Half-Logistic Odd Power Generalized Weibull-inverse Lindley: Properties, Characterizations, Applications and Covid-19 Data

ARBËR QOSHJA, ARTUR STRINGA, KLODIANA BANI, REA SPAHO
Department of Applied Mathematics, Department of Mathematics,
Tirana University,
Sheshi "Nënë Tereza", Tirana,
ALBANIA

Abstract: - Using the Half-Logistic Odd Power Generalised Weibull-G family distributions, this article constructed a novel distribution termed the Half-Logistic Odd Power Generalised Weibull-inverse Lindley. Some of its statistical features are derived by us. Selecting the most efficient estimators is among the basic issues in parameter estimation theory. We are employing maximum likelihood estimation, moment estimation, least squares estimation, weighted least estimation, L-moment estimation, Maximum Product Spacing estimation, and techniques of minimum distances for the parameter estimation for the distribution. We will examine simulation research that compares the various estimators' levels of efficiency using the Kolmogorov-Smirnov test. Lastly, an analysis is done on an actual COVID-19 data set to demonstrate the adaptability of our suggested model in comparison to the fit obtained by several other competing distributions.

Key-Words: - Inverse Lindley distribution, Statistical Properties, Estimation, Kolmogorov-Smirnov test

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1 Introduction

One method for characterizing a process or device's lifespan that may be used in a variety of domains, such as biology, engineering, and medicine, is the Lindley distribution. As a counter-example of fiducial statistics, Lindley developed a blend of $\mathrm{Exp}(\theta)$ and $\mathrm{Gamma}(2,\theta)$ distribution with mixing percentage $(\theta/(\theta+1))$. This was done in the context of Bayesian statistics. One notable advancement in the literature on Lindley distribution is the two-parameter weighted Lindley distribution, [1], which is found to be highly beneficial when modelling biological data derived from mortality studies. The proposal of a generalized Poisson Lindley distribution has been made, [2]. However, an exponential geometric (EG) distribution was introduced, [3], in contrast to the extended Lindley (EL) distribution that was demonstrated, [4]. An article, [5], describes a two-parameter Lindley distribution. The authors of, [6], propose a novel two-parameter lifespan distribution model and characteristics. The Lindley distribution convolution was initially proposed by, [7]. The estimation of the dependability of a stress-strength system through the utilisation of power Lindley distribution was the subject of some science, [8]. There has been a recent proposal for an extended Lindley distribution, [9]. An attempt was made to extend the Lindley distribution using the Transmuted Lindley Distribution, which is a quadratic rank transmutation map, [10].

Definition 1.1. If the probability density function of a random variable X has the following definition, [11], [12], then the variable is said to have a Lindley distribution with parameter θ.

$$p_X(x;\theta) = \frac{\theta^2}{(1+\theta)}(1+x)e^{-\theta x}, \quad x > 0, \theta > 0. \tag{1}$$

and cumulative distribution function

$$F_X(x) = 1 - \frac{e^{-\theta x}(1+\theta+\theta x)}{1+\theta}, \quad x > 0, \theta > 0. \tag{2}$$

The inverse Lindley distribution, [13], was due to the broad use of inverse distributions. Its probability density function (pdf) and cumulative distribution function (cdf) are provided by:

$$g(x;\theta) = \frac{\theta^2}{1+\theta}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}, \quad x > 0, \theta > 0. \tag{3}$$

$$G_X(x;\theta) = \left[1 + \frac{\theta}{(1+\theta)x}\right]e^{-\frac{\theta}{x}}, \quad x > 0, \theta > 0. \tag{4}$$

Some researchers, [14], presented a generator of a continuous distribution known as the half-logistic odd power generalized Weibull-G family of distributions, where the pdf and cdf are provided by:

$$p_X\left(x;\alpha,\beta,\underline{\xi}\right) = 2\alpha\beta\left[1+\left(\frac{G\left(x;\underline{\xi}\right)}{\overline{G}\left(x;\underline{\xi}\right)}\right)^{\alpha}\right]^{\beta-1} exp\left\{1-\left[1+\left(\frac{G\left(x;\underline{\xi}\right)}{\overline{G}\left(x;\underline{\xi}\right)}\right)^{\alpha}\right]^{\beta}\right\}\times\left(G\left(x;\underline{\xi}\right)\right)^{\alpha-1}\left(\overline{G}\left(x;\underline{\xi}\right)\right)^{-\alpha+1}\left(1+exp\left\{1-\left[1+\left(\frac{G\left(x;\underline{\xi}\right)}{\overline{G}\left(x;\underline{\xi}\right)}\right)^{\alpha}\right]^{\beta}\right\}\right)^{-2} g\left(x;\underline{\xi}\right) \quad (5)$$

and

$$F_X\left(x;\alpha,\beta,\underline{\xi}\right) = \frac{1-exp\left\{1-\left[1+\left(\frac{G\left(x;\underline{\xi}\right)}{\overline{G}\left(x;\underline{\xi}\right)}\right)^{\alpha}\right]^{\beta}\right\}}{1+exp\left\{1-\left[1+\left(\frac{G\left(x;\underline{\xi}\right)}{\overline{G}\left(x;\underline{\xi}\right)}\right)^{\alpha}\right]^{\beta}\right\}} \quad (6)$$

respectively, for $\alpha > 0, \beta > 0$ and parameter vector $\underline{\xi}$.

This work is aimed at examining the inverse Lindley distributions (3) and (4), also known as the half-logistic odd power generalized Weibull-inverse Lindley distribution, as baseline functions to (5) and (6).

Definition 1.2. If the probability density function of a random variable X is described as follows, it is said to have a half-logistic odd power generalized Weibull-inverse Lindley distribution with a vector parameter (α, β, θ).

$$p_X(x;\alpha,\beta,\theta) = 2\,\alpha\,\beta\left(\left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta\,x+x}-1\right)^{-\alpha}+1\right)^{\beta-1}\times exp\left(1-\left(\left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta\,x+x}-1\right)^{-\alpha}+1\right)^{\beta}\right)\times\left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta\,x+x}-1\right)^{1-\alpha}\frac{\frac{\theta^2}{1+\theta}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}}{\left(exp\left(1-\left(\left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta\,x+x}-1\right)^{-\alpha}+1\right)^{\beta}\right)+1\right)^{2}} \quad (7)$$

and cumulative distribution function

$$F_X(x;\alpha,\beta,\theta) = \frac{1-exp\left(1-\left(\left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta\,x+x}-1\right)^{-\alpha}+1\right)^{\beta}\right)}{1+exp\left(1-\left(\left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta\,x+x}-1\right)^{-\alpha}+1\right)^{\beta}\right)} \quad (8)$$

respectively, for $\alpha > 0, \beta > 0, \theta > 0$.

Figure 1 and Figure 2 illustrate some of the possible shapes of the (Cdf) and (Pdf) of the Half-Logistic Odd Power Generalized Weibull-Inverse Lindley (HLOPGW-ILD) distribution for selected values of the parameters α, β and θ, respectively.

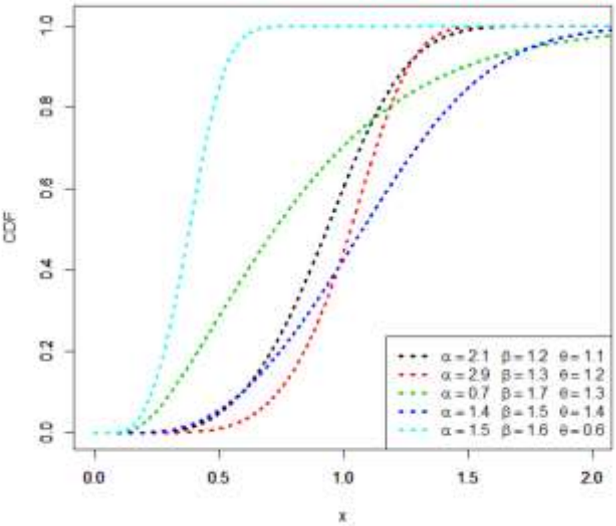

Fig. 1: Cumulative Density Function Of The The Half-Logistic Odd Power Generalized Weibull-Inverse Lindley distribution

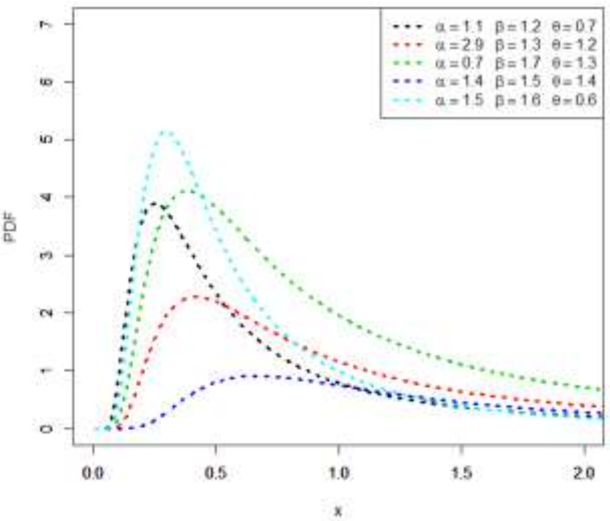

Fig. 2: Probability Density Function Of The The Half-Logistic Odd Power Generalized Weibull-Inverse Lindley distribution

2 Mathematical Properties

2.1 Survival Function

The Half-Logistic Odd Power Generalised Weibull-Inverse Lindley (HLOPGW-ILD) distribution's survival function, or reliability function, is as follows:

$$R(x;\alpha,\beta,\theta) = 1 - F_X(x;\alpha,\beta,\theta) = 1 - \frac{1-exp\left(1-\left(\left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta\,x+x}-1\right)^{-\alpha}+1\right)^{\beta}\right)}{1+exp\left(1-\left(\left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta\,x+x}-1\right)^{-\alpha}+1\right)^{\beta}\right)} \quad (9)$$

2.2 Hazard Function

The Half-Logistic Odd Power Generalised Weibull-Inverse Lindley (HLOPGW-ILD) distribution's hazard rate function, often known as the failure rate, is provided by:

$$h(x;\alpha,\beta,\theta)=\frac{p_X(x;\alpha,\beta,\theta)}{R(x;\alpha,\beta,\theta)} \tag{10}$$

For certain values of the parameters α,β and θ, respectively, Figure 3 and Figure 4 show several potential forms of the Reliability and Hazard functions of the Half-Logistic Odd Power Generalised Weibull-Inverse Lindley (HLOPGW-ILD) distribution.

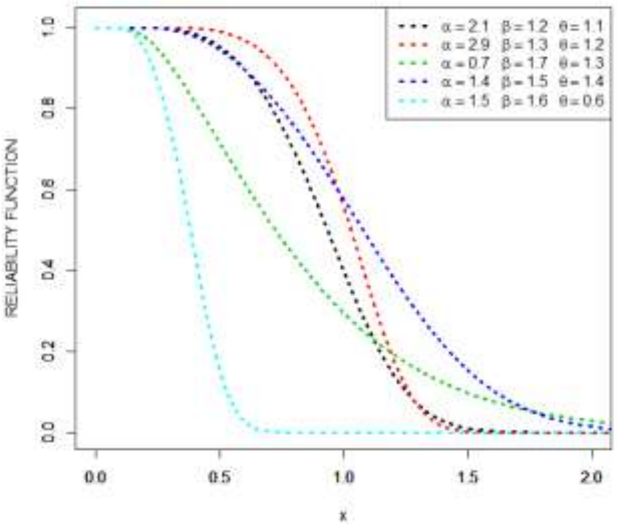

Fig. 3: Reliability Function Function Of The The Half-Logistic Odd Power Generalized Weibull-Inverse Lindley distribution

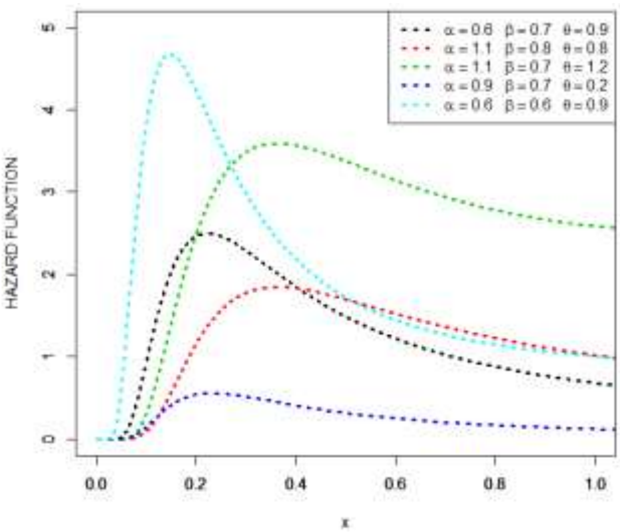

Fig. 4: Hazard Function Function Of The The Half-Logistic Odd Power Generalized Weibull-Inverse Lindley distribution

2.3 Quantiles

The quantile of any distribution is given by solving the equation $G(x_p)=p$, for $0<p<1$. The quantile function of the Half-Logistic Odd Power Generalized Weibull-Inverse Lindley (HLOPGW-ILD) distribution is:

$$x(p)=G^{-1}\left[\left(\left([1+ln(1+p)-ln(1-p)]^{1/\beta}-1\right)^{-1/\alpha}+1\right)^{-1}\right] \tag{11}$$

2.4 Some Useful Expression

Taking generalized binomial expansion, [11], the pdf (5) of X may be written as:

$$p_X(x;\alpha,\beta,\theta)=2\alpha\beta\sum_{n=1}^{\infty}\sum_{q,k,m,l}^{\infty}\binom{-\alpha\left((m+1)+1\right)}{l}\binom{\beta\left((k+1)+1\right)}{m}\times\frac{n^{q+1}}{q!}\binom{q}{k}(-1)^{l+k+n-1}\left(\left(G(x;\theta)\right)\right)^{l+\alpha(m+1)-1}g(x;\theta)$$

By using this methodology, the pdf (7) of the HLOPGW-ILD distribution is:

$$p_X(x;\alpha,\beta,\theta)=2\alpha\beta\sum_{n=1}^{\infty}\sum_{q,k,m,l=0}^{\infty}\binom{-\alpha\left((m+1)+1\right)}{l}\binom{\beta(k+1)-1}{m}\times\frac{n^{q+1}}{q!}\binom{q}{k}(-1)^{l+k+n-1}\left(\left(\left(1+\frac{\theta}{(1+\theta)x}\right)e^{-\frac{\theta}{x}}\right)\right)^{l+\alpha(m+1)-1}\frac{\theta^2}{1+\theta}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}$$

Alternatively, we may represent the Half-Logistic Odd Power Generalised Weibull-Inverse Lindley as a linear combination of Exp-Inverse Lindley densities because, following a series definition:

$$p_X(x;\alpha,\beta,\theta)=\sum_{p=0}^{\infty}s_p t_p(x,\theta)$$

where,

$$s_p=2\alpha\beta\sum_{n=1}^{\infty}\sum_{q,k,m,l=0}^{\infty}\binom{-\alpha\left((m+1)+1\right)}{l}\binom{\beta(k+1)-1}{m}\times\frac{n^{q+1}}{q!}\binom{q}{k}\frac{(-1)^{l+k+n-1}}{p+\alpha(m+1)}$$

for $\beta(k+1)>1$, $\beta>1$ and

$$t_p(x,\theta)=p\frac{\theta^2}{1+\theta}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}\left(\left(1+\frac{\theta}{(1+\theta)x}\right)e^{-\frac{\theta}{x}}\right)^{p-1}.$$

This form may be shown to be a linear combination of Lindley densities with Exp-Inverse Lindley densities and a power parameter p.

2.5 Order Statistics

For $X_1, X_2, \dots, X_n$ i.i.d. continuous random variables with pdf (7) and cdf (8) the density of the maximum order is:

$$p_{(n)}(x) = np(x)F(x)^{n-1}$$
$$= 2\,\alpha\,\beta n \left(\left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta x+x} - 1\right)^{-\alpha} + 1\right)^{\beta-1} \times exp\left(1 - \left(\left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta x+x} - 1\right)^{-\alpha} + 1\right)^{\beta}\right) \times \left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta x+x} - 1\right)^{1-\alpha} \times \frac{\frac{\theta^2}{1+\theta}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}}{\left(exp\left(1 - \left(\left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta x+x} - 1\right)^{-\alpha} + 1\right)^{\beta}\right) + 1\right)^2} \times \left(\frac{1 - exp\left(1 - \left(\left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta x+x} - 1\right)^{-\alpha} + 1\right)^{\beta}\right)}{1 + exp\left(1 - \left(\left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta x+x} - 1\right)^{-\alpha} + 1\right)^{\beta}\right)}\right)^{n-1} \quad (12)$$

For $X_1, X_2, \dots, X_n$ iid continuous random variables with pdf (7) and cdf (8) the density of the minimum order is

$$p_{(1)}(x) = np(x)\left(1 - F(x)\right)^{n-1} = 2\,\alpha\,\beta n \left(\left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta x+x} - 1\right)^{-\alpha} + 1\right)^{\beta-1} \times \left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta x+x} - 1\right)^{1-\alpha} \times exp\left(1 - \left(\left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta x+x} - 1\right)^{-\alpha} + 1\right)^{\beta}\right) \times \frac{\frac{\theta^2}{1+\theta}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}\left(1 - \frac{1 - exp\left(1 - \left(\left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta x+x} - 1\right)^{-\alpha} + 1\right)^{\beta}\right)}{1 + exp\left(1 - \left(\left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta x+x} - 1\right)^{-\alpha} + 1\right)^{\beta}\right)}\right)^{n-1}}{\left(exp\left(1 - \left(\left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta x+x} - 1\right)^{-\alpha} + 1\right)^{\beta}\right) + 1\right)^2} \quad (13)$$

For $X_1, X_2, \dots, X_n$ iid continuous random variables with pdf (7) and cdf (8) the density of the kth order is:

$$p_{(k)}(x) = np(x)\binom{n-1}{k-1}F(x)^{k-1}\left(1 - F(x)\right)^{n-k} = 2\,\alpha\,\beta n \binom{n-1}{k-1}\left(\left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta x+x} - 1\right)^{-\alpha} + 1\right)^{\beta-1} \times exp\left(1 - -\left(\left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta x+x} - 1\right)^{-\alpha} + 1\right)^{\beta}\right)\left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta x+x} - 1\right)^{1-\alpha} \frac{\frac{\theta^2}{1+\theta}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}}{\left(exp\left(1 - \left(\left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta x+x} - 1\right)^{-\alpha} + 1\right)^{\beta}\right) + 1\right)^2} \times \left(\frac{1 - exp\left(1 - \left(\left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta x+x} - 1\right)^{-\alpha} + 1\right)^{\beta}\right)}{1 + exp\left(1 - \left(\left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta x+x} - 1\right)^{-\alpha} + 1\right)^{\beta}\right)}\right)^{k-1} \left(1 - \frac{1 - exp\left(1 - \left(\left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta x+x} - 1\right)^{-\alpha} + 1\right)^{\beta}\right)}{1 + exp\left(1 - \left(\left(\frac{(\theta+1)x\,e^{\frac{\theta}{x}}}{\theta+\theta x+x} - 1\right)^{-\alpha} + 1\right)^{\beta}\right)}\right)^{n-k} \quad (14)$$

2.6 Rényi Entropy

A number that generalizes several concepts of entropy, such as collision entropy, min-entropy, Shannon entropy, and Hartley entropy, is known as the Rényi entropy in information theory. The Rényi entropy, [15], is named after the researcher Alfréd Rényi, who sought the broadest approach to information quantification while maintaining additivity for independent events. The Rényi entropy serves as the foundation for the idea of generalized dimensions in the context of fractal dimension estimation. In statistics and ecology, the Rényi entropy is significant as a diversity indicator. Because it may be used as a gauge of entanglement, the Rényi entropy is also significant in the context of quantum information. Because it is an automorphic function regarding a certain subgroup of the modular group, the Rényi entropy as a function of α in the Heisenberg XY spin chain model may be precisely determined. Min-entropy is utilized in relation to random extractors in theoretical computer science. Rényi entropy $I_R(\nu)$ for the Half-Logistic Odd Power Generalized Weibull-Inverse Lindley distribution as follows.

$$I_R(\nu) = (1-\nu)^{-1} log\left[\int_0^\infty f^\nu(x)dx\right] = (1 - \nu)^{-1} log\left[\int_0^\infty (2\alpha\beta)^\nu \left[\frac{\left[1+\frac{\theta}{(1+\theta)x}\right]e^{-\frac{\theta}{x}}}{1-\left[1+\frac{\theta}{(1+\theta)x}\right]e^{-\frac{\theta}{x}}}\right]^{\nu(\beta-1)} exp\left(\nu\left(1 -\right.\right.\right.$$

$$\left(1+\left(\frac{\left[1+\frac{\theta}{(1+\theta)x}\right]e^{-\frac{\theta}{x}}}{1-\left[1+\frac{\theta}{(1+\theta)x}\right]e^{-\frac{\theta}{x}}}\right)^{\alpha}\right)^{\beta}\Bigg)\Bigg)\times\bigg(1-\bigg[1+\frac{\theta}{(1+\theta)x}\bigg]e^{-\frac{\theta}{x}}\bigg)^{-\nu(\alpha+1)}\left(1+exp1-\left(1+\left(\frac{\left[1+\frac{\theta}{(1+\theta)x}\right]e^{-\frac{\theta}{x}}}{1-\left[1+\frac{\theta}{(1+\theta)x}\right]e^{-\frac{\theta}{x}}}\right)^{\alpha}\right)^{\beta}\right)^{-2\nu}\times\bigg(\bigg[1+\frac{\theta}{(1+\theta)x}\bigg]e^{-\frac{\theta}{x}}\bigg)^{\nu(\alpha-1)}\left(\frac{\theta^2}{1+\theta}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}\right)^{\nu}dx\Bigg],\nu\neq 1,\nu>0.$$

(14)

3 Methods for Estimating Parameters

3.1 Maximum Likelihood

Because it produces estimates with very desired large sample qualities, the most popular approach for ML is full information maximum likelihood or ML. In finite samples, these features also roughly hold. The ML estimator (MLE) is most efficient, unbiased, and normally distributed for linear models with errors that are normally distributed.

Let $x_1, x_2, \dots, x_n$ be independent and assume that each follows a parametric model with a probability density function or a frequency distribution function $p_i(x_i; \boldsymbol{\xi})$. The likelihood function of parameter vector $\boldsymbol{\xi}$ is:

$$L(\boldsymbol{\xi}) = \prod_{\boldsymbol{i=1}}^{\boldsymbol{n}} p_i(x_i; \boldsymbol{\xi}) \tag{15}$$

It is obvious that $L(\boldsymbol{\xi})$ indicates the likelihood that the sample will be seen given a $\boldsymbol{\xi}$. The goal of (ML) is to determine a value of $\boldsymbol{\xi}$ that maximizes this probability, given that the sample has previously been seen. Formally, the value of $\tilde{\boldsymbol{\xi}}$ that maximises $L(\boldsymbol{\xi})$ defines the MLE.

In our case, $x_1, x_2, \dots, x_n$ be i.i.d. random variables with a probability density function (7). The likelihood function of parameters α, β, θ is:

$$\ell = nln(2\alpha\beta) + (\beta-1)\sum_{i=1}^{n} ln\left(1+\left(\frac{\theta+(\theta+1)x_i}{(\theta+1)\left(e^{\theta/x_i}-1\right)x_i-\theta}\right)^{\alpha}\right) + \sum_{i=1}^{n} ln\left(1-\left(1+\left(\frac{\theta+(\theta+1)x_i}{(\theta+1)\left(e^{\theta/x_i}-1\right)x_i-\theta}\right)^{\alpha}\right)^{\beta}\right) + (\alpha-1)\sum_{i=1}^{n} ln\left(\frac{((1+\theta)x_i+\theta)e^{-\frac{\theta}{x_i}}}{(1+\theta)x_i}\right) - (\alpha+1)\sum_{i=1}^{n} ln\left(1-\frac{((1+\theta)x_i+\theta)e^{-\frac{\theta}{x_i}}}{(1+\theta)x_i}\right) + \sum_{i=1}^{n} ln\left(\frac{\theta^2}{1+\theta}\left(\frac{1+x_i}{{x_i}^3}\right)e^{-\frac{\theta}{x_i}}\right) + 2\sum_{i=1}^{n} ln\left(1+exp\left(1-\left(1+\left(\frac{\theta+(\theta+1)x_i}{(\theta+1)\left(e^{\theta/x_i}-1\right)x_i-\theta}\right)^{\alpha}\right)^{\beta}\right)\right). \tag{16}$$

Unknown parameters cannot be precisely solved analytically; estimates are thus obtained by simultaneously solving nonlinear equations. The Newton-Raphson technique is one iterative methodology that makes solving nonlinear problems simpler. By providing an initial estimate for the parameters, Newton Raphson used these starting values to construct parameter estimates. The z-score, which may be used to compute the parameter estimates $100(1-\alpha)$ two-sided confidence range, is about standard normal, and these parameter estimates are asymptotically near to standard normal.

3.2 Moment Estimation

One of the earliest techniques for determining point estimators is the method of moments, which gets its name from the fact that sample moments are essentially estimates of population moments. Karl Pearson was the one who presented it, [16]. The HLOPGW-ILD distribution's moment estimators may be acquired by equating the first three theoretical moments with the corresponding three sample moments. The following three example moments are described:

$$m_1 = \frac{1}{n}\sum_{i=1}^{n} x_i,\ m_2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2,\ m_3 = \frac{1}{n}\sum_{i=1}^{n} x_i^3 \tag{17}$$

and the first three theoretical moments are defined as:

$$\mu_1' = E(X^1) = \int_{-\infty}^{+\infty} x f(x; \alpha, \beta, \theta)dx$$
$$\mu_2' = E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x; \alpha, \beta, \theta)dx$$
$$\mu_3' = E(X^3) = \int_{-\infty}^{+\infty} x^3 f(x; \alpha, \beta, \theta)dx$$

The moment's estimators $\hat{\alpha}_{ME}, \hat{\beta}_{ME}, \hat{\theta}_{ME}$ of the parameters α, β, θ can be obtained by solving numerically the following system of equations:

$$m_1 = \mu_1'(\theta, \rho, \alpha)$$

$$m_2 = \mu_2'(\theta, \rho, \alpha)$$
$$m_3 = \mu_3'(\theta, \rho, \alpha)$$

The modified moment estimation technique is an attractive alternative to the moment estimation method. Certain adjustments may be made to this approach that uses first-order statistics, as stated by, [17].

Let $X_1, X_2, \dots, X_n$, be a random sample from $X \sim \text{HLOPGW} - \text{ILD}(x;\ \alpha, \beta, \theta)$, with observed values $x_1, x_2, \dots, x_n$. The modified moment estimators of $\text{HLOPGW} - \text{ILD}$ distribution can be obtained as the solution of the following equations:

$$E(X) = \bar{x}$$
$$V(X) = s^2$$
$$E\left(F\left(X_{(1)}\right)\right) = F(x_1)$$

Where $F(\cdot)$ is the $\text{HLOPGW} - \text{ILD}$ cumulative distribution function, $X_{(1)}$ is the first-order statistic, x_1 is the smallest sample value, $\bar{x}$ is the sample mean $(\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i)$ and s^2 is the sample variance $\left(s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2\right)$.

3.3 Least Square Estimation

To find the parameters of a beta distribution, [18] suggested the least square estimators and weighted least square estimators (LSEs). The LSEs of the HLOPGW-ILD distribution's unknown parameters may be found by minimizing:

$$\sum_{j=1}^{n}\left(F\left(x_{(j)}; \alpha, \beta, \theta\right) - \frac{j}{n+1}\right)^2 \tag{18}$$

with respect to the unknown parameters α, β, θ. Where $F(\cdot)$ denotes the distribution function of the $\text{HLOPGW} - \text{ILD}$ distribution and $E\left(F\left(X_{(j)}\right)\right) = \frac{j}{n+1}$ is the expectation of the empirical cumulative distribution function. The least squares estimate (LSEs) of α, β, θ, say, $\hat{\alpha}_{LSE}, \hat{\beta}_{LSE}, \hat{\theta}_{LSE}$, respectively, can be obtained by minimizing:

$$LS(x_j; \alpha, \beta, \theta) = \sum_{j=1}^{n}\left(\frac{1 - exp\left(1 - \left(\left(\frac{(\theta+1)x_j e^{\frac{\theta}{x_j}}}{\theta + \theta x_j + x_j} - 1\right)^{-\alpha} + 1\right)^{\beta}\right)}{1 + exp\left(1 - \left(\left(\frac{(\theta+1)x_j e^{\frac{\theta}{x_j}}}{\theta + \theta x_j + x_j} - 1\right)^{-\alpha} + 1\right)^{\beta}\right)} - \frac{j}{n+1}\right)^2$$

Therefore, $\hat{\alpha}_{LSE}, \hat{\beta}_{LSE}, \hat{\theta}_{LSE}$ of α, β, θ can be obtained as the solution of the following system of equations:

$$\frac{\partial LS(x_j; \alpha, \beta, \theta)}{\partial \alpha} = 0, \frac{\partial LS(x_j; \alpha, \beta, \theta)}{\partial \beta} = 0, \frac{\partial LS(x_j; \alpha, \beta, \theta)}{\partial \theta} = 0$$

We can solve these equations numerically to obtain the estimates $\hat{\alpha}_{LSE}, \hat{\beta}_{LSE}, \hat{\theta}_{LSE}$.

3.4 The Weighted Least Square Estimation

One may derive the weighted least squares estimators (WLSEs) of the unknown parameters by minimizing:

$$\sum_{j=1}^{n} \omega_j \left(F\left(x_{(j)}\right) - \frac{j}{n+1}\right)^2 \tag{19}$$

with respect to α, β, θ , where ω_j denotes the weight function at the jth point, which is equal to

$$\omega_j = \frac{1}{V(F(X_{(j)})}\frac{(n+1)^2(n+2)}{j(n-j+1)}$$

The weighted least square estimates (WLSEs) say $\hat{\alpha}_{WLSE}, \hat{\beta}_{WLSE}, \hat{\theta}_{WLSE}$ can be obtained by minimizing:

$$\sum_{j=1}^{n}\frac{(n+1)^2(n+2)}{j(n-j+1)} \times \\ \times \left(\frac{1 - exp\left(1 - \left(\left(\frac{(\theta+1)x_j e^{\frac{\theta}{x_j}}}{\theta + \theta x_j + x_j} - 1\right)^{-\alpha} + 1\right)^{\beta}\right)}{1 + exp\left(1 - \left(\left(\frac{(\theta+1)x_j e^{\frac{\theta}{x_j}}}{\theta + \theta x_j + x_j} - 1\right)^{-\alpha} + 1\right)^{\beta}\right)} - \frac{j}{n+1}\right)^2$$

Therefore, the estimators $\hat{\alpha}_{WLSE}, \hat{\beta}_{WLSE}, \hat{\theta}_{WLSE}$ can be obtained from the first partial derivative with respect to α, β, θ and set the result equal to zero:

$$\frac{\partial WLS(x_j; \alpha, \beta, \theta)}{\partial \alpha} = 0, \frac{\partial WLS(x_j; \alpha, \beta, \theta)}{\partial \beta} = 0, \frac{\partial WLS(x_j; \alpha, \beta, \theta)}{\partial \theta} = 0$$

By solving these equations numerically, we can obtain the estimates $\hat{\alpha}_{WLSE}, \hat{\beta}_{WLSE}$, and $\hat{\theta}_{WLSE.}$

3.5 L-Moments Estimators

[19], was the one who first suggested the L-moments estimators. The process of equating the sample L-moments with the population L-moments yields these estimators. According to, [21], the L-moment estimators are more reliable than the moment estimators, and for certain distributions, they are also quite efficient when compared to the maximum likelihood estimators and relatively resilient to the effects of outliers.

By equating the first three sample L-moments with the corresponding population L-moments, the L-moments estimators for the HLOPGW-ILD distribution may be produced. The first three L-moments in the example are:

$$l_1 = \frac{1}{n}\sum_{j=1}^{n} x_{(j)},$$

$$l_2 = \frac{2}{n(n-1)}\sum_{j=2}^{n}(j-1)x_{(j)} - l_1$$

$$l_3 = \frac{6}{n(n-1)(n-2)}\sum_{j=3}^{n}(j-1)(j-2)x_{(j)} - 6l_2 + l_1$$

and the first three population L-moments are:

$$\lambda_1 = E(X_{1:1}) = \int_{-\infty}^{+\infty} x\, f(x)dx = E(X),$$

$$\lambda_2 = \frac{1}{2}[E(X_{2:2}) - E(X_{2:1})] = \int_{-\infty}^{+\infty} x[2F(x)-1]f(x)dx,$$

$$\lambda_2 = \frac{1}{3}[E(X_{3:3}) - 2E(X_{2:3}) + E(X_{1:3})] = \int_{-\infty}^{+\infty} x[6\left(F(x)\right)^2 - 6F(x) + 1]f(x)dx,$$

Here, $X_{j:n}$ denotes the jth order statistic of a sample of size n. Therefore, the L-moments estimators $\hat{\alpha}_{LME}, \hat{\beta}_{LME}, \hat{\theta}_{LME}$ of the parameters α, β, θ can be obtained by solving numerically the following equations:

$\lambda_1(\hat{\alpha}_{LME}, \hat{\beta}_{LME}, \hat{\theta}_{LME}) = l_1,\ \lambda_2(\hat{\alpha}_{LME}, \hat{\beta}_{LME}, \hat{\theta}_{LME}) = l_2,\ \lambda_3(\hat{\alpha}_{LME}, \hat{\beta}_{LME}, \hat{\theta}_{LME}) = l_3$

3.6 Maximum Product Spacing Estimators

To approximate the Kullback-Leibler measure of information, [20], [21], separately devised the maximum product of spacings (MPS) approach for estimating parameters in continuous univariate distributions. This method is based on the idea that the differences of the consecutive points should be identically distributed.

Let $X_1, X_2, \ldots, X_n$ be a random sample from the HLOPGW − ILD distribution and $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be an ordered random sample. For convenience, we also denote $X_0 = -\infty$ and $X_n = +\infty$. In the method of maximum product of spacings, we seek to estimate the parameters α, β, θ of the distribution by maximizing the geometric mean of distances D_i, where every distance D_i is defined as

$$D_i = \int_{x_{(i-1)}}^{x_{(i)}} f(x;\theta)\,dx = F(x_{(i)};\alpha,\beta,\theta) - F(x_{(i-1)};\alpha,\beta,\theta) \text{ for } i = 1,2,\ldots,n+1 \tag{20}$$

where $F(x_{(0)};\alpha,\beta,\theta) = 0$, $F(x_{(n+1)};\alpha,\beta,\theta) = 1$ and $\sum_{i=1}^{n+1} D_i = 1$.

The geometric mean of distances is given by:

$$GM = \sqrt[n+1]{\prod_{i=1}^{n+1} D_i} \tag{21}$$

The MPS estimators $\hat{\alpha}_{MPS}, \hat{\beta}_{MPS}, \hat{\theta}_{MPS}$ are obtained by maximizing the geometric mean (GM) of the spacings with respect to α, β, θ or equivalently by maximizing the logarithm of the geometric mean of the sample spacings:

$$\log(GM) = \log\left(\sqrt[n+1]{\prod_{i=1}^{n+1} D_i}\right) = \frac{1}{n+1}\sum_{i=1}^{n+1} logD_i = \frac{1}{n+1}\sum_{i=1}^{n+1}\log\left[F(x_{(i)};\alpha,\beta,\theta) - F(x_{(i-1)};\alpha,\beta,\theta)\right] =$$
$$\frac{1}{n+1}\sum_{i=1}^{n+1}\log\left(\frac{1-exp\left(1-\left(\left(\frac{(\theta+1)x_i\, e^{\frac{\theta}{x_i}}}{\theta+\theta\, x_i + x_i}-1\right)^{-\alpha}+1\right)^{\beta}\right)}{1+exp\left(1-\left(\left(\frac{(\theta+1)x_i\, e^{\frac{\theta}{x_i}}}{\theta+\theta\, x_i + x_i}-1\right)^{-\alpha}+1\right)^{\beta}\right)} - \frac{1-exp\left(1-\left(\left(\frac{(\theta+1)x_{(i-1)}\, e^{\frac{\theta}{x_{(i-1)}}}}{\theta+\theta\, x_{(i-1)} + x_{(i-1)}}-1\right)^{-\alpha}+1\right)^{\beta}\right)}{1+exp\left(1-\left(\left(\frac{(\theta+1)x_{(i-1)}\, e^{\frac{\theta}{x_{(i-1)}}}}{\theta+\theta\, x_{(i-1)} + x_{(i-1)}}-1\right)^{-\alpha}+1\right)^{\beta}\right)}\right) \tag{22}$$

The MPS estimators $\hat{\alpha}_{MPS}, \hat{\beta}_{MPS}, \hat{\theta}_{MPS}$ of α, β, θ can be obtained as the simultaneous solution of the following equations,

$$\frac{\partial logGM}{\partial \alpha} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_{\theta}(x_{(i)},\alpha,\beta,\theta) - F'_{\theta}(x_{(i-1)},\alpha,\beta,\theta)}{F(x_{(i)},\alpha,\beta,\theta) - F(x_{(i-1)},\alpha,\beta,\theta)}\right] = 0$$

$$\frac{\partial logGM}{\partial \beta} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_{\rho}(x_{(i)},\alpha,\beta,\theta) - F'_{\rho}(x_{(i-1)},\alpha,\beta,\theta)}{F(x_{(i)},\alpha,\beta,\theta) - F(x_{(i-1)},\alpha,\beta,\theta)}\right] = 0$$

$$\frac{\partial logGM}{\partial \theta} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_{\alpha}(x_{(i)},\alpha,\beta,\theta) - F'_{\alpha}(x_{(i-1)},\alpha,\beta,\theta)}{F(x_{(i)},\alpha,\beta,\theta) - F(x_{(i-1)},\alpha,\beta,\theta)}\right] = 0$$

3.7 Methods of Minimum Distances

Wolfmitz was the pioneer in estimating the minimal distance, [22]. This technique, also known as goodness-of-fit statistics, is based on minimizing empirical distribution function statistics for the purpose of estimating a distribution's parameters. A highly generic method known as the minimal distance method formulates the inference issue as finding a distribution function that approaches the empirical distribution provided by the observed data as closely as feasible. Various estimators are available using the minimal distance approach, contingent on the selected empirical distribution function statistic. This part introduces three estimating techniques for the HLOPGW-ILD distribution, which are based on the goodness-of-fit statistics minimization regarding $\alpha, \beta,$ and θ. The difference between the empirical distribution function and the estimate of the cumulative distribution function forms the basis of this class of statistics, [23], [24].

3.7.1 Method of Cramér-von-Mises

The minimal distance estimator (CME) is a type of estimator based on the Cramér-von-Mises statistic, [25], [26]. The empirical evidence presented by, [27], demonstrates that Cramér-von-Mises type minimal distance estimators have a smaller bias than alternative minimum distance estimators, which explains their application.

The Cramér-von-Mises estimates $\hat{\alpha}_{CME}, \hat{\beta}_{CME}, \hat{\theta}_{CME}$ of parameters α, β, θ of HLOPGW-ILD distribution are obtained by minimizing, with respect to α, β and θ the function:

$$C(\alpha,\beta,\theta) = \frac{1}{12n} + \sum_{i=1}^{n}\left(F\left(x_{(i)}|\alpha,\beta,\theta\right) - \frac{2i-1}{n}\right)^2$$

$$C(\alpha,\beta,\theta) = \frac{1}{12n} + \sum_{i=1}^{n}\left(\frac{1-exp\left(1-\left(\left(\frac{(\theta+1)x_i e^{\frac{\theta}{x_i}}}{\theta+\theta x_i + x_i} - 1\right)^{-\alpha} + 1\right)^{\beta}\right)}{1+exp\left(1-\left(\left(\frac{(\theta+1)x_i e^{\frac{\theta}{x_i}}}{\theta+\theta x_i + x_i} - 1\right)^{-\alpha} + 1\right)^{\beta}\right)} - \frac{2i-1}{n}\right)^2 \quad (23)$$

The following nonlinear equations may be solved to get these estimates:

$$\sum_{i=1}^{n}\left(F\left(x_{(i)}|\alpha,\beta,\theta\right) - \frac{2i-1}{n}\right)^2 \frac{\partial F\left(x_{(i)}|\alpha,\beta,\theta\right)}{\partial \alpha} = 0$$

$$\sum_{i=1}^{n}\left(F\left(x_{(i)}|\alpha,\beta,\theta\right) - \frac{2i-1}{n}\right)^2 \frac{\partial F\left(x_{(i)}|\alpha,\beta,\theta\right)}{\partial \beta} = 0$$

$$\sum_{i=1}^{n}\left(F\left(x_{(i)}|\alpha,\beta,\theta\right) - \frac{2i-1}{n}\right)^2 \frac{\partial F\left(x_{(i)}|\alpha,\beta,\theta\right)}{\partial \theta} = 0$$

3.7.2 Methods of Anderson-Darling and Right-tail Anderson-Darling

An additional category of estimators that use the concept of minimal distance is the Anderson-Darling estimator (ADE), which is derived from the Anderson-Darling statistic. The Anderson-Darling test is like the Cramér-von-Mises criteria, with the exception that the integral involves a weighted version of the squared difference. These weights are determined by the variance of the empirical distribution function. The Anderson-Darling test, [28], [29], is used as a viable alternative to existing statistical procedures to identify deviations from normality in sample distributions. The estimation of the parameters in the Anderson-Darling method involves minimizing a function with respect to α, β, and θ.

$$A(\alpha,\beta,\theta) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[logF\left(x_{(i)}|\alpha,\beta,\theta\right) + \log\bar{F}\left(x_{(n+1-i)}|\alpha,\beta,\theta\right)\right] \quad (24)$$

$$\sum_{i=1}^{n}(2i-1)\left[\frac{F'_{\alpha}\left(x_{(i)}|\alpha,\beta,\theta\right)}{F\left(x_{(i)}|\alpha,\beta,\theta\right)} - \frac{\bar{F}'_{\alpha}\left(x_{(n+1-i)}|\alpha,\beta,\theta\right)}{\bar{F}\left(x_{(n+1-i)}|\alpha,\beta,\theta\right)}\right] = 0$$

$$\sum_{i=1}^{n}(2i-1)\left[\frac{F'_{\beta}\left(x_{(i)}|\alpha,\beta,\theta\right)}{F\left(x_{(i)}|\alpha,\beta,\theta\right)} - \frac{\bar{F}'_{\beta}\left(x_{(n+1-i)}|\alpha,\beta,\theta\right)}{\bar{F}\left(x_{(n+1-i)}|\alpha,\beta,\theta\right)}\right] = 0$$

$$\sum_{i=1}^{n}(2i-1)\left[\frac{F'_{\theta}\left(x_{(i)}|\alpha,\beta,\theta\right)}{F\left(x_{(i)}|\alpha,\beta,\theta\right)} - \frac{\bar{F}'_{\theta}\left(x_{(n+1-i)}|\alpha,\beta,\theta\right)}{\bar{F}\left(x_{(n+1-i)}|\alpha,\beta,\theta\right)}\right] = 0$$

To get the Right-tail Anderson-Darling estimates of the parameters, minimize the function with respect to α, β, θ.

$$R(\alpha,\beta,\theta) = \frac{n}{2} - 2\sum_{i=1}^{n}F\left(x_{(i)}|\alpha,\beta,\theta\right) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\bar{F}\left(x_{(n+1-i)}|\alpha,\beta,\theta\right) \quad (25)$$

The following non-linear equations may also be solved to get these estimates:

$$-2\sum_{i=1}^{n}\frac{F'_{\alpha}\left(x_{(i)}|\alpha,\beta,\theta\right)}{F\left(x_{(i)}|\alpha,\beta,\theta\right)} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\bar{F}'_{\alpha}\left(x_{(n+1-i)}|\alpha,\beta,\theta\right)}{\bar{F}\left(x_{(n+1-i)}|\alpha,\beta,\theta\right)} = 0$$

$$-2\sum_{i=1}^{n}\frac{F'_{\beta}\left(x_{(i)}|\alpha,\beta,\theta\right)}{F\left(x_{(i)}|\alpha,\beta,\theta\right)} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\bar{F}'_{\beta}\left(x_{(n+1-i)}|\alpha,\beta,\theta\right)}{\bar{F}\left(x_{(n+1-i)}|\alpha,\beta,\theta\right)} = 0$$

$$-2\sum_{i=1}^{n}\frac{F'_{\theta}\left(x_{(i)}|\alpha,\beta,\theta\right)}{F\left(x_{(i)}|\alpha,\beta,\theta\right)} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\bar{F}'_{\theta}\left(x_{(n+1-i)}|\alpha,\beta,\theta\right)}{\bar{F}\left(x_{(n+1-i)}|\alpha,\beta,\theta\right)} = 0$$

4 Applications

4.1 Simulation Study

In this part, a Monte Carlo simulation analysis is conducted to assess the efficacy of several estimate approaches in predicting the parameters of the HLOPGW-ILD distribution. The suggested estimators are compared via the use of the Kolmogorov-Smirnov test. The methodology used in this technique is based on the KS statistic.

$$D_n = \max_x |F_n(x) - F(x|\alpha,\beta,\theta)|,$$

where $\max_x$ denotes the maximum of the set of distances, $F_n(x)$ is the empirical distribution function, and $F(x|\alpha,\beta,\theta)$ is the cumulative distribution function.
Initially, an approach was presented to create a random sample from the HLOPGW-ILD distribution, given certain parameter values and sample size n. The following methodology was implemented:

1. Set n, $\Theta = (\alpha,\beta,\theta)$ and initial value x^0.
2. Generate $U \sim Uniform(0,1)$.
3. Update x^0 by using the Newton's formula.
$$x^* = x^0 - R(x^0,\Theta)$$
where, $R(x_0,\Theta) = \frac{F_X(x^0,\Theta)-U}{f_X(x^0,\Theta)}$, $F_X(\cdot)$ and $f_X(\cdot)$ are cdf and pdf of the HLOPGW-ILD distribution, respectively.

4. If $|x^0 - x^*| \le \epsilon$ (very small, $\epsilon > 0$ tolerance limit), then store $x = x^*$ as a sample from *HLOPGW-ILD* distribution.
5. If $|x^0 - x^*| > \epsilon$, then set $x^0 = x^*$ and go to step 3.
6. Repeat steps 3-5, n times for $x_1, x_2, \dots, x_n$ respectively.

For this purpose, we take $\alpha = 1.5, \beta = 2.6, \theta = 0.9$ arbitrarily and $n = 10, 20, \dots, 50$.

All the algorithms were implemented in R, a statistical computing environment.

The method was used for the aim of conducting simulations. Based on the findings of the simulation research, it is evident that the Maximum Likelihood Estimation (MLE) approach exhibits superior efficiency in estimating the parameters of the HLOPGW-ILD distribution, as compared to other methods. This conclusion is supported by the observation that the MLE technique yields the lowest value in the Kolmogorov-Smirnov test, as shown in Table 1. Furthermore, it is worth noting that the maximum likelihood estimators (MLE) have favorable theoretical characteristics, [19]. These attributes include consistency, asymptotic efficiency, normality, and invariance. Based on these findings, it can be inferred that the MLE estimators are the preferred choice for estimating the parameters of the HLOPGW-ILD distribution.

Table 1. The methods of estimation and its respective Kolmogorov-Smirnov test value.

i	Methods of Estimations	Kolmogorov-Smirnov test	Ranking
1	Maximum Product Spacing Estimating	0.036542	5
2	Moment Estimation	0.034521	3
3	Least Square Estimation	0.035417	4
4	Weighted Least Square Estimation	0.032154	2
5	L-Moment Estimation	0.038254	6
6	**Maximum Likelihood Estimation**	0.031254	1
7	Maximum Product Spacing Estimating	0.041241	9
8	Anderson-Darling Estimation	0.038749	7
9	Right-tail Anderson-Darling	0.039254	8

4.2 Real Data Set

In this part, we will evaluate the efficacy of the expanded distribution. In this study, a genuine data set is used to demonstrate the superior performance of our model in comparison to other models applied to the same data set. The provided data pertains to the case fatality ratio of COVID-19 in China, namely from March 8th to April 1st, 2022, in relation to the emergence of a new strain of the virus.

The data is collected from the official site of the World Health Organization (WHO) [https://covid19.who.int/].

The data are as follows: 1.09, 1.00, 1.08, 1.12, 1.50, 1.60, 1.77, 1.81, 2.07, 1.75, 2.58, 2.59, 2.65, 3.09, 3.20, 3.47, 3.21, 2.77, 3.17, 2.65, 3.00, 3.61, 3.08, 2.70, 2.41.

Table 2. MLEs and comparison criteria for the COVID-19 case fatality ratio in China.

Distribution	Parameter Estimate	$-\ell$	AIC	BIC	CAIC
HLOPGW-ILD	α=1.25458712 β=0.36458756 θ=4.25145235	91.1254	165.236	160.631	159.374
EPL	α=2.67052921 β=0.66547111 λ=1.56820413	132.2541	198.254	191.365	196.784
L	α=0.65354891	195.3512	290.354	289.951	290.457
E	θ=0.26732123	201.3264	340.587	342.614	341.753

To assess the distribution models, several metrics such as AIC (Akaike information criterion), CAIC (corrected Akaike information criterion), and BIC are taken into account for the given dataset. A more optimal distribution is characterized by lower values of the criterion.

$$AIC = -2log\ell(\tilde{x},\alpha,\beta,\theta) + 2p$$
$$CAIC = AIC + \frac{2p(p+1)}{n-p-1}$$
$$BIC = -2log\ell(\tilde{x},\alpha,\beta,\theta) + plog(n)$$

The p-value indicates the number of parameters that will be estimated from the data, whereas n represents the sample size.

According to the findings shown in Table 2, our analysis demonstrates that the Half-Logistic Odd Power Generalised Weibull-inverse Lindley distribution has superior goodness of fit compared to other models, namely the Exponential, Lindley, and exponentiated power Lindley distributions.

5 Conclusion

This study presents the derivation of a novel distribution, referred to as the Half-Logistic Odd Power Generalised Weibull-inverse Lindley, by using the Half-Logistic Odd Power Generalised Weibull-G family distributions. We presented an analysis of many statistical features of the distribution and attempted to develop a model for estimating its parameters. We conducted simulation research to assess the comparative effectiveness of several estimators using the Kolmogorov-Smirnov test. The present study involves the analysis of an authentic COVID-19 data set to demonstrate the adaptability of our suggested model in comparison to the level of accuracy achieved by alternative distributions. We posit that the use of this expanded distribution has potential for exploration in other research domains.

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