Partial-Sum Matrix and its Rank

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Abstract: A partial-sum matrix is a matrix whose entries are partial sums of a seires associate with a sequence. The rank of a partial-sum matrix associate with any recurrence sequence can be related to the rank of an associate recurrence matrix, a matrix whose entries are from the same recurrence sequence. In particular, we find ranks of partial-sum matrices associated with arithmetic series and geometric series.

Key-Words: Partial Sum, Matrix, Rank, recurrence relation.

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1 Introduction

A matrix is one of the well-known tools that has been used to solve a system of linear equations in many areas, for example, economics, statistics, engineering, or computational science, e.g., see [1], [2], to find values of unknown varibales. The basic approach for finding a solution of a system of linear eqautions is a Gaussian elimination which is an algorithm to modify a corresponding augmented matrix of the system to be in a reduced row echelon form. However, a complication of coefficients of linear equations can lead to a difficulty in solving a system of the linear equations. In such cases, in stead of getting a solution, the direction is changed to determine only the existence of solutions. Finding a rank of an augmented matrix and a coefficient matrix then plays an important role in verifying whether the system has a solution. This shows that a rank of a matrix is an interesting topic to be studied. In particular, a rank of a matrix is one of fundamental characteristics of the matrix itself. Its importance leads to many research studies on rank of a matrix and its applications, e.g., see [3], [4], [5]. In this work, we are interested in finding a rank of some special matrices which can be associated with a system of linear equations with coefficients in the form of partial sums of series.

A partial-sum matrix is a matrix whose entries are partial sums of a series (read left-to-right, row-by-row). For example, the repunit sequence $\{1, 11, 111, ...\}$, which can be considered as a sequence of partial sums of a geometric series with the common ratio 10, forms a 4×3 partial-sum matrix as follows:

$$\begin{bmatrix} 1 & 11 & 111 \\ 1111 & 11111 & 111111 \\ 1111111 & 11111111 & 111111111 \\ 1111111111 & 11111111111 & 111111111111 \end{bmatrix}.$$

The rank of this partial-sum matrix is equal to 2. Generally speaking, it can be shown that any $m \times n$ partial-sum matrix of the repunit sequence is equal to 2 for any $m, n \geq 2$. In general, the rank of a partial-sum matrix of any recurrence sequence can be related to the rank of an associate recurrence matrix, a matrix whose entries are from the same recurrence sequence. Our approach is to apply row operations on a partial-sum matrix to determine its rank.

Definition 1.1. For a sequence (a_j) of complex numbers where $j \in \mathbb{N}$, a *partial sum S_j associated with* (a_j) is defined by $S_j = \sum_{i=1}^{j} a_i$ for any $j \in \mathbb{N}$. For $m, n \in \mathbb{N}$, we define an $m \times n$ *partial-sum matrix* of (a_j), written by $S_{mn}(a_j)$, to be the matrix

$$\begin{bmatrix} S_1 & S_2 & S_3 & \dots & S_n \\ S_{n+1} & S_{n+2} & S_{n+3} & \dots & S_{2n} \\ S_{2n+1} & S_{2n+2} & S_{2n+3} & \dots & S_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ S_{(m-1)n+1} & S_{(m-1)n+2} & S_{(m-1)n+3} & \dots & S_{mn} \end{bmatrix}.$$

To study the rank of a partial-sum matrix, we use the fact that the rank of any matrix is the same as the rank of its transpose, see [6] (p.114). Given a complex-valued sequence (a_j), the transpose of an $m \times n$ partial-sum matrix is denoted by $S^T_{mn}((a_j))$ such that

$$S^T_{mn}((a_j)) = \begin{bmatrix} S_1 & S_{n+1} & S_{2n+1} & \dots & S_{(m-1)n+1} \\ S_2 & S_{n+2} & S_{2n+2} & \dots & S_{(m-1)n+2} \\ S_3 & S_{n+3} & S_{2n+3} & \dots & S_{(m-1)n+3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ S_n & S_{2n} & S_{3n} & \dots & S_{mn} \end{bmatrix}.$$

Since $S_{j+1} = S_j + a_{j+1}$, we apply the row oporations by adding each of the j^{th} row to the multiple of the $(j+1)^{th}$ row (multiplied by -1) starting from $j = n-1, n-2, n-3, ..., 1$, respectively to obtain that $S_{mn}^T((a_j))$ is row equivalent to

$$\begin{bmatrix} S_1 & S_{n+1} & S_{2n+1} & \cdots & S_{(m-1)n+1} \\ a_2 & a_{n+2} & a_{2n+2} & \cdots & a_{(m-1)n+2} \\ a_3 & a_{n+3} & a_{2n+3} & \cdots & a_{(m-1)n+3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_n & a_{2n} & a_{3n} & \cdots & a_{mn} \end{bmatrix}. \tag{1.1}$$

Let $A_{mn}((a_j))$ be the matrix

$$\begin{bmatrix} a_2 & a_3 & a_4 & \cdots & a_n \\ a_{n+2} & a_{n+3} & a_{n+4} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(m-1)n+2} & a_{(m-1)n+3} & a_{(m-1)n+4} & \cdots & a_{mn} \end{bmatrix}$$

and let $S_1((a_j))$ denote the matrix

$$\begin{bmatrix} S_1 & S_{n+1} & S_{2n+1} & \cdots & S_{(m-1)n+1} \end{bmatrix}.$$

The row-reduced matrix in (1.1) implies that the rank of $S_{mn}((a_j))$ is the same as the rank of $\begin{bmatrix} S_1((a_j)) \\ A_{mn}^T((a_j)) \end{bmatrix}$.

In particular, if the row of $S_1((a_j))$ is linearly independent to all rows of $A_{mn}^T((a_j))$, then

$$\begin{aligned} \text{rank } S_{mn}((a_j)) &= \text{rank } A_{mn}^T((a_j)) + 1 \\ &= \text{rank } A_{mn}((a_j)) + 1. \end{aligned} \tag{1.2}$$

Then finding the rank of $A_{mn}((a_j))$ becomes the key to our goal. However, the matrix $A_{mn}((a_j))$ is related to the matrix of the sequence (a_j) , as defined in [7] and [8], for which we shall explain.

Definition 1.2. For a sequence (a_j) of complex numbers where $j \in \mathbb{N}$ and $m, n \in \mathbb{N}$, we define an $m \times n$ *matrix of the sequence* (a_j), denoted by $M_{mn}(a_j)$, to be the matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_{n+1} & a_{n+2} & a_{n+3} & \cdots & a_{2n} \\ a_{2n+1} & a_{2n+2} & a_{2n+3} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(m-1)n+1} & a_{(m-1)n+2} & a_{(m-1)n+3} & \cdots & a_{mn} \end{bmatrix}.$$

For any $k = 1, 2, 3, ..., n$, let $C_k((a_j))$ be the matrix $\begin{bmatrix} a_k \\ a_{n+k} \\ a_{2n+k} \\ \vdots \\ a_{(m-1)n+k} \end{bmatrix}$. Then we rewrite $M_{mn}((a_j))$ to be $M_{mn}((a_j)) = [C_1((a_j)) \quad A_{mn}((a_j))]$ where
$A_{mn}((a_j)) = [C_2((a_j)) \quad C_3((a_j)) \quad ... \quad C_n((a_j))]$. Clearly, rank($M_{mn}((a_j))$) is the same as rank $[A_{mn}((a_j)) \quad C_1((a_j))]$. Moreover, we can consider $[A_{mn}((a_j)) \quad C_1((a_j))]$ as the augmented matrix of the system of the following linear equations

$$\begin{aligned} a_2x_2 + a_3x_3 + \cdots + a_nx_n &= a_1 \\ a_{n+2}x_2 + a_{n+3}x_3 + \cdots + a_{2n}x_n &= a_{n+1} \\ &\vdots \\ a_{(m-1)n+2}x_2 + a_{(m-1)n+3}x_3 + \cdots + a_{mn}x_n & \\ &= a_{(m-1)n+1}. \end{aligned} \tag{1.3}$$

Notice that the system (1.3) has a solution if and only if the rank of $[A_{mn}((a_j)) \quad C_1((a_j))]$ is the same as the rank of $A_{mn}((a_j))$. That is, the system (1.3) has a solution if and only if $M_{mn}((a_j))$ has the same rank as $A_{mn}((a_j))$.

From the above result and (1.2) , if $S_1((a_j))$ is linearly independent to $C_2((a_j)), C_3((a_j)), ..., C_n((a_j))$ and the system (1.3) has a solution, then

$$\text{rank } S_{mn}((a_j)) = \text{rank } M_{mn}((a_j)) + 1. \tag{1.4}$$

We will apply (1.4) to study the rank of some partial-sum matrices associated with special series in the next section.

2 Rank of partial-sum matrices in some types

2.1 Partial-sum matrices of arithmetic series

Let (a_j) be an arithmetic sequence with an initial value a_1 and a nonzero common difference d. We have that $a_{j+k} = a_k + (j-1)d$ for all $j, k \in \mathbb{N}$. The system (1.3) can be written in the following system

$$
\begin{aligned}
&a_1(x_2 + x_3 + \cdots + x_n) \\
&\ + d(x_2 + 2x_3 + \cdots + (n-1)x_n) = a_1 \\
&a_{n+1}(x_2 + x_3 + \cdots + x_n) \\
&\ + d(x_2 + 2x_3 + \cdots + (n-1)x_n) = a_{n+1} \\
&\ \vdots \\
&a_{(m-1)n+1}(x_2 + x_3 + \cdots + x_n) \\
&\ + d(x_2 + 2x_3 + \cdots + (n-1)x_n) = a_{(m-1)n+1}
\end{aligned} \tag{2.1}
$$

which is equivalent to the system

$$
\begin{aligned}
x_2 + x_3 + \cdots + x_n &= 1 \\
x_2 + 2x_3 + \cdots + (n-1)x_n &= 0.
\end{aligned} \tag{2.2}
$$

The system (2.2) is a consistent system. Therefore, (2.1) is also consistent. We can conclude that rank $A_{mn}((a_j))$ = rank $M_{mn}((a_j))$. C. Lee and V. Peterson showed in [8] that rank $M_{mn}((a_j)) = 2$ when $m, n \geq 2$, and so rank $A_{mn}((a_j)) = 2$ when $m, n \geq 2$. Next, we will show that the row of $S_1((a_j))$ is linearly independent to all rows of $A^T_{mn}((a_j))$.
In this case, $A_{mn}((a_j))$ is column equivalent to

$$
\begin{bmatrix}
a_1 + d & d & 0 & \ldots & 0 \\
a_1 + (n+1)d & d & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \\
a_1 + ((m-1)n+1)d & d & 0 & \ldots & 0
\end{bmatrix}.
$$

We have that the partial sum is $S_k = ka_1 + \frac{k(k-1)}{2}d$ for any k. Then $S_1((a_j))$ is linearly independent to both
$\left[\, a_1+d \;\; a_1+(n+1)d \;\; \ldots \;\; a_1+((m-1)n+1)d \,\right]$ and

$$
\begin{bmatrix} d & d & d & \ldots & d \end{bmatrix}.
$$

As a result, rank $S_{mn}((a_j))$ = rank $M_{mn}((a_j)) + 1 = 3$ when $m, n \geq 2$. We have proved the following theorem.

Theorem 2.1. *Let (a_j) be an arithmetic sequence with an initial value a_1 and a nonzero common difference d. Then a partial-sum matrix $S_{mn}((a_j))$ with $m, n \geq 3$ has rank 3.*

2.2 Partial-sum matrices of geometric series

Let (a_j) be a geometric sequence with a nonzero initial value a_1 and a nonzero common ratio r. Accordingly, $a_{j+k} = r^j a_k$ for all $j, k \in \mathbb{N}$. We rewrite the system (1.3) to be

$$
\begin{aligned}
&ra_1x_2 + r^2a_1x_3 + \cdots + r^{n-1}a_1x_n \\
&\ = a_1 \\
&ra_{n+1}x_2 + r^2a_{n+1}x_3 + \cdots + r^{n-1}a_{n+1}x_n \\
&\ = a_{n+1} \\
&\ \vdots \\
&ra_{(m-1)n+1}x_2 + r^2a_{(m-1)n+1}x_3 + \cdots \\
&\ + r^{n-1}a_{(m-1)n+1}x_n = a_{(m-1)n+1}
\end{aligned}
$$

which is equivalent to

$$
rx_2 + r^2x_3 + \cdots + r^{n-1}x_n = 1.
$$

The latter equation always has a solution, for example, let $x_2 = \frac{1}{r}, x_3 = x_4 = \cdots = x_n = 0$. By the same reason as in the case of an arithmetic sequence, we derive that $S_{mn}((a_j))$ = rank $M_{mn}((a_j)) + 1 = 2$, referring to [8] that rank $M_{mn}((a_j)) = 1$ when $m, n \geq 2$ for a geometric sequence (a_j). The following theorem is now proved.

Theorem 2.2. *Let (a_j) be a geometric sequence with a nonzero initial value a_1 and a nonzero common ratio r. Then a partial-sum matrix $S_{mn}((a_j))$ with $m, n \geq 2$ has rank 2.*

2.3 Partial-sum matrices of linear recurrence relations

Let (a_j) be a homogeneous linear recurrence sequence of order k such that for all $j \geq k$

$$
a_j = \alpha_1 a_{j-1} + \alpha_2 a_{j-2} + \alpha_3 a_{j-3} + \cdots + \alpha_k a_{j-k} \tag{2.3}
$$

with initial values $a_1, a_2, \cdots, a_k$ and $\alpha_1, \alpha_2, \cdots, \alpha_k$ where $\alpha_k \neq 0$. An geometric sequence is an example of a homogeneous linear recurrence sequence of order 1 because $a_j = ra_{j-1}$. Since $\alpha_k \neq 0$, we arrange (2.3) to be that for any $j \in \mathbb{N}$

$$
\begin{aligned}
a_j = \ & \frac{1}{\alpha_k}a_{j+k} - \frac{\alpha_1}{\alpha_k}a_{j+k-1} - \frac{\alpha_2}{\alpha_k}a_{j+k-2} - \cdots \\
& - \frac{\alpha_{k-1}}{\alpha_k}a_{j+1}.
\end{aligned} \tag{2.4}
$$

Let $m, n \geq k+1$. We derive the following equations afterward.

$$\begin{aligned}
a_1 = & \frac{1}{\alpha_k}a_{k+1} - \frac{\alpha_1}{\alpha_k}a_k - \frac{\alpha_2}{\alpha_k}a_{k-1} - \cdots - \\
& \frac{\alpha_{k-1}}{\alpha_k}a_2 \\
a_{n+1} = & \frac{1}{\alpha_k}a_{n+1+k} - \frac{\alpha_1}{\alpha_k}a_{n+k} - \frac{\alpha_2}{\alpha_k}a_{n+k-1} - \\
& \cdots - \frac{\alpha_{k-1}}{\alpha_k}a_{n+2} \\
& \vdots \\
a_{(m-1)n+1} = & \frac{1}{\alpha_k}a_{(m-1)n+1+k} - \frac{\alpha_1}{\alpha_k}a_{(m-1)n+k} - \\
& \frac{\alpha_2}{\alpha_k}a_{(m-1)n+k-1} - \cdots - \\
& \frac{\alpha_{k-1}}{\alpha_k}a_{(m-1)n+2}.
\end{aligned}$$

Therefore, the system (1.3) has a solution, that is, rank $M_{mn}((a_j))$ = rank $A_{mn}((a_j))$. In this linear recurrence sequence, the row R_1 of $S_{mn}((a_j))$ is linearly independent to all rows of $A^T_{mn}((a_j))$, and so $S_{mn}((a_j))$ = rank $M_{mn}((a_j)) + 1$. Here $M_{mn}((a_j))$ must have a rank drop. To determine the rank of $M_{mn}((a_j))$, we shall recall the occurrence of a rank drop in a recurrence matrix, which is first mentioned in [8] and further studied in[7].

2.3.1 Order rank drops

The order rank drops in $M_{mn}((a_j))$ occurs when (a_j) also satisfies a recurrence relation with order less than k. For example, the recurrence sequence $a_j = 4a_{j-1} - 3a_{j-2}$ with $a_1 = 1$ and $a_2 = 3$. Thus $a_j = 3^{j-1}$, and hence $a_j = 3a_{j-1}$. By Theorem 2.2, rank $S_{mn}((a_j)) = 2$ since $M_{mn}((a_j)) = 1$. This means that $S_{mn}((a_j))$ has a rank drop if $m, n \geq 3$. We also call that $S_{mn}((a_j))$ has an *order rank drop*. To clarify the order of a recurrence sequence in this case, let *the minimal order of* (a_j) be the smallest order satisfied by (a_j).

2.3.2 Width rank drops

Let (a_j) be a recurrence sequence satisfying $a_j = a_{j-3}$ with $a_1 = 1, a_2 = 0, a_3 = 0$.
Then $(a_j) = (1,0,0,1,0,0,1,0,0,\dots)$ and $(S_j) = (1,1,1,2,2,2,3,3,3,\dots)$. However, both

$$S_{44}((a_j)) = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & 4 & 4 & 4 \\ 5 & 5 & 5 & 6 \end{bmatrix} \text{ and } S_{55}((a_j)) = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 \\ 2 & 3 & 3 & 3 & 4 \\ 4 & 4 & 5 & 5 & 5 \\ 6 & 6 & 6 & 7 & 7 \\ 7 & 8 & 8 & 8 & 9 \end{bmatrix}$$ have rank 4. The difference between these two matrix is that $S_{44}((a_j))$ does not have a drop in rank whereas $S_{55}((a_j))$ does. The rank drop depending on the width (or the number of columns) of a matrix is called a *width rank drop*.

From the two cases of rank drops in recurrence matrices, S. Bozlee derives the exact rank of a recurrence matrix by using a characteristic polynomial of a recurrence relation. For more details, readers may consult [7].

Theorem 2.3. *[7] Let (a_j) be a recurrence sequence with the minimal order k and q distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_q$ with multiplicities $k_1, k_2, \dots, k_q$, respectively. If $m, n \geq k$, let Λ_n be the set of all distinct values taken by $\lambda_1^n, \lambda_2^n, \dots, \lambda_q^n$. Then*

$$\textit{rank } M_{mn}((a_j)) = \sum_{a \in \Lambda_n} \max\{k_l \;:\; \lambda_l = a\}$$

Then the rank of $S_{mn}((a_j))$ can be derived consequently.

Corollary 2.4. *Let (a_j) be a homogeneous recurrence sequence with the minimal order k and q distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_q$ with multiplicities $k_1, k_2, \dots, k_q$, respectively. If $m, n \geq k+1$, let Λ_n be the set of all distinct values taken by $\lambda_1^n, \lambda_2^n, \dots, \lambda_q^n$. Then*

$$\textit{rank } S_{mn}((a_j)) = \sum_{a \in \Lambda_n} \max\{k_l \;:\; \lambda_l = a\} + 1$$

In particular, Theorem 5.1 in [7] shows that if (a_j) is a homogeneous linear recurrence sequence of order two such that $a_j = \alpha a_{j-1} + \beta a_{j-2}$ with given initial values a_1 and a_2, then for $m, n \geq 2$,

rank $M_{mn}((a_j)) =$

$$\begin{cases} 0 & \text{if } a_1 = a_2 = 0, \\ 1 & \text{if } a_2^2 - \alpha a_1 a_2 - \beta a_1^2 = 0, \\ 1 & \text{if } \alpha^2 + 4c_2^2 \neq 0 \text{ and } \left(\frac{\alpha + \sqrt{\alpha^2 + 4\beta}}{\alpha - \sqrt{\alpha^2 + 4\beta}}\right)^n = 1, \\ 2 & \text{otherwise} \end{cases}$$

and hence, for $m, n \geq 3$, rank $S_{mn}((a_j)) =$

$$\begin{cases} 0 & \text{if } a_1 = a_2 = 0, \\ 2 & \text{if } a_2^2 - \alpha a_1 a_2 - \beta a_1^2 = 0, \\ 2 & \text{if } \alpha^2 + 4\beta^2 \neq 0 \text{ and } \left(\frac{\alpha + \sqrt{\alpha^2 + 4\beta}}{\alpha - \sqrt{\alpha^2 + 4\beta}}\right)^n = 1, \\ 3 & \text{otherwise.} \end{cases} \tag{2.5}$$

The Fibonacci sequence $(1, 1, 2, 3, 5, \dots)$ is homogeneous linear recurrence sequence of order two satisfying $F_j = F_{j-1} + F_{j-2}$ with $a_1 = 1$ and $a_2 = 1$. This sequence does not satisfy the first three conditions in (2.5). That is, an $m \times n$ partial-sum matrix of the Fibonacci sequence has rank 3 whenever $m, n \geq 3$. Clearly, the $m \times 1$ and $1 \times n$ partial-sum matrices of the Fibonacci sequence has rank 1. For the remaining case, it is easy to see that $m \times 2$ and $2 \times m$ partial-sum matrices of the Fibonacci sequence for $m \geq 2$ has rank 2, e.g., see $S_{24}((a_j)) = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 5 & 8 & 13 & 21 \end{bmatrix}$.

In the case of an inhomogeneous recurrence relation (a_j) of order k such that for all $j \geq k$

$$a_j = \alpha_1 a_{j-1} + \alpha_2 a_{j-2} + \alpha_3 a_{j-3} + \cdots + \alpha_k a_{j-k} \tag{2.6}$$

with initial values $a_1, a_2, \cdots, a_k$ and $\alpha_1, \alpha_2, \cdots, \alpha_k$ with $\alpha_k \neq 0$.
We can rewrite (2.6) to be the homogeneous recurrence relation of order $k + 1$

$$\begin{aligned} a_{j+1} &= (\alpha_1 + 1)a_j + (\alpha_2 - \alpha_1)a_{j-1} + \cdots + \\ &\quad (\alpha_k - \alpha_{k-1})a_{j-k+1} + \alpha_k a_{j-k}. \end{aligned}$$

Therefore, one can also derive the rank of $S_{mn}((a_j))$ by referring to the homogeneous case. We shall leave it to reader to work on the details.

3 Conclusion

We have defined a partial-sum matrix and provide the rank of this type of matrices of some special cases. The process we have applied is from Linear Algebra. However, in analysis, the sequence of partial sums of a sequence will lead to a series, then we may question whether a sequence of $m \times n$ partial-sum matrices (for either fixed m or n) can be related to the associated series to the partial sums in some ways. This can be an open problem for further study.

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