\nabla f(y) - \nabla f(y^*)\| \\ &\le M\varrho_1 e^{-\iota_1 t}\|y - y^*\|. \end{aligned} \tag{29}$$

$v(t)$ satisfies

$$\begin{aligned} \|v(t)\| &= \|((\Xi^{-1} - Z_N^{-1}) \otimes I_d)\nabla f(y^*)\| \\ &\le \max|\xi_i^{-1} - (z_i^i)^{-1}| \cdot \|\nabla f(y^*)\| \\ &\le \varrho_1 \|\nabla f(y^*)\| e^{-\iota_1 t}. \end{aligned} \tag{30}$$

In light of (29), (30), it is evident that $\lim_{t\to\infty} g(t,\theta) = \mathbf{0}$ and $\lim_{t\to\infty} v(t) = \mathbf{0}$. As already indicated above, one has $\theta(t)$ asymptotically converges to θ^* as $t \to \infty$, which means that the proposed algorithm is effective for solving the problem (3). This completes the proof.

5 Numerical Examples

Two numerical examples are provided to demonstrate the effectiveness of the presented algorithm in this section.

Example 1 We consider a heterogeneous multi-agent system consisting of six agents in $\mathbb{R}^2$ inspired by [23], where $A_1 = A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $A_3 = A_4 = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$, $A_5 = A_6 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$, $B_1 = B_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $B_3 = B_4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $B_5 = B_6 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $C_1 = C_2 = [1 \quad 1]$, $C_3 = C_4 = [3 \quad 1]$, $C_5 = C_6 = \begin{bmatrix} \frac{1}{2} & -1 & 0 \end{bmatrix}$. The local objective functions referred to [11] and the local convex inequality functions of six agents are given by

$$\begin{aligned} f_1(y_1) &= \frac{1}{2}y_{11}^2 + \frac{1}{2}y_{12}^2, \\ f_2(y_2) &= \frac{1}{2}(y_{21}+1)^2 + \frac{1}{2}y_{22}^2, \\ f_3(y_3) &= \frac{1}{2}y_{31}^2 + \frac{1}{2}(y_{32}+1)^2, \\ f_4(y_4) &= \frac{1}{2}(y_{41}+1)^2 + \frac{1}{2}(y_{42}+1)^2, \\ f_5(y_5) &= \frac{1}{4}y_{51}^4 + \frac{1}{4}y_{52}^4, \\ f_6(y_6) &= \frac{1}{4}(y_{61}+1)^4 + \frac{1}{4}y_{62}^4, \\ g_i(y_i) &= (y_1 - 2)^2 + (y_2 - 2)^2 - 5, i = 1, \ldots, 6. \end{aligned} \tag{31}$$

Obviously, all of the local objective functions and inequality functions are continuously differentiable. Due to the strong convexity of function $f(y) = \sum_{i=1}^N f_i(y_i)$, the global minimizer y^* is unique. And the minimum value $f(y) = 5.2347$ and the optimal solution $y^* = [0.2945, 0.5539]^\top$ can be acquired through some simple and easy calculations.

In addition, the feedback gain matrixes of each agent can be selected by $\gamma_1 = \gamma_2 = [-2 \quad -1]$, $\gamma_3 = \gamma_4 = [-1 \quad 2]$, $\gamma_5 = \gamma_6 = [1 \quad -1 \quad 2]$. $\Psi_1 = \Psi_2 = [-1]$, $\Psi_3 = \Psi_4 = \left[\frac{1}{2}\right]$, $\Psi_5 = \Psi_6 = [2]$. The initial states of $x_i(0), \eta_i(0) \in \mathbb{R}^2$ are chosen randomly and the initial value $\mu_i(0) = 0$ is set.

The communication network among these six agents is unbalanced and depicted in Figure 2 (Appendix) setting the weight of all edges as 1.

Let $\alpha_1 = 7$ and $\alpha_2 = 10$, the state trajectories of these agents by executing algorithm (6) are given in Figure 3(Appendix), it can be seen that the outputs of all agents converge uniformly to the global optimal solution y^* of optimization problem 2. Meanwhile, compared with the two subplots in Figure 3 (Appendix), we can see that, the curve corresponding to gain parameter $\alpha = 7$ converges around $t = 20$ s and the curve corresponding to $\alpha = 10$ converges around at the time $t = 15$ s, namely, the convergence rate is improved when α is increased from 7 to 10. This indicates that the convergence rate can be faster by choosing larger constant control parameters.

Example 2 In this example, we also consider a system consisting of six agents in $\mathbb{R}^2$. Assume that the coefficient matrixes and the feedback gain matrixes are the same as that chosen in Example 1, so is the communication graph. Each agent i has the following objective function $f_i, i = 1, \ldots, 6$ and convex inequality function $g_i, i = 1, \ldots, 6$.

$$\begin{aligned}
f_1(y_1) &= \frac{y_{11}^2}{\ln(y_{11}^2 + 6)}, \\
f_2(y_2) &= \frac{1}{2}(y_{21} + 1)^2 \ln(y_{11}^2 + 3) + y_{22}^2, \\
f_3(y_3) &= \frac{y_{31}^2}{\sqrt{y_{31}^2 + 1}} + \frac{1}{5}y_{32}^2, \\
f_4(y_4) &= y_{41}^4 + 2y_{42}^2 + 5, \\
f_5(y_5) &= \frac{1}{10}y_{51}^2 + e^{\frac{1}{10}y_{52}}, \\
f_6(y_6) &= \frac{1}{2}e^{-\frac{1}{2}y_{61}} + \frac{2}{5}e^{\frac{3}{10}y_{62}}, \\
g_i(y_i) &= (y_1 - 3)^2 + (y_2 - 5)^2 - 4, i = 1, \ldots, 6.
\end{aligned} \tag{32}$$

Besides, configuring the initial states of $x_i(0), \eta_i(0) \in \mathbb{R}^2$ randomly and setting the initial value to $\mu_i(0) = 0, \alpha_i(0) = 10$. Figure 4 (Appendix) depicts the evolution trajectories of the six agents by executing algorithm (6), and it is clear to observe that the output decision of each agent also effectively converges to the exact optimal value $y^* = [1.606, 3.566]^\top$.

6 Conclusion

To handle the distributed optimization problem with inequality constraints of heterogeneous linear multi-agent systems over weight-unbalanced digraphs, a distributed continuous-time optimization algorithm is presented. Additionally, a variable is designed to evaluate the left eigenvector associated with the zero eigenvalue of the Laplacian matrix, which removes the imbalance of the communication graph. When the local cost functions are assumed to be strongly convex with global Lipschitz gradients and the local constraint functions are assumed to be convex, by resorting to the Lyapunov stability, it is proved that the state of agents will asymptotically converge consensus at an optimal solution of the given optimization problem.

In the future, we will focus on the nonsmooth constrained optimization problems with communication delays and practical safety constraints of each agent against a collision.

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Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

APPENDIX

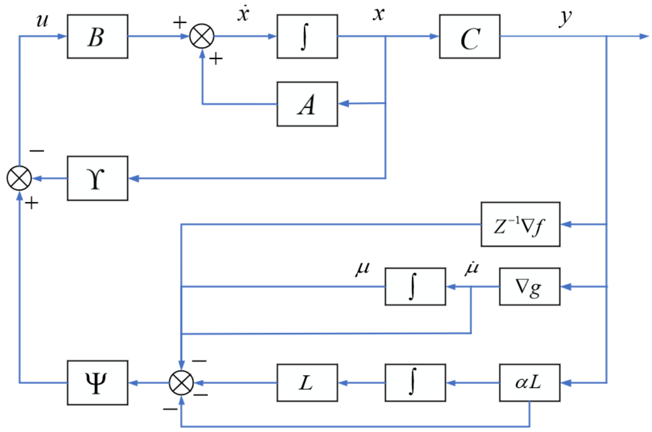

Figure 1: The controller to solve the optimization problem (3).

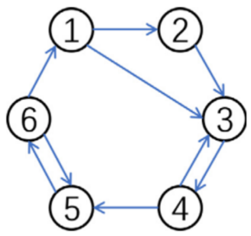

Figure 2: Communication graph among six agents.

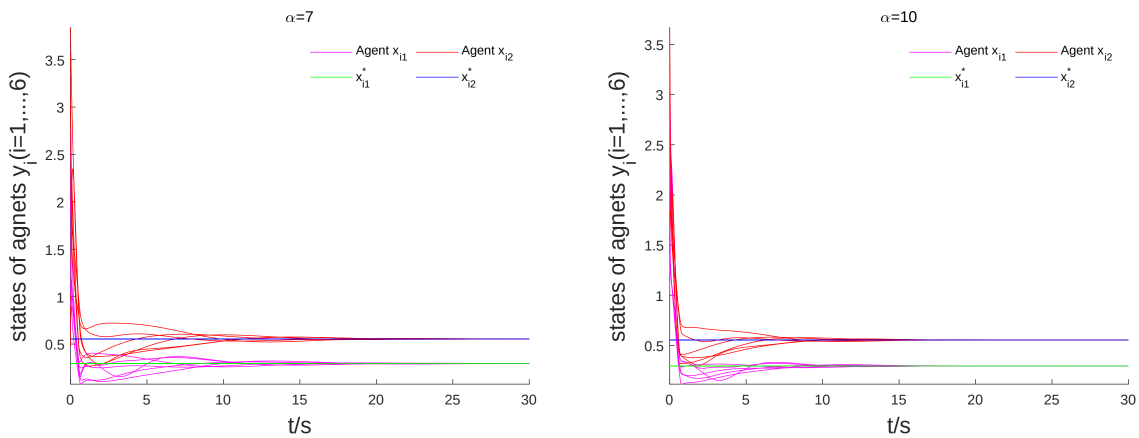

Figure 3: The state trajectories of output variable y_i with respect to time t for different values of α by executing algorithm (6).

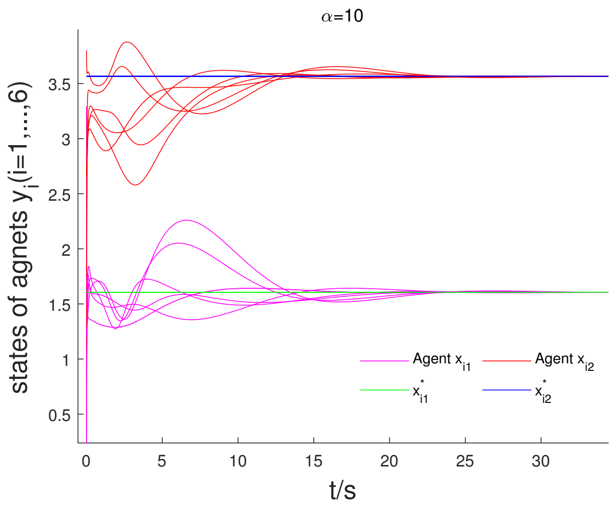

Figure 4: The state trajectories of output variable y_i respect to time t by executing algorithm (6).