Notes on subsemigroups of $\mathbb{N}_0^m(+)$

BARBORA BATÍKOVÁ
Faculty of Engineering
Czech University of Life Sciences
Kamýcká 129 165 21 Praha 6 - Suchdol
CZECH REPUBLIC

ANTONÍN JANČAŘÍK
Faculty of Education
Charles University
M. Rettigove 4 116 39 Praha 1
CZECH REPUBLIC

PETR NĚMEC
Faculty of Engineering
Czech University of Life Sciences
Kamýcká 129 165 21 Praha 6 - Suchdol
CZECH REPUBLIC

TOMÁŠ KEPKA
Faculty of Education
Charles University
M. Rettigove 4 116 39 Praha 1
CZECH REPUBLIC

Abstract: The paper summarizes and extends the knowledge of various subsemigroups of $\mathbb{N}_0^m(+)$ $(= \mathbb{N}_0(+)^m)$. It creates a theoretical basis for further study in this area and applications in other areas, such as the investigation of context-free languages. The last chapter introduces the notion of pure subsemigroups and presents one construction of a pure subsemigroup to a chosen semisubgroup of $\mathbb{N}_0^m(+)$.

Key-Words: integer, semigroup, pure, $\mathbb{N}_0^m(+)$, subsemigroups, pure subsemigroups

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1 Introduction

This note presents some results on various subsemigroups of $\mathbb{N}_0^m(+)$ $(= \mathbb{N}_0(+)^m)$. The results showcased in this article may encompass both new findings and components of previously published results. However, consolidating them into a singular output is a theoretical basis for further research.

The motivation for investigating such problems stems from several parts of mathematics. Holes in subsemigroups of $\mathbb{N}_0^m(+)$ (if A is a subsemigroup of $\mathbb{N}_0^m(+)$ then an element h of the pure subsemigroup generated by A is said to be a hole in A if $h \notin A$) were applied to transportation problems in [1]. There also is a close connection with the theory of semirings. For example, the problem whether every commutative parasemifield (i.e., an algebraic structure with two commutative and associative binary operations such that the multiplication is a group and distributes over addition) finitely generated as a semiring is additively idempotent was in [2] affirmatively answered in 2-generated case by transferring the problem to subsemigroups of $\mathbb{N}_0^m(+)$ with special properties; this method was further developed in [3], [4]. It also seems that the investigation of finitely generated cones could be useful in the investigation of context-free languages (see e.g. [5], [6], [7]).

2 Preliminaries (a)

The following notation will be used throughout the note:

$\mathbb{N}$ ... the semiring of positive integers;
$\mathbb{N}_0$ ... the semiring of non-negative integers;
$\mathbb{Z}$ ... the ring of integers;
$\mathbb{Q}^+$ ... the parasemifield of positive rationals;
$\mathbb{Q}_0^+$ ... the semifield of non-negative rationals;
$\mathbb{R}_0^+$ ... the semifield of non-negative reals.

Throughout this section, let $m \in \mathbb{N}$ and let A be a subset of $\mathbb{Q}^m$ (the ring of ordered m-tuples of rationals, where the operations of addition and multiplication are defined componentwise), where we denote $0 = (0, \dots, 0) \in \mathbb{Q}^m$ (no confusion can arise). We put

$\mathrm{conv(A)} =$
$= \{ \sum_{i=1}^n q_i a_i \mid n \in \mathbb{N}, a_i \in A, q_i \in \mathbb{Q}^+, \sum q_i = 1 \}$
and $\mathrm{cone(A)} = \{ \sum_{i=1}^n q_i a_i \mid n \in \mathbb{N}, a_i \in A, q_i \in \mathbb{Q}_0^+ \}$.

Lemma 1.1. (i) $A \subseteq \mathrm{conv(A)}$.
(ii) $\mathrm{conv(conv(A))} = \mathrm{conv(A)}$.
(iii) *If* $A \subseteq (\mathbb{Q}_0^+)^m$ *then* $\mathrm{conv(A)} \subseteq (\mathbb{Q}_0^+)^m$ *and, moreover,* $0 \in \mathrm{conv(A)}$ *if and only if* $0 \in A$.
(iv) *If* $A \subseteq (\mathbb{Q}^+)^m$ *then* $\mathrm{conv(A)} \subseteq (\mathbb{Q}^+)^m$ *and* $0 \notin \mathrm{conv(A)}$.

Proof. The assertions (i), (iii) and (iv) are quite easy. As concerns (ii), take $a \in \mathrm{conv(conv(A))}$. Then $a = \sum_{i=1}^n q_i a_i$, $n \in \mathbb{N}$, $a_i \in \mathrm{conv(A)}$, $q_i \in \mathbb{Q}^+$ and $\sum_i q_i = 1$. Furthermore, $a_i = \sum_{j=1}^{k_i} q_{ij} a_{ij}$, $k_i \in \mathbb{N}$, $a_{ij} \in A$, $q_{ij} \in \mathbb{Q}^+$ and $\sum_j q_{ij} = 1$, and hence $a = \sum_i \sum_j q_i q_{ij} a_{ij}$. However, $\sum_i \sum_j q_i q_{ij} = \sum_i (q_i \sum_j q_{ij}) = \sum_i q_i = 1$. Thus $a \in \mathrm{conv(A)}$. □

Lemma 1.2. (i) $A \subseteq \mathrm{conv(A)} \subseteq \mathrm{cone(A)} = \mathrm{conv(B)}$, *where* $B = \{ qa \mid a \in A, q \in \mathbb{Q}_0^+ \}$.
(ii) *If* $A \neq \emptyset$ *then* $0 \in \mathrm{cone(A)}$.
(iii) $\mathrm{cone(cone(A))} = \mathrm{conv(cone(A))} = \mathrm{cone(conv(A))} = \mathrm{cone(A)}$.
(iv) *If* $A \subseteq (\mathbb{Q}_0^+)^m$ *then* $\mathrm{cone(A)} \subseteq (\mathbb{Q}_0^+)^m$.

Proof. It is easy. □

Lemma 1.3. $\mathrm{cone(A)} = \mathrm{A}$ *if and only if* $\mathrm{conv(A)} = \mathrm{A}$ *and* $qA \subseteq A$ *for every* $q \in \mathbb{Q}_0^+$.

Proof. It is easy. □

Lemma 1.4. *If* $A \neq \emptyset$ *then* $\mathrm{cone(A)}$ *is just the subsemimodule of the* $\mathbb{Q}_0^+$*-semimodule* $\mathbb{Q}^m$ *generated by the set* A.

Proof. It is easy. □

Lemma 1.5. $\mathrm{cone(A)} = \mathrm{A}$ *if and only if either* $A = \emptyset$ *or* A *is a subsemimodule of the* $\mathbb{Q}_0^+$*-semimodule* $\mathbb{Q}^m$.

Proof. See 1.3 and 1.4. □

Lemma 1.6. *Let* $n \in \mathbb{N}$, $n \geq m+2$, $q_1, \dots, q_n \in \mathbb{Q}_0^+$, $a_1, \dots, a_n \in \mathbb{Q}^m$ *and* $a = \sum q_i a_i$. *Then there are* $r_1, \dots, r_n \in \mathbb{Q}_0^+$ *such that* $a = \sum r_i a_i$ *and* $r_{i_1} = 0$ *for at least one* i_1, $1 \leq i_1 \leq n$.

Proof. Clearly, there are $s_1, \dots, s_{n-1} \in \mathbb{Q}$ such that $\sum_{i=1}^{n-1} s_i(a_i - a_n) = 0$ and $s_{i_0} \neq 0$ for at least one i_0, $1 \leq i_0 \leq n-1$. Then $\sum_{i=1}^n s_i a_i = 0$, where $s_n = -s_1 - \dots - s_{n-1}$, and we have $\sum s_i = 0$. Consequently, the set $J = \{ j \mid s_j > 0 \}$ is a non-empty subset of the set $\{1, \dots, n\}$ and we find $j_0 \in J$ such that $t = q_{j_0}/s_{j_0} \leq q_j/s_j$ for every $j \in J$. Clearly, $t \in \mathbb{Q}_0^+$. Now, put $r_i = q_i - t s_i$ for all $i = 1, \dots, n$. Then $r_i \in \mathbb{Q}_0^+$ and $r_{j_0} = 0$. Finally, $\sum r_i a_i = a$. □

Lemma 1.7. $\mathrm{cone(A)} = \bigcup \mathrm{cone(B)}$, $B \subseteq A$, $|B| \leq m+1$.

Proof. Let $a = \sum_{i=1}^n q_i a_i$, $n \in \mathbb{N}$, $q_i \in \mathbb{Q}_0^+$, $a_i \in A$. If n is the smallest possible number with this property then $n \leq m+1$ by 1.6. The rest is clear. □

Lemma 1.8. $\mathrm{cone(A)} = \{ \sum_{i=1}^{m+1} q_i a_i \mid a_i \in A, q_i \in \mathbb{Q}_0^+ \}$.

Proof. See 1.7. □

Lemma 1.9. *Assume that* A *is a subsemigroup of* $\mathbb{Q}^m(+)$ *that is generated by a non-empty set* B. *Then* $\mathrm{cone(A)} = \mathrm{cone(B)}$.

Proof. Since $B \subseteq A$, we have $\mathrm{cone(B)} \subseteq \mathrm{cone(A)}$. On the other hand, $\mathrm{cone(B)}$ is a subsemigroup of $\mathbb{Q}^m(+)$ by 1.4, and hence $A \subseteq \mathrm{cone(B)}$. Consequently, $cone(A) \subseteq \mathrm{cone(cone(B))} = \mathrm{cone(B)}$ by 1.2(iii). □

Lemma 1.10. *Assume that* A *is a subsemigroup of* $\mathbb{Q}^m(+)$. *Then:*
(i) *For every* $a \in \mathrm{cone(A)}$, $a \neq 0$, *there is* $k \in \mathbb{N}$ *with* $ka \in A$ *(*$\subseteq \mathrm{conv(A)}$*).*
(ii) *Either* $0 \notin A$ *and* $\mathrm{cone(A)} = \{0\} \cup \bigcup_{k \in \mathbb{N}} \mathrm{A/k}$ *or* $0 \in A$ *and* $\mathrm{cone(A)} = \bigcup_{k \in \mathbb{N}} \mathrm{A/k}$.

Proof. We have $a = \sum_{i=1}^n q_i a_i$, $n \in \mathbb{N}$, $a_i \in A$, $q_i \in \mathbb{Q}^+$. If $k \in \mathbb{N}$ is such that $kq_i \in \mathbb{N}$ for every i then $ka = \sum kq_i a_i \in A$. □

Lemma 1.11. *If* A *is a subsemigroup of* $\mathbb{Q}^m(+)$ *then* $\mathrm{cone(A)} = \{0\} \cup \bigcup_{k \in \mathbb{N}} \mathrm{conv(A/k)}$.

Proof. See 1.10. □

Lemma 1.12. *Assume that* A *is a subsemimodule of the* $\mathbb{Q}^+$*-semimodule* $\mathbb{Q}^m$. *Then:*
(i) *If* $0 \in A$ *then* $\mathrm{conv(A)} = \mathrm{cone(A)} = \mathrm{A}$.
(ii) *If* $0 \notin A$ *then* $\mathrm{conv(A)} = \mathrm{A}$ *and* $\mathrm{cone(A)} = \mathrm{A} \cup \{0\}$.

Proof. It is easy. □

Lemma 1.13. *If* $\mathrm{cone(A)} = \mathbb{Q}^{\mathrm{m}}$ *then* $0 \in \mathrm{conv(A)}$.

Proof. There are $n \in \mathbb{N}$, $a_i \in A$ and $q_i, r_i \in \mathbb{Q}_0^+$ such that $(-1, 0, \dots, 0) = \sum q_i a_i$, $(1, 0, \dots, 0) = \sum r_i a_i$, $q = \sum q_i \in \mathbb{Q}^+$ and $r = \sum r_i \in \mathbb{Q}^+$. Now, $0 = \sum (q_i + r_i) a_i$, and so $\sum \frac{q_i + r_i}{q + r} a_i = 0$ as well. But $\sum \frac{q_i + r_i}{q + r} = 1$ and it follows easily that $0 \in \mathrm{conv(A)}$. □

Lemma 1.14. *Let A be a subsemigroup of $\mathbb{Q}^m(+)$ and let $a \in A$, $b \in \mathbb{Q}^m$ and $k, l \in \mathbb{N}_0$, $k + l \geq 1$, be such that $kb - la \in A$ ($A \cup \{0\}$, resp.). Then $(k-1)a + kb \in A$ ($A \cup \{0\}$, resp.).*

Proof. We have $(k-1)a + kb = (kb - la) + (k + l - 1)a \in A$. □

Lemma 1.15. *Let $a_1, \dots, a_n \in \mathrm{conv(A)}$, $n \in \mathbb{N}$. Then $(a_1 + \dots + a_n)/n \in \mathrm{conv(A)}$.*

Proof. It is easy. □

3 Preliminaries (b)

For $a = (q_1, \dots, q_n) \in \mathbb{Q}^m$, we put $||a|| = \sqrt{\sum q_i^2} \in \mathbb{R}_0^+$.

Remark 2.1. Let $\varepsilon = \{e_1, \dots, e_m\}$ be a basis of the vector $\mathbb{Q}$-space $\mathbb{Q}^m$. For every $a \in \mathbb{Q}^m$ there are uniquely determined $\pi_i(a) \in \mathbb{Q}$ such that $a = \sum \pi_i(a) e_i$. Then $\pi_i : \mathbb{Q}^m \to \mathbb{Q}$ are projections of the vector $\mathbb{Q}$-spaces and these projections are continuous. Consequently, for every $q \in \mathbb{Q}^+$ there is $t \in \mathbb{Q}^+$ such that $|\pi_i(a)| < q$ for all $i = 1, \dots, m$, whenever $a \in \mathbb{Q}^m$ is such that $||a|| < t$.

Lemma 2.2. *Let $a_0, \dots, a_m \in \mathbb{Q}^m$ be such that the elements $a_1 - a_0, \dots, a_m - a_0$ are linearly $\mathbb{Q}$ independent (then they form a basis of $\mathbb{Q}^m$). Take $q_0, \dots, q_m \in \mathbb{Q}^+$ such that $\sum_{j=0}^m q_j = 1$ and put $a = \sum_j q_j a_j$. Then there is $t \in \mathbb{Q}^+$ such that $a + b \in \mathrm{conv}(\{a_0, \dots, a_m\})$ for every $b \in \mathbb{Q}^m$ with $||b|| < t$.*

Proof. Put $q = \min(\{ q_j/m \mid j = 0, 1, \dots, m \}) \in \mathbb{Q}^+$. By 2.1, where $\varepsilon = \{a_1 - a_0, a_2 - a_0, \dots, a_m - a_0\}$, there is $t \in \mathbb{Q}^+$ such that $|\pi_i(b)| < q$ for all $i = 1, \dots, m$, whenever $b \in \mathbb{Q}^m$ is such that $||b|| < t$. Moreover, if we put $\pi_0(b) = -\sum_{i=1}^m \pi_i(b)$ then $b = \sum_{j=0}^m \pi_j(b) a_j$, $|\pi_0(b)| < q$, $\sum_{j=0}^m \pi_j(b) = 0$, $\sum_{j=0}^m (q_i + \pi_j(b)) = 1$ and $q_j + \pi_j(b) > 0$ for every $j = 0, 1, \dots, m$. Thus $a + b = \sum_{j=0}^m (q_j + \pi_j(b)) a_i \in \mathrm{conv}(\{a_0, \dots, a_m\})$. □

Lemma 2.3. *Let $a_1, \dots, a_m \in \mathbb{Q}^m$ be linearly independent elements (then they form a basis of $\mathbb{Q}^m$). Take $q_1, \dots, q_m \in \mathbb{Q}^+$ and put $a = \sum_{i=1}^m q_i a_i$. Then there is $t \in \mathbb{Q}^+$ such that $a + b \in \mathrm{cone}(\{a_1, \dots, a_m\})$ for every $b \in \mathbb{Q}^m$ with $||b|| < t$.*

Proof. First, find $r \in \mathbb{Q}^+$ such that $rq < 1$, where $q = \sum_{i=1}^m q_i \in \mathbb{Q}^+$, and put $a'_0 = 0$, $a'_i = a_i/r$ and $q'_i = rq_i$ for $i = 1, \dots, m$. If $q'_0 = 1 - rq$ then $q'_j \in \mathbb{Q}^+$ for every $j = 0, 1, \dots, m$, $\sum_{j=0}^m q'_j = 1$ and $a = \sum_{j=0}^m q'_j a'_j$. Now, by 2.2, there is $t \in \mathbb{Q}^+$ such that $a + b \in \mathrm{conv}(\{a'_0, a'_1, \dots, a'_m\}) \subseteq \mathrm{cone}(\{a_1, \dots, a_m\})$ for every $b \in \mathbb{Q}^m$ with $||b|| < t$. □

Lemma 2.4. *Let $a_1, \dots, a_m \in \mathbb{Q}^m$ be linearly independent and let $a = \sum_{i=1}^m q_i a_i$, $q_i \in \mathbb{Q}^+$. Then, for every $b \in \mathbb{Q}^m$, there is $n \in \mathbb{N}$ with $na + b \in \mathrm{cone}(\{a_1, \dots, a_n\})$.*

Proof. By 2.3, there is $t \in \mathbb{Q}^+$ such that $a + c \in \mathrm{cone}(\{a_1, \dots, a_m\})$ whenever $||c|| < t$. Now, $||b/n|| < t$ for some $n \in \mathbb{N}$, and hence $a + b/n \in \mathrm{cone}(\{a_1, \dots, a_m\})$. Then $na + b \in \mathrm{cone}(\{a_1, \dots, a_m\})$. □

Lemma 2.5. *Let $a_1, \dots, a_m \in \mathbb{Q}^m$ be linearly independent and let $a = \sum_{i=1}^m q_i a_i$, $q_i \in \mathbb{Q}^+$. Then, for every $b \in \mathbb{Q}^m$, there are $r, s \in \mathbb{Q}^+$ with $ra + sb \in \mathrm{conv}(\{a_1, \dots, a_m\})$.*

Proof. By 2.4, $na + b \in \mathrm{cone}(\{a_1, \dots, a_m\})$ for some $n \in \mathbb{N}$. Our result is clear for $na + b \neq 0$. If $na + b = 0 \neq a$ then $0 \neq (n+1)a + b \in \mathrm{cone}(\{a_1, \dots, a_m\})$. Finally, if $a = 0 = na + b$ then $b = 0$, $0 \in \mathrm{conv}(\{a_1, \dots, a_m\})$ and we can put $r = 1 = s$. □

4 Preliminaries (c)

In this section, the structure of these subsets is described in more detail. Attention is paid to the case when A is a subsemigroup of $\mathbb{Q}^m(+)$.

Let A be a subset of $\mathbb{Q}^m$, $m \in \mathbb{N}$. We put $\alpha(A) = \{ a \in \mathbb{Q}^m \mid (\exists b \in \mathbb{Q}^m)(\forall r, s \in \mathbb{Q}^+)\ ra + sb \notin \mathrm{cone(A)} \}$, $\beta(A) = \alpha(A) \cap \mathrm{cone(A)}$, $\gamma(A) = \alpha(A) \cap \mathrm{conv(A)}$ and $\delta(A) = \alpha(A) \cap A$.

Lemma 3.1. *The following conditions are equivalent for $a \in \mathbb{Q}^m$:*

(i) $a \in \alpha(A)$.

(ii) *There is at least one $c \in \mathbb{Z}^m$ such that $ra + sc \notin \mathrm{cone(A)}$ for all $r, s \in \mathbb{Q}^+$.*

(iii) *There is at least one $d \in \mathbb{Q}^m$ such that $ra + sd \notin \mathrm{conv(A)}$ for all $r, s \in \mathbb{Q}^+$.*

(iv) *There is at least one $e \in \mathbb{Z}^m$ such that $ra + se \notin \mathrm{conv(A)}$ for all $r, s \in \mathbb{Q}^+$.*

Proof. Clearly, the conditions (i) and (ii) are equivalent, the conditions (iii) and (iv) are equivalent and (i) implies (iii). It remains to show that (iii) implies (i). Assume first that $ra+sd \in$ cone(A) and $ra+sd \neq 0$. Then there are $n \in \mathbb{N}$, $a_i \in A$ and $q_i \in \mathbb{Q}^+$ such that $ra + sd = \sum_{i=1}^{n} q_i a_i$ and $\sum q_i = q \in \mathbb{Q}^+$. Now, $(r/q)a + (s/q)d = \sum(q_i/q)a_i$, $\sum q_i/q = 1$ and $(r/q)a+(s/q)d \in$ conv(A), a contradiction with (iii). Consequently, if $ra + sd \neq 0$ for all $r, s \in \mathbb{Q}^+$ then $ra + sb \notin$ cone(A) and $a \in \alpha(A)$.

Next, assume that $r_1a+s_1d = 0$ for some $r_1, s_1 \in \mathbb{Q}^+$. Then $0 \notin$ conv(A), $d = (-r_1/s_1)a$ and $ra + sd = (r - ((r_1s/s_1))a \notin$ conv(A) for all $r, s \in \mathbb{Q}^+$. If $a \in$ cone(A), $a \neq 0$, then $r_2a \in$ conv(A) for suitable $r_2 \in \mathbb{Q}^+$ and, setting $r = r_2 + (r_1s/s_1)$, we get $ra+sd = r_2a \in$ conv(A), a contradiction. It follows that $a \in (\mathbb{Q}^m \setminus$ cone(A)$) \cup \{0\}$.

If $a \notin$ cone(A) then, setting $b = 0$, we get $ra + sb = ra \notin$ cone(A) for all $r, s \in \mathbb{Q}^+$ and $a \in \alpha(A)$. Finally, if $a = 0$ then, choosing $b \in \mathbb{Q}^m \setminus$ cone(A) (see 1.13), we get $ra + sb = sb \notin$ cone(A) for all $r, s \in \mathbb{Q}^+$. Again, we obtain $a \in \alpha(A)$. □

Lemma 3.2. (i) *$q\alpha(A) \subseteq \alpha(A)$ and $q\beta(A) \subseteq \beta(A)$ for every $q \in \mathbb{Q}^+$.*
(ii) *For every $a \in \beta(A)$, $a \neq 0$, there is $r \in \mathbb{Q}^+$ with $ra \in \gamma(A)$.*

Proof. It is easy. □

Lemma 3.3. *The following conditions are equivalent:*

(i) $0 \in \alpha(A)$.

(ii) $0 \in \beta(A)$.

(iii) $\beta(A) \neq \emptyset$.

(iv) $\alpha(A) \neq \emptyset$.

(v) cone(A) $\neq \mathbb{Q}^{\mathrm{m}}$.

Proof. Clearly, (i) implies (ii), (ii) implies (iii), (iii) implies (iv) and (iv) implies (v). If $b \notin$ cone(A) then $r0 + sb = sb \notin$ cone(A) for all $r, s \in \mathbb{Q}^+$. Thus $0 \in \alpha(A)$ and (v) implies (i). □

Lemma 3.4. *If $A \subseteq (\mathbb{Q}^+)^m$ then $0 \in \beta(A)$.*

Proof. We have cone(A) $\subseteq (\mathbb{Q}_0^+)^{\mathrm{m}}$ and $0 \in \beta(A)$ by 3.3. □

Lemma 3.5. $\alpha(A) = (\mathbb{Q}^m \setminus$ cone(A)$) \cup \beta$(A).

Proof. If $a \notin$ cone(A) then $ra \notin$ cone(A) for every $r \in \mathbb{Q}^+$, and hence $a \in \alpha(A)$. □

Lemma 3.6. *Let $a_1, \dots, a_m \in \mathbb{Q}^m$ be linearly independent and let $a = \sum_{i=1}^{m} q_i a_i$, $q_i \in \mathbb{Q}^+$. Then $a \notin \alpha(A)$.*

Proof. See 2.4 (or 2.5). □

Lemma 3.7. (i) α(cone(A)) $= \alpha$(A).
(ii) β(cone(A)) $= \gamma$(cone(A)) $= \delta$(cone(A)) $= \beta$(A).

Proof. The equalities follow easily from the definitions of the sets involved and from 1.2(iii). □

Lemma 3.8. *Assume that A is a subsemigroup of $\mathbb{Q}^m(+)$. Then for every $a \in \beta(A)$, $a \neq 0$, there is $k \in \mathbb{N}$ with $ka \in \delta(A)$.*

Proof. We have $a = \sum_i q_i a_i$ for some $a_i \in A$, $q_i \in \mathbb{Q}^+$, $i = 1, \dots, n$, $n \in \mathbb{N}$. Choosing $k \in \mathbb{N}$ such that $kq_i \in \mathbb{N}$ for every i, we get $ka \in A$. But $ka \in \beta(A)$ by 3.2(i), and hence $ka \in \delta(A)$. □

Corollary 3.9. *If A is a subsemigroup of $\mathbb{Q}^m(+)$ then $\beta(A) = \bigcup_{k\in\mathbb{N}} \delta(A)/k$ or $\beta(A) = \{0\} \cup \bigcup_{k\in\mathbb{N}} \delta(A)/k$.* □

Lemma 3.10. *Assume that A is a subset of $(\mathbb{Q}_0^+)^m$. Then $\mathbb{Q}^m \setminus (\mathbb{Q}_0^+)^m \subseteq \alpha(A)$.*

Proof. We have cone(A) $\subseteq (\mathbb{Q}_0^+)^{\mathrm{m}}$, and hence $\mathbb{Q}^m \setminus (\mathbb{Q}_0^+)^m \subseteq \alpha(A)$ by 3.5. On the other hand, if $a = (q_1 \dots, q_m) \in (\mathbb{Q}_0^+)^m \setminus (\mathbb{Q}^+)^m$ then $q_i = 0$ for at least one i and, setting $b = (0, \dots, 0, -1, 0, \dots, 0)$ where -1 is the i-th coordinate, we have $ra + sb \notin (\mathbb{Q}_0^+)^m$ for all $r, s \in \mathbb{Q}^+$. Thus $a \in \alpha(A)$. □

Lemma 3.11. *Let $a_1, \dots, a_m \in \mathbb{Q}^m$ be linearly independent and let $a = \sum_{i=1}^{m} q_i a_i$, $q_i \in \mathbb{Q}^+$. Then $a \notin \alpha(\{a_1, \dots, a_m\})$.*

Proof. See 2.5. □

Lemma 3.12. *The following conditions are equivalent for $a \in \mathbb{Q}^m$:*

(i) *$a \notin \alpha(A)$.*

(ii) *For every $b \in \mathbb{Q}^m$ there are $k, l \in \mathbb{N}$ with $ka + lb \in$ cone(A).*

Moreover, if A is a subsemigroup of $\mathbb{Q}^m(+)$ with $0 \in A$ then these two conditions are equivalent to

(iii) *For every $b \in \mathbb{Q}^m$ there are $k_1, l_1 \in \mathbb{N}$ with $k_1a + l_1b \in A$.*

Proof. Clearly, (iii) implies (ii) and (ii) implies (i). If $a \notin \alpha(A)$ then $ra+sb \in$ cone(A) for some $r, s \in \mathbb{Q}^+$ and we have $tra + tsb \in$ cone(A), where $t \in \mathbb{N}$ is such that $tr, ts \in \mathbb{N}$. Finally, if A is a subsemigroup of $\mathbb{Q}^m(+)$ and $0 \in A$ then (ii) implies (iii) by 1.10(i). □

Lemma 3.13. *If $b \in \mathbb{Q}^m \setminus \alpha(A)$, $a \in A$ and $q \in \mathbb{Q}_0^+$ then $qa + b \notin \alpha(A)$.*

Proof. The result follows easily from 3.12(ii). □

Lemma 3.14. *Assume that A is a subsemigroup of $\mathbb{Q}^m(+)$. Then either $\delta(A) = A$ or $A \setminus \alpha(A)$ is a subsemigroup of $\mathbb{Q}^m(+)$.*

Proof. If $\delta(A) \neq A$ then $A \nsubseteq \alpha(A)$ and $B = A \setminus \alpha(A) \neq \emptyset$. Now, the fact that B is a subsemigroup of $\mathbb{Q}^m(+)$ follows from 3.13. □

Lemma 3.15. *Assume that A is a subsemigroup of $\mathbb{Q}^m(+)$. Then, for all $a \in A$ and $b \in \mathbb{Q}^m \setminus \alpha(A)$, there is $n \in \mathbb{N}$ with $(n-1)a + nb \in A$.*

Proof. By 3.12(iii), $k_1 b - l_1 a \in A \cup \{0\}$ for some $k_1, l_1 \in \mathbb{N}$ (notice that cone(A) = cone(A∪{0})) and $\alpha(A) = \alpha(A \cup \{0\})$. Now, by 1.14, $(n-1)a + nb \in A \cup \{0\}$ for some $n \in \mathbb{N}$. If $(n-1)a + nb = 0$ then $(2n-1)a + 2nb = a \in A$. □

5 Subsemigroups of $\mathbb{N}_0^m(+)$ (a)

In this section, let A be a subsemigroup of $\mathbb{N}_0^m(+)$. We denote by $\varepsilon(A)$ the set of the elements $a \in A$ such that $b - a \in A$, whenever $b \in A$ and $b - a \in \mathbb{N}_0^m$. Equivalently, $\varepsilon(A) = \{\, a \in A \mid A + a = (\mathbb{N}_0^m + a) \cap A \,\}$.

Lemma 4.1. *The following conditions are equivalent:*

(i) $0 \in A$.

(ii) $0 \in \varepsilon(A)$.

(iii) $\varepsilon(A) \neq \emptyset$.

Proof. Clearly, (i) implies (ii) and (ii) implies (iii). It remains to show that (iii) implies (i). If $a \in \varepsilon(A)$ then $a = a + 0$, and so $a \in (\mathbb{N}_0^m + a) \cap A = A + a$. Then $a = b + a$ for some $b \in A$ and, of course, $b = 0$. □

Lemma 4.2. *Either $0 \notin A$ and $\varepsilon(A) = \emptyset$ or $0 \in \varepsilon(A) \subseteq A$ and $\varepsilon(A)$ is a subsemigroup of A.*

Proof. It is easy (use 4.1). □

In the rest of the section, we will assume that $0 \in A$ (see 4.1 and 4.2).

Lemma 4.3. (i) *If $b, a \in \varepsilon(A)$ are such that $b - a \in \mathbb{N}_0^m$ then $b - a \in \varepsilon(A)$.*
(ii) $(\varepsilon(A) - \varepsilon(A)) \cap \mathbb{N}_0^m = \varepsilon(A)$.
(iii) $\varepsilon(\varepsilon(A)) = \varepsilon(A)$.

Proof. (i) Since $a \in \varepsilon(A)$ and $b \in A$, we have $b - a \in A$. Now, if $c \in A$ is such that $c - (b-a) = (c+a) - b \in \mathbb{N}_0^m$ then $c + a \in A$ and $c - (b-a) \in A$, since $b \in \varepsilon(A)$. Thus $b - a \in \varepsilon(A)$.
(ii) and (iii). Use (i). □

Lemma 4.4. *$\varepsilon(A) - \varepsilon(A) \subseteq \mathbb{Z}^m$ and $\varepsilon(A) - \varepsilon(A)$ is just the difference (sub)group of the (sub)semigroup $\varepsilon(A)$.*

Proof. It is easy. □

Lemma 4.5. *If $A + a \in \varepsilon(A)$ for at least one $a \in \mathbb{N}_0^m$ then $\varepsilon(A) = A$.*

Proof. First, $a = 0 + a \in a + A \subseteq \varepsilon(A)$, and so $a \in \varepsilon(A)$. Now, $A = ((A + a) - a) \cap \mathbb{N}_0^m \subseteq (\varepsilon(A) - \varepsilon(A)) \cap \mathbb{N}_0^m = \varepsilon(A)$ by 4.3(ii). Thus $\varepsilon(A) = A$. □

Corollary 4.6. *If $\varepsilon(A) \neq A$ then for every $a \in \mathbb{N}_0^m$ there is at least one $b \in A$ such that $a + b \notin \varepsilon(A)$.* □

For every $a \in A$, put $\varphi(A, a) = \{\, b \in \mathbb{N}_0^m \mid a + b \in A \,\}$, $\psi(A, a) = \{\, c \in \mathbb{N}_0^m \mid (n-1)a + nc \in A$ for some $n \in \mathbb{N} \,\}$ and $\xi(A, a) = \varphi(A, a) \cap \psi(A, a)$.

Lemma 4.7. (i) $A \cup \{0\} \subseteq \xi(A, a)$.
(ii) $\mathbb{N}_0^m \setminus \alpha(A) \subseteq \psi(A, a)$.
(iii) $\mathbb{N}_0^m \cap \mathbb{Q}a \subseteq \psi(A, a)$.

Proof. (i) It is obvious.
(ii) This is ensured by 3.15.
(iii) We can assume that $a \neq 0$. Take $b \in \mathbb{N}_0^m \cap \mathbb{Q}a$, $b \neq 0$. We have $b = ra/b$ for some $r \in \mathbb{Z}$, $s \in \mathbb{N}$, and hence $sb = ra$ and we conclude easily that $r \in \mathbb{N}$. Now, $sb - ra = 0$ and $(s-1)a + sb \in A$ (see 1.14). □

Lemma 4.8. *If $a \in \varepsilon(A)$ then both $\varphi(A, a)$ and $\xi(A, a)$ are subsemigroups of $\mathbb{N}_0^m(+)$.*

Proof. First, if $a + b_1 \in A$ and $a + b_2 \in A$ for some $b_1, b_2 \in \mathbb{N}_0^m$ then $2a + b_1 + b_2 \in A$ and $a + b_1 + b_2 \in \mathbb{N}_0^m$. Since $a \in \varepsilon(A)$, we get $a + b_1 + b_2 \in A$. Similarly, if $c_1, c_2 \in \xi(A, a)$ then $(n-1)a + nc_i \in A$ for some $n \in \mathbb{N}$, $n \geq 3$, $(2n-2)a + n(c_1 + c_2) \in A$ and, since $a \in \varepsilon(A)$, we conclude that $(n-1)a + n(c_1 + c_2) \in A$. □

6 Subsemigroups of $\mathbb{N}_0^m(+)$ (b)

Define a relation $\leq$ on $\mathbb{N}_0^m$ by $a \leq b$ if and only if $b - a \in \mathbb{N}_0^m$. Clearly, the relation $\leq$ is reflexive, antisymmetric and transitive, and hence it is an ordering. This ordering is stable under both addition and multiplication. Notice that it satisfies the descending chain condition, but not the ascending one.

Lemma 5.1. *There is no infinite set of pair-wise incomparable m-tuples in $\mathbb{N}_0^m$.*

Proof. The assertion is clear for $m = 1$ and we proceed by induction for $m \geq 2$.

Let A be a set of pair-wise incomparable elements from $\mathbb{N}_0^m$. For $k \in \mathbb{N}_0$, put $B_k = \{\, b \in \mathbb{N}_0^{m-1} \mid a \leq$

(b,k) for at least one $a \in A\}$. Then $B_0 \subseteq B_1 \subseteq B_2 \subseteq \dots$ and all the sets B_k are filters of the ordered set N_0^{m-1}. Then $B = \bigcup B_k$ is a filter and we denote by C the set of minimal elements of B. The elements of C are pair-wise incomparable, and hence C is a finite set by induction. Consequently, $C \subseteq B_{k_0}$ for some k_0 and it follows that $B_{k_0} = B_{k_0+1} = \dots = B$. Now, if $a = (b_1, k_1) \in A$, $b_1 \in \mathbb{N}_0^{m-1}$, $k_1 \in \mathbb{N}_0$, then $b_1 \in B_{k_1} \subseteq B = B_{k_0}$, and therefore $a_1 \le (b_1, k_0)$ for some $a_1 \in A$. If $k_0 \le k_1$ then $a_1 \le a$, and hence $a_1 = a$ and $k_1 = k_0$. It means that $k_1 \le k_0$ anyway. Furthermore, if C_i denotes the set of minimal elements from B_i, $i = 0, 1, \dots, k_0$, then C_i is a finite set (and $C_{k_0} = C$). Since $b_1 \in B_{k_1}$, there is $c_1 \in C_{k_1}$ with $c_1 \le b_1$ and $(c_1, k_1) \le a$. According to the definition of B_{k_1}, we can find $a_2 \in A$ such that $a_2 \le (c_1, k_1)$. Then $a_2 \le a$, and therefore $a_2 = a$ and $b_1 = c_1$. We have proved that $A \subseteq \bigcup_{i=0}^{k_0} (C_i \times \{i\})$ and it follows immediately that A is finite. □

Example 5.2. Put $m = 2$. Then $A_n = \{(0,n), (1, n-1), \dots, (n-1, 1), (n, 0)\}$ $(\subseteq \mathbb{N}_0^2)$, $n \in \mathbb{N}_0$, is a set of incomparable elements and $|A_n| = n + 1$.

Proposition 5.3. *Let A_1 be a non-empty finite subset of $\mathbb{N}_0^m$ and let A be a subsemigroup of $\mathbb{N}_0^m(+)$ such that $A_1 \subseteq A \subseteq \mathrm{cone}(A_1)$. Then A is a finitely generated semigroup.*

Proof. Let $A_1 = \{a_1, \dots, a_n\}$, $n \in \mathbb{N}$, and $B = \{\sum_{i=1}^n q_i a_i \mid q_i \in \mathbb{Q}_0^+, q_i \le 1\} \cap \mathbb{N}_0^m$. Clearly, $A_1 \subseteq A \subseteq \mathrm{cone}(A_1) \cap \mathbb{N}_0^m$ and B is a finite subset of $\mathbb{N}_0^m$.

If $a \in A$ then $a = \sum r_i a_i$, $r_i \in \mathbb{Q}_0^+$, and we have $r_i = l_i + s_i$ for suitable $l_i \in \mathbb{N}_0$ and $s_i \in \mathbb{Q}_0^+$, $s_i \le 1$. From this, $a = b_1 + \sum l_i a_i$, where $b_1 = \sum s_i a_i$. Since $a \in \mathbb{N}_0^m$ and $\sum l_i a_i \in \mathbb{N}_0^m$, we get $b_1 \in B$. Moreover, either $\sum l_i a_i = 0$ or $\sum l_i a_i \in A$.

For every $b \in B$, put $N_b = \{(k_1, \dots, k_n) \in \mathbb{N}_0^n \mid b + \sum k_i a_i \in A\}$. Then N_b is a filter of the ordered set $\mathbb{N}_0^m$ (see 5.1 and its proof) and the set M_b of minimal elements from N_b is finite. Of course, $N_b = \{d \in \mathbb{N}_0^m \mid c \le d \text{ for some } c \in M_b\}$ and the set $C = \{a_1, \dots, a_n\} \cup \bigcup_{b \in B} \{b + k_i a_i \mid (k_1, \dots, k_n) \in M_b\}$ is a finite subset of A. Taking into account the preceding steps, one sees easily that the semigroup A is generated by the finite subset C. □

Theorem 5.4. *The following conditions are equivalent for a subsemigroup A of $\mathbb{N}_0^m(+)$:*

(i) *A is a finitely generated semigroup.*

(ii) $\mathrm{cone}(A) = \mathrm{cone}(A_1)$ *for a non-empty finite subset A_1 of A.*

(iii) $\mathrm{cone}(A) = \mathrm{cone}(A_2)$ *for a non-empty finite subset A_2 of $(\mathbb{Q}_0^+)^m$.*

Proof. (i) implies (ii) by 1.9, (ii) implies (iii) trivially and (ii) implies (i) by 5.3. It remains to show that (iii) implies (ii).

Since A_2 is a finite subset of cone(A), there is a finite subset A_1 of A such that $A_2 \subseteq \mathrm{cone}(A_1)$. Then $\mathrm{cone}(A) = \mathrm{cone}(A_2) \subseteq \mathrm{cone}(\mathrm{cone}(A_1)) = \mathrm{cone}(A_1) \subseteq \mathrm{cone}(A)$. Thus $\mathrm{cone}(A) = \mathrm{cone}(A_1)$. □

Corollary 5.5. *Let A be a finitely generated subsemigroup of $\mathbb{N}_0^m(+)$. Then every subsemigroup A' of $\mathbb{N}_0^m(+)$ such that $A \subseteq A' \subseteq \mathrm{cone}(A)$ is finitely generated. In particular, $\mathrm{cone}(A) \cap \mathbb{N}_0^m$ is a finitely generated subsemigroup of $\mathbb{N}_0^m(+)$.* □

Remark 5.6. It is easy to see that 5.4 remains true for subsemigroups of $(\mathbb{Q}_0^+)^m(+)$. Indeed, let B be such a subsemigroup and assume that $\mathrm{cone}(B) = \mathrm{cone}(B_1)$ for a non-empty finite subset B_1 of B. We have $B_1 = \{k_1/l_1, \dots, k_n/l_n\}$ for some $n \in \mathbb{N}$, $k_i \in \mathbb{N}_0$ and $l_i \in \mathbb{N}$. If $l = l_1 \cdots l_n$ then $A = lB$ is a subsemigroup of $\mathbb{N}_0^m(+)$ and $\mathrm{cone}(A) = \mathrm{cone}(B) = \mathrm{cone}(B_1) = \mathrm{cone}(A_1)$, where $A_1 = lB_1$. By 5.4, A is a finitely generated semigroup and the same is true for B, since the mapping $a \mapsto a/l$, $a \in A$, is an isomorphism of A onto B.

7 Pure subsemigroups of $\mathbb{N}_0{}^m(+)$

In this section, let A be a subsemigroup of $\mathbb{N}_0^m(+)$.

Lemma 6.1. *The following conditions are equivalent:*

(i) $nA = A \cap n\mathbb{N}_0^m$ *for every $n \in \mathbb{N}$.*

(ii) *If $a \in \mathbb{N}_0^m$ and $n \in \mathbb{N}$ are such that $na \in A$ then $a \in A$.*

(iii) *If $a \in \mathbb{N}_0^m$ and $q \in \mathbb{Q}^+$ are such that $qa \in A$ then $a \in A$.*

Proof. It is easy. □

If these equivalent conditions are satisfied then A is called *pure subsemigroup* of $\mathbb{N}_0^m(+)$. In the remaining part of this section(except for 6.12), we will assume that A is a pure subsemigroup.

Lemma 6.2. $A \cup \{0\}$ *is a pure subsemigroup of $\mathbb{N}_0^m(+)$.*

Proof. It is easy. □

Lemma 6.3. *If $0 \in A$ then $\varepsilon(A)$ is a pure subsemigroup of $\mathbb{N}_0^m(+)$.*

Proof. By 4.2, $\varepsilon(A)$ is a subsemigroup of A. Let $a = nb$, where $n \in \mathbb{N}$, $a \in \varepsilon(A)$ and $b \in \mathbb{N}_0^m$. Since A is pure, we have $b \in A$. Moreover, if $c \in A$ is such that $c - b \in \mathbb{N}_0^m$ then $nc - a = n(c - b) \in \mathbb{N}_0^m$, and so $n(c - b) \in A$, since $a \in \varepsilon(A)$. Using again the fact that A is pure, we get $c - b \in A$. Thus $b \in \varepsilon(A)$ and $\varepsilon(A)$ is pure. □

Lemma 6.4. *Let $n \in \mathbb{N}$, $a_1, \dots, a_n \in A$ and $q_1, \dots, q_n \in \mathbb{Q}^+$ be such that $a = \sum q_i a_i \in \mathbb{N}_0^m$. Then $a \in A$.*

Proof. We have $q_i = r_i/s_i$ for some $r_i, s_i \in \mathbb{N}$. If $s = s_1 \cdots s_n$ then $sq_i \in \mathbb{N}$, $b_i = sq_i a_i \in A$ and $sa = \sum b_i \in A$. Now, $a \in A$ by 6.1. □

Lemma 6.5. *If $a \in \mathbb{N}_0^m$ and $a \notin \alpha(A)$ then $a \in A$.*

Proof. By 3.3, $a \neq 0$. If $b \in A$ then there are $r, s \in \mathbb{Q}^+$ with $ra - sb \in \mathrm{cone(A)}$ and we get $ra \in \mathrm{cone(A)} + \mathrm{sb} \subseteq \mathrm{cone(A)}$. Thus $a \in \mathrm{cone(A)} \cap \mathbb{N}_0^{\mathrm{m}}$ and $a \in A$ by 6.4. □

Proposition 6.6. (i) $\mathrm{cone(A)} \cap \mathbb{N}_0^{\mathrm{m}} = \mathrm{A} \cup \{0\}$.
(ii) *If* $0 \in A$ *then* $A = \mathrm{cone(A)} \cap \mathbb{N}_0^{\mathrm{m}}$.
(iii) *If* $\mathrm{cone(A)} = (\mathbb{Q}_0^+)^{\mathrm{m}}$ *then* $A \cup \{0\} = \mathbb{N}_0^m$.
(iv) $\alpha(A) \cap \mathbb{N}_0^m = (\mathbb{N}_0^m \setminus A) \cup \delta(A)$.
(v) $\mathbb{N}_0^m = (\alpha(A) \cap \mathbb{N}_0^m) \cup A$.
(vi) $\xi(A, a) = \psi(A, a)$ *for every* $a \in A$.

Proof. (i), (ii) and (iii). Use 6.4.
(iv) and (v). Use 6.5.
(vi) If $(n - 1)a + nc \in A$ then $n(a + c) \in A$ and $a + c \in A$. Thus $\psi(A, a) \subseteq \varphi(A, a)$ and $\xi(A, a) = \psi(A, a)$. □

Remark 6.7. Let $k \in \mathbb{N}$ and $a, b_1, \dots, b_k \in \mathbb{N}_0^m$ be such that $a + b_i \in A$ for every $i = 1, \dots, k$ (e.g., $a \in A$ and $b_i \in \varphi(A, a)$). Furthermore, assume that there are $n_i \in \mathbb{N}$ such that $(n_i - 1)a + n_i b_i \in A$ for every i (e.g., $a \in A$ and $b_i \in \psi(A, a)$). If $n \in \mathbb{N}$, $n \geq \max(n_i)$ then $(n - 1)a + nb_i \in A$ for all i. In particular, if $t_i \in \mathbb{N}_0$ are such that $t = \sum t_i \geq \max(n_i)$ then $(t - 1)a + tb_i \in A$ and $(t - 1)a + \sum t_i b_i = \sum((t_i/t)((t-1)a+tb_i)$, $(t-1)a+\sum t_i b_i \in \mathbb{N}_0^m$. Now, by 6.4, we get $(t - 1)a + \sum t_i b_i \in A$.

Lemma 6.8. *If $a \in A$ and $b \in \mathbb{N}_0^m$ are such that $a + b \notin \alpha(A)$ then $a + b \in A$ and $b \in \xi(A, a)$.*

Proof. Since $a + b \notin \alpha(A)$, $a + b \in A$ by 6.6(v) and there are $r, s \in \mathbb{Q}^+$ such that $r(a+b)-sa \in \mathrm{cone(A)}$. We have $r = k/n$, $s = l/n$ for suitable $k, l, n \in \mathbb{N}$ and $c/n = r(a + b) - sa \in \mathrm{cone(A)}$, where $c = (k - l)a + kb$. Then $c \in \mathbb{Z}^m \cap \mathrm{cone(A)} = \mathbb{Z}^{\mathrm{m}} \cap (\mathbb{Q}_0^+)^{\mathrm{m}} \cap \mathrm{cone(A)} = \mathbb{N}_0^{\mathrm{m}} \cap \mathrm{cone(A)} = \mathrm{A} \cup \{0\}$ (see 6.6(i)), so that $c \in A \cup \{0\}$ and $d = (k-1)a+kb = c+(l-1)a \in A \cup \{0\}$. If $d \neq 0$ then $d \in A$ and $b \in \xi(A, a)$ (see 6.6(vi)). If $d = 0$ then $b = 0 \in \xi(A, a)$. □

Lemma 6.9. *Let $a \in A$ be such that $b \in \psi(A, a)$ for every $b \in \mathbb{N}_0^m$ with $a + b \in \delta(A)$ (then $b \in \alpha(A)$). Then $\varphi(A, a) = \psi(A, a) = \xi(A, a)$.*

Proof. We have $\psi(A, a) = \xi(A, a) \subseteq \varphi(A, a)$ by 6.6(vi). If $b \in \varphi(A, a)$ then $a + b \in A$ and if $a + b \notin \alpha(A)$ then $b \in \xi(A, a)$ by 6.8. On the other hand, if $a + b \in \alpha(A)$ then $a + b \in \alpha(A) \cap A = \delta(A)$ and $b \in \psi(A, a)$ by assumption. □

Lemma 6.10. *Let $a \in A$ be such that $\delta(A) \subseteq \mathbb{Q}a$. Then $\varphi(A, a) = \psi(A, a) = \xi(A, a)$.*

Proof. In view of 6.9, if $b \in \mathbb{N}_0^m$ is such that $a + b \in \delta(A)$ then $a + b \in \mathbb{Q}a$, $b \in \mathbb{Q}a$ and $b \in \psi(A, a)$ by 4.7(iii). □

Lemma 6.11. *The following conditions are equivalent for all $a \in A$ and $b \in \mathbb{N}_0^m$:*

(i) *$b \in \psi(A, a)$ ($b \in \xi(A, a)$, resp.).*

(ii) *$r(a + b) - sa = \sum_{i=1}^{k} q_i a_i$ for some $r, s \in \mathbb{Q}$, $k \in \mathbb{N}$, $a_i \in A$ and $q_i \in \mathbb{Q}^+$.*

Proof. See the proof of 6.8. □

Construction 6.12. Let A be a subsemigroup of $\mathbb{N}_0^m(+)$. Put $\mathrm{p}(A) = \mathbb{N}_0^m \cap \left(\bigcup_{n\in\mathbb{N}} A/n\right)$. It is easy to check that $\mathrm{p}(A)$ is a pure subsemigroup of $\mathbb{N}_0^m(+)$. It is the smallest pure subsemigroup containing A. Clearly, $A \subseteq \mathrm{p}(A) \subseteq \mathrm{cone(A)}$, and so $\mathrm{cone(p(A))} = \mathrm{cone(A)}$. Now, according to 5.4, $\mathrm{p}(A)$ is finitely generated if and only if A is so. Finally, notice that $\mathbb{N}_0^m \cap \left(\bigcup_{n\in\mathbb{N}} \varepsilon(A)/n\right) \subseteq \varepsilon(\mathrm{p}(A))$. In particular, $\varepsilon(A) \subseteq \varepsilon(\mathrm{p}(A))$, and so $\varepsilon(\mathrm{p}(A)) \neq \emptyset$, provided that $\varepsilon(A) \neq \emptyset$.

8 Conclusion

In this paper, the properties of subsemigroups and pure subsemigroups of $\mathbb{N}_0^m(+)$ ($= \mathbb{N}_0(+)^m$) are investigated. A theoretical and reference basis for further research in this area has been established. It can be used for further research of finitely generated cones, e.g. in connection with the investigation of context-free languages. Our further research will be directed to a deeper description of pure subsemigroups of $\mathbb{N}_0^m$ for a finite m, in particular $m = 2$.

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Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

B. Batíková – formal analysis and investigation.
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