The Limited Validity of the Fractional Euler Finite Difference Method and an Alternative Definition of the Caputo Fractional Derivative to Justify Modification of the Method

DOMINIC P. CLEMENCE-MKHOPE, ZACHARY DENTON
Department of Mathematics and Statistics,
North Carolina A&T State University,
1601 East Market Street, Greensboro, NC 27411,
USA

Abstract: - A method, advanced as the fractional Euler finite difference method (FEFDM), a general method for the finite difference discretization of fractional initial value problems (IVPs) for $0 < \alpha \leq 1$ for the Caputo derivative, is shown to be valid only for $\alpha = 1$. This is accomplished by establishing, through a recently proposed generalized difference quotient representation of the fractional derivative, that the FEFDM is valid only if a property of the Mittag-Leffler function holds that has only been shown to be valid only for $\alpha = 1$. It is also shown that the FEFDM is inconsistent with the exact discretization of the IVP for the Caputo fractional relaxation equation. The generalized derivative representation is also used to derive a modified generalized Euler's method, its nonstandard finite difference alternative, their improved Euler versions, and to recover a recent result by Mainardi relating the Caputo and conformable derivatives.

Key-Words: - Caputo fractional derivative, fractional Euler finite difference method (FEFDM), modified FEFDM, fractional initial value problem, fractional relaxation equation

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1 Introduction

The Caputo fractional derivative (FD) is defined, [1], as

$$ {}_0^C D_t^\alpha (f(t)) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-x)^{-\alpha} \frac{d}{dx} f(x) dx \qquad (1)$$

for $0 < \alpha \leq 1$, with which this paper is concerned. It is one of the classical fractional derivatives, [2], upon which most theory on the subject has been developed by various authors and it has been widely applied across many areas of science and engineering, [3], and references therein for example).

Since most fractional differential equations systems do not have exact analytic solutions, numerical approximation methods must be developed. Among the many numerical methods, [4], [5], [6], [7], [8], [9], that have been proposed to solve initial value problems (IVPs) for differential equations with the Caputo FD,

$$D_t^\alpha y(t) = f(t; y(t)), y(t_0) = y_0, \ a \leq t \leq b \qquad (2)$$

the finite difference method

$$y_{k+1} - y_k = \frac{1}{\Gamma(\alpha+1)} h^\alpha f\big(t_k, y(t_k)\big), \ 0 \leq t \leq N,$$
where $h = \frac{b-a}{N}$, (3)

was proposed in, [5], which has been widely cited. Method (3), which we will refer to as the fractional Euler finite difference method (FEFDM) (and is often referred to as the generalized Euler method), has been used in direct applications, [10], [11], [12], [13], as well as in developing other methods, [14], [15], [16], [17], [18], for the discretization of the fractional IVP (2). It is justified by applying Taylor's expansion for the Caputo FD, which is proposed in, [19], and is used to develop an algorithm (4) below for solving the IVP (2):

$$\begin{aligned} y(t_k) &= \frac{h^\alpha}{\Gamma(\alpha+2)}\big((k-1)^{\alpha+1} - (k-\alpha-1)k^\alpha\big) f\big(t_0, y(t_0)\big) + y(t_0) \\ &+ \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{i=1}^{k-1} \big((k-i+1)^{\alpha+1} - 2(k-i)^{\alpha+1} + (k-i-1)^{\alpha+1}\big) f\big(t_i, y(t_i)\big) \\ &+ \frac{h^\alpha}{\Gamma(\alpha+2)} f\left(t_k, y(t_{k-1}) + \frac{h^\alpha}{\Gamma(\alpha+1)} f\big(t_{k-1}, y(t_{k-1})\big)\right), \end{aligned} \qquad (4)$$

which will be referred to as the Odibat-Momani finite difference method (OMFDM).

The purpose of this article is to show that the FEFDM, method (3), as a model for (2) is valid only for $\alpha = 1$. Therefore, any algorithm based on it, such as the OMFM (4) wherein the FEFDM (3) is used as a predictor and a modified trapezoidal rule as corrector, is consequently valid only at $\alpha = 1$. A modified fractional Euler finite difference method (MFEFDM

$$y_{k+1} - y_k = \frac{1}{\alpha}\Big(E_\alpha(-{t_k}^\alpha)/ E_{\alpha,\alpha}(-{t_k}^\alpha)\Big)\left(({t_{k+1}})^\alpha - {t_k}^\alpha\right) f\big(t_k, y(t_k)\big), \tag{5}$$

as well as its improved Euler counterpart, are proposed, where $E_\alpha(.)$ and $E_{\alpha,\beta}(.)$ are respectively the one- and two-parameter Mittag-Leffler (ML) functions. This modification is accomplished using the expression (6) below, recently proposed in, [20], of the Caputo FD in terms of the ML function:

$${}_0^C D_t^\alpha f(t) = \lim_{h\to 0} \frac{y(t+h) - y(t)}{\mu(h,t,\alpha)},$$

where $\mu(h,t,\alpha) = \left(1 - \frac{\mathrm{E}_\alpha(-\,(t+h)^\alpha)}{\mathrm{E}_\alpha(-t^\alpha)}\right)$ (6)

The rest of this article is organized as follows. In the next section, the ESDDFD method for difference quotient representations for non-integer derivatives, generalized fractional derivatives, and subsequent Euler method extensions are recalled. Section 3 recalls the derivation of the FEM from, [1], and it is shown that it is valid only for $\alpha = 1$. In Section 4, the ESDDFD method difference quotient representations and generalized fractional derivatives for the Caputo FD, and subsequent Euler method extensions are presented. Numerical experiments are presented in Section 5 assessing the accuracy, against analytic solutions, of the FEFDM and MFEFDM for two examples. A discussion in Section 6 of the theoretical and experimental results presented, as well as recommendations based on those results, concludes the article.

2 ESDDFD Difference Quotient Representation

The exact spectral derivative discretization finite difference (ESDDFD) method was introduced in, [20], in which it is generally assumed that the IVP (2) is being discretized on intervals of the form $[0, b]$ and the following were presented.

Definition 2.1. For a given definition of an FD, let $\mho(t, \alpha; y_0)$ denote the analytic solution of IVP for the fractional relaxation equation (FRE):

$$D_t^\alpha y(t) = -y(t), y(0) = y_0,$$
$$0 \le t \le b;\ 0 < \alpha \le 1. \tag{7}$$

Then a corresponding difference quotient representation (DQR) of the Caputo type consistent with that derivative is

$${}_0^{GC}\Delta_t^\alpha[y(t)] = \frac{y(t+h) - y(t)}{\left(1 - \mho(t+h,\alpha;y_0)/\mho(t,\alpha;y_0)\right)}$$

Taking the limit as $h \to 0$ in the equation above yields the following alternative definition of the derivative associated with $\mho(t, \alpha; y_0)$:

Definition 2.2. Given a real-valued function on $[0, \infty)$, the generalized fractional derivative (GFD) associated with $\mho(t, \alpha; y_0)$ has the following alternative definition:

$${}_0^{GC}D_t^\alpha[f(t)] \equiv \lim_{h\to 0} {}_0^{GC}\Delta_t^\alpha[y(t)],$$

where ${}_0^{GC}D_t^\alpha[f(0)]$ is understood to mean ${}_0^{GC}D_t^\alpha[f(0)] = \lim_{t\to 0^+} {}_0^{GC}D_t^\alpha[f(t)]$.

The identifications

$$t \longrightarrow t_k, t + h \longrightarrow t_{k+1},$$
$$y(t+h) \longrightarrow y_{k+1}, y(t) \longrightarrow y_k \tag{8}$$

applied in Definitions 2.1 and 2.2 yields the following discretization rule for $D_t^\alpha y(t)$ as a corollary.

Corollary 2.1. Let $\mho(t, \alpha; y_0)$ be as in Definitions 2.1. Then the following are consistent discrete Euler and nonstandard finite difference (NSFD) representations of ${}_0^{GC}D_t^\alpha y(t)$:

Generalized Fractional Euler:

$${}_0^{GC}D_t^\alpha y(t) \longrightarrow \frac{y_{k+1} - y_k}{\lim_{h\to 0} \mu(h, t_k, \alpha)},$$

Generalized Fractional NSFD:

$${}_0^{GC}D_t^\alpha y(t) \longrightarrow \frac{y_{k+1} - y_k}{\mu(h, t_k, \alpha)},$$

where $\mu(h, t_k, \alpha)$ is defined as follows:

$$\mu(h, t_k, \alpha) = \big(1 - \mho(t_{k+1}, \alpha; y_0)/\mho(t_k, \alpha; y_0)\big)$$

The denominator function $\mu(h, t_k, \alpha)$ in Corollary 2.1 is a complex function of both the step size $h = t_{k+1} - t_k$ and lattice point t_k, and is described in [20], as a fractional generalization of the nonstandard finite difference (NSFD) denominator, [21].

Clear corollaries to the foregoing are the following Euler discretization rules for the IVP (2), which provide justified extensions of the Euler method to the GFD:

Corollary 2.2. The following discrete representations are fractional generalizations of the Euler finite difference method for the GFD valid for $\alpha \in (0,1]$:

Generalized FEFDM:

$$\frac{y_{k+1}-y_k}{\lim_{h\to 0}\mu(h,t_k,\alpha)} = f(t_k,y_k)$$

Generalized Fractional NSFD Method:

$$\frac{y_{k+1}-y_k}{\mu(h,t_k,\alpha)} = f(t_k,y_k)$$

The following alternate definition of the conformable fractional derivative (CFD), [22], given in, [23], will be used to arrive at the MFEFDM (5). It can be obtained by setting $\mho(t,\alpha) = \exp\left(-\frac{1}{\alpha}t^\alpha\right)$, in Definitions 2.1 and 2.2 above.

Definition 2.3. Given a real-valued function on $[0,\infty)$, the conformable fractional derivative has the following alternative definition:

$$ {}_0^C T_t^\alpha[f(t)] \equiv \lim_{h\to 0} {}^{CFD}_{\ 0}\Delta_t^\alpha[y(t)] = \alpha \lim_{h\to 0}\frac{y(t+h)-y(t)}{[(t+h)^\alpha - t^\alpha]}$$

where ${}_0^C T_t^\alpha[f(0)]$ is understood to mean ${}_0^C T_t^\alpha[f(0)] = \lim_{t\to 0^+} {}_0^C T_t^\alpha[f(t)]$.

The following, which gives the relationship between the integer derivative and NIDs, will also be used to arrive at the MFEFDM, given by (5).

Proposition 2.1 If $f(t)$ and $\mho(t_0,t,y_0)$ are both first-order differentiable, then the following also holds:

$${}^{GC}_{\ 0}D_t^\alpha[f(t)] = -\lambda\mho(t_0,t,y_0)\frac{1}{\frac{d\mho(t_0,t,y_0)}{dt}}\frac{df(t)}{dt}$$

Proof

The proof follows directly from Definitions 2.1 and 2.2:

$${}^{GC}_{\ 0}D_t^\alpha[f(t)] = \lim_{h\to 0}\frac{f(t+h)-f(t)}{\frac{1}{\lambda}\left(1-\mho(t_0,t+h,y_0)/\mho(t_0,t,y_0)\right)} = \lim_{h\to 0}\frac{f(t+h)-f(t)}{\frac{1}{\lambda}\left[\left(\frac{\mho(t_0,t,y_0)-\mho(t_0,t+h,y_0)}{\mho(t_0,t,y_0)}\right)\right]}$$

$$= -\mho(t_0,t,y_0)\lim_{h\to 0}\frac{1}{\frac{1}{\lambda}\frac{\mho(t_0,t+h,y_0)-\mho(t_0,t,y_0)}{h}}\frac{f(t+h)-f(t)}{h} = \frac{-\lambda\mho(t_0,t,y_0)}{\mho'(t_0,t,y_0)}\frac{df(t)}{dt}$$

3 The Generalized Fractional Euler Finite Difference Method (FEFDM)

In this section, justification of the generalized fractional Euler finite difference method is recalled, and proof is presented of its limited validity.

3.1 Justification of the FEFDM

The FEFDM, given by (3), for the IVP (2), is obtained in, [5], by considering a Caputo FD power series expansion, [19], as follows. It is assumed that for each t there exists c_1 so that the following is true:

$$y(t) - y(t_0) = \frac{1}{\Gamma(\alpha+1)}t^\alpha (D_t^\alpha y)(t_0) + \frac{1}{\Gamma(2\alpha+1)}t^{2\alpha}(D_t^{2\alpha}y)(c_1). \tag{9}$$

Letting $y(t_{k+1}) - y(t_k) \longrightarrow y_{k+1} - y_k$ and substituting $(D_t^\alpha y)(t_k) = f(t_k,y_k)$ into (9) results in

$$y_{k+1} - y_k = \frac{1}{\Gamma(\alpha+1)}h^\alpha f(t_k,y_k) + \frac{1}{\Gamma(2\alpha+1)}h^{2\alpha}(D_t^{2\alpha}y)(c_1),$$

or, equivalently

$$\Gamma(\alpha+1)\frac{y_{k+1}-y_k}{h^\alpha} = f(t_k,y_k) + \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}h^\alpha(D_t^{2\alpha}y)(c_1). \tag{10}$$

For h small enough, ignoring the second term on the right-hand side of (10) yields the fractional Euler finite difference method (FEFDM) given in Eqn. (3):

$$y_{k+1} - y_k = \frac{1}{\Gamma(\alpha+1)}h^\alpha f(t_k,y_k),$$

which reduces to the usual Euler's method for $\alpha = 1$.

3.2 Limited Validity of the FEFDM

Next, the following is proved:

Proposition 3.1 The FEFDM as given by Eqn. (3) is valid only for $\alpha = 1$.

Proof
Substituting $\mho(t_0, t, y_0) = \mathrm{E}_\alpha(-t^\alpha)$, the solution of the FRE for the Caputo FD, into Definition 2.2 results in the generalized representation:

$$ {}_0^C D_t^\alpha y(t) = \lim_{h\to 0} \frac{y(t+h)-y(t)}{\left(1-\mho(t_0,t+h,y_0)/\mho(t_0,t,y_0)\right)} = \lim_{h\to 0} \frac{y(t+h)-y(t)}{[1-\mathrm{E}_\alpha(-(t+h)^\alpha)/\mathrm{E}_\alpha(-t^\alpha)]}. \quad (11)$$

For h small enough, therefore, and making the identifications (8) in Eqn. (11), or using Corollary 2.1, results in the following discrete representation of the Caputo FD, ${}_0^C D_t^\alpha y(t)$:

$$ {}_0^C D_t^\alpha y(t) = \lim_{h\to 0} \frac{y(t+h)-y(t)}{\left(1-\mathrm{E}_\alpha(-(t+h)^\alpha)/\mathrm{E}_\alpha(-t^\alpha)\right)} \to \frac{y_{k+1}-y_k}{\lim_{h\to 0}\left(1-\mathrm{E}_\alpha(-(t_{k+1})^\alpha)/\mathrm{E}_\alpha(-{t_k}^\alpha)\right)}.$$

Now, if the FEFDM is valid, then we have the following two equivalent discrete representations of the IVP (7) with Caputo FD ${}_0^C D_t^\alpha y(t)$:

$$\frac{y_{k+1}-y_k}{1-\mathrm{E}_\alpha(-(t_{k+1})^\alpha)/\mathrm{E}_\alpha(-{t_k}^\alpha)} = -y_k \text{ and}$$

$$\Gamma(\alpha+1)\frac{y_{k+1}-y_k}{h^\alpha} = -y_k. \quad (12)$$

Therefore, from the equivalence of the left-hand sides in (12) above we conclude the following:

$$ {}_0^C D_t^\alpha y(t) \to \frac{y_{k+1}-y_k}{\lim_{h\to 0}\left(1-\mathrm{E}_\alpha(-(t_{k+1})^\alpha)/\mathrm{E}_\alpha(-{t_k}^\alpha)\right)} = \Gamma(\alpha+1)\lim_{h\to 0}\frac{y_{k+1}-y_k}{h^\alpha}. \quad (13)$$

However, the only way that Eqn. (13) holds that the following identity holds,

$$\lim_{v\to 0}\left[1-\frac{\mathrm{E}_\alpha(-(u+v)^\alpha)}{\mathrm{E}_\alpha(-u^\alpha)}\right] = \lim_{v\to 0}[1-\mathrm{E}_\alpha(-v^\alpha)], \quad (14)$$

so that there follows

$$\lim_{h\to 0}\left[1-\frac{\mathrm{E}_\alpha(-(t_k+h)^\alpha)}{\mathrm{E}_\alpha(-{t_k}^\alpha)}\right] = \lim_{v\to 0}[1-\mathrm{E}_\alpha(-h^\alpha)] = \frac{1}{\Gamma(\alpha+1)}\lim_{v\to 0} h^\alpha.$$

Since the left-hand side of the first representation in (12) is an exact discretization of the RE for the Caputo FD for $0 < \alpha \le 1$, it is consistent with the representation of the Caputo FD for $0 < \alpha \le 1$. Since the identity (14) has been shown by example, [24], to not hold for $\alpha \neq 1$, we conclude therefore that (13) also holds only for $\alpha = 1$. From the preceding statements, we conclude therefore that the right-hand side (RHS) of (13) is consistent with the representation of the Caputo FD only for $\alpha = 1$, and hence that the FEFDM, since it derives from the RHS of (13), is valid only for $\alpha = 1$.

4 Alternative Definition of the Caputo FD and Justification of the MFEM

In this section, an alternative definition of the Caputo derivative is presented and used to derive a modified fractional Euler finite difference method (MFEFDM).

4.1 Alternative Definition of the Caputo FD

The derivation of a modified FEFDM is based on the exact discretization of the initial value problem for the FRE, obtained from using the solution of the FRE for the Caputo FD, $\mho(t_0, t, y_0) = \mathrm{E}_\alpha(-t^\alpha)$ in Definitions 2.1 and 2.2, and leads to the following DQR and GFD for the Caputo FD:

Definition 4.1. The Caputo fractional derivative has the following difference quotient representation (DQR):

$$ {}_0^C\Delta_t^\alpha[y(t)] = \frac{y(t+h)-y(t)}{\mu(h,t,\alpha)}, \text{ where } \mu(h,t,\alpha) = \left(1-\frac{\mathrm{E}_\alpha(-(t+h)^\alpha)}{\mathrm{E}_\alpha(-t^\alpha)}\right)$$

and associated generalized Caputo derivative as given by (6):

$$ {}_0^C D_t^\alpha f(t) = \lim_{h\to 0} {}_0^C\Delta_t^\alpha[y(t)] = \lim_{h\to 0}\frac{y(t+h)-y(t)}{\mu(h,t,\alpha)}.$$

The following result about the basic properties of ${}_0^C D_t^\alpha$ is a particular case of Theorem 2.1.6 of, [20].

Theorem 4.1. Let $\alpha \in (0,1]$ and the functions f, g be α-differentiable at a point $t \in [0,\infty)$. Then, for all real-valued constants A, B, K, p, the following properties hold:

(1). ${}_0^C D_t^\alpha[Af+Bg] = A\,{}_0^C D_t^\alpha[f] + B\,{}_0^C D_t^\alpha[g]$
(2). ${}_0^C D_t^\alpha[fg] = g\,{}_0^C D_t^\alpha[f] + f\,{}_0^C D_t^\alpha[g]$
(3). ${}_0^C D_t^\alpha\left[\frac{f}{g}\right] = \frac{1}{g^2}\left[g\,{}_0^C D_t^\alpha[f] - f\,{}_0^C D_t^\alpha[g]\right]$
(4). ${}_0^C D_t^\alpha[t^p] = \frac{E_\alpha(-t^\alpha)}{E_{\alpha,\alpha}(-t^\alpha)} p t^{p-\alpha}$
(5). ${}_0^C D_t^\alpha[K] = 0$
(6). If $f(t)$ is first-order differentiable, then the following also holds:

$$ {}_{0}^{C}D_t^{\alpha}[f(t)] = \frac{E_\alpha(-t^\alpha)}{E_{\alpha,\alpha}(-t^\alpha)}\, t^{1-\alpha}\frac{df(t)}{dt} $$

Proof (of (6))

The proofs are elementary proofs that are omitted here: 1,2,3,5 follow directly from Definition 3.1 while 4),6) follow directly from Proposition 2.1, with $\mho(t_0,t,y_0) = \mathrm{E}_\alpha(-t^\alpha)$ and use of the identity $\frac{d\mathrm{E}_\alpha(-t^\alpha)}{dt} = -t^{\alpha-1}\mathrm{E}_{\alpha\alpha,}(-t^\alpha)$

Remark: The formula in Theorem 4.1 (6) is the same as that recently obtained by, [25], from consideration of the relationship between relaxation equations of integer order and those of fractional order. The ESDDFD therefore offers, through Proposition 2.1, a generalization of the Mainardi result as well as the relationship between the Caputo FD and the integer derivative.

From Definition 4.1, we have the following:

Corollary 4.2. The Caputo derivative has the following generalized fractional derivative (GFD):

$$ {}_{0}^{C}D_t^{\alpha}[f(t)] = \lim_{h\to 0}\frac{y(t+h)-y(t)}{\psi(h;\alpha,t)}, $$

where $\psi(h;\alpha,t)$ is defined as follows:

$$ \psi(h;\alpha,t) = \frac{1}{\alpha}\Big(E_{\alpha,\alpha}(-t^\alpha)/E_\alpha(-t^\alpha)\Big)\,[\,(t+h)^\alpha - t^\alpha]. $$

Proof

The proof follows directly from Theorem 4.1 (6), noting that

$$ \frac{E_\alpha(-t^\alpha)}{E_{\alpha,\alpha}(-t^\alpha)}\, t^{1-\alpha}\frac{df(t)}{dt} = \frac{E_\alpha(-t^\alpha)}{E_{\alpha,\alpha}(-t^\alpha)}\, {}_{0}^{C}T_t^{\alpha}[f(t)], $$

where ${}_{0}^{C}T_t^{\alpha}$ denotes the CFD, and the use of the alternate definition of the CFD given in Definition 2.3.

4.2 Modification of the FEFDM and Its NSFD Alternative

The proposed modification of the FEFDM and its alternative follow directly from the GFD in Corollary 4.2 and the DQR in Definition 4.1, with the identifications in Eqn. (8), which yields the following possible discretizations for the Caputo FD:

Caputo FEFDM:

$$ {}_{0}^{C}D_t^{\alpha}y(t) \to \alpha\frac{E_\alpha(-t_k{}^\alpha)}{E_{\alpha,\alpha}(-t_k{}^\alpha)}\frac{y_{k+1}-y_k}{((t_{k+1})^\alpha-(t_k)^\alpha)} $$

Caputo FNSFD: $ {}_{0}^{C}D_t^{\alpha}y(t) \to \frac{y_{k+1}-y_k}{1-\mathrm{E}_\alpha(-\,(t_{k+1})^\alpha)/\mathrm{E}_\alpha(-t_k{}^\alpha)}$

Since the OMFDM may be viewed as a two-step improved version of the FEFDM, an improved version of the MFEFDM may be constructed for comparison. A corollary to the foregoing, and given Corollary 2.2, are the following Euler discretization rules for the IVP (2) for the Caputo FD, which justifies the modified FEFDM (MFEFDM) and its NSFD alternative as extensions of the Euler method to the Caputo FD:

Corollary 4.3. The following discrete representations are generalizations of the (forward) Euler method and its two-step improved versions for the Caputo FD valid for $\alpha \in (0,1]$

Modified FEFDM (MFEFDM):

$$ \alpha\frac{E_\alpha(-t_k{}^\alpha)}{E_{\alpha,\alpha}(-t_k{}^\alpha)}\frac{y_{k+1}-y_k}{(t_{k+1})^\alpha-(t_k)^\alpha} = f(t_k,y_k) $$

Improved Modified FEFDM (IMFEFDM):

$$ \alpha\frac{E_\alpha(-t_k{}^\alpha)}{E_{\alpha,\alpha}(-t_k{}^\alpha)}\frac{y_{k+1}-y_k}{(t_{k+1})^\alpha-(t_k)^\alpha} = \frac{1}{2}[f(t_k,y_k) + f(t_{k+1},y_{k+1}^*)] $$

Fractional NSFD (FNSFD):

$$ \frac{y_{k+1}-y_k}{[1-\mathrm{E}_\alpha(-\,(t_{k+1})^\alpha)/\mathrm{E}_\alpha(-t_k{}^\alpha)]} = f(t_k,y_k) $$

Improved Fractional NSFD (IFNSFD)::

$$ \frac{y_{k+1}-y_k}{[1-\mathrm{E}_\alpha(-\,(t_{k+1})^\alpha)/\mathrm{E}_\alpha(-t_k{}^\alpha)]} = \frac{1}{2}[f(t_k,y_k) + f(t_{k+1},y_{k+1}^*)] $$

5 Numerical Experiments

To further demonstrate that the FEFDM is not a viable extension of the Euler method to the Caputo FD for $\alpha \in (0,1)$ and to validate the suggested alternatives, two examples are presented.

Example 1

$$ D_t^\alpha y(t) = -y(t), y(0) = 1,\ 0 \le t \le 1;\ 0 < \alpha \le 1. $$

Example 1 is used to justify the FEFDM in, [5], and to validate the OMFDM in, [26], and those results are in agreement with those of, [27]. The OMFM model for Example 1,

$$ y(t_k) = 1 - \frac{h^\alpha}{\Gamma(\alpha+2)}((k-1)^{\alpha+1} - (k-\alpha-1)k^\alpha) - \frac{h^\alpha}{\Gamma(\alpha+2)}\sum_{i=1}^{k-1}((k-i+1)^{\alpha+1} - 2(k-1)^{\alpha+1} + (k-i-1)^{\alpha+1})y(t_i) + \frac{h^\alpha}{\Gamma(\alpha+2)}\left(\frac{h^\alpha}{\Gamma(\alpha+1)} - 1\right)y(t_{k-1}), \quad (15) $$

is compared to the, respectively, FEFDM, MFEFDM, IMFEFDM, FNSFD, and IFNSFD models:

$$y_{k+1} = \left(1 - \frac{1}{\Gamma(\alpha+1)} h^{\alpha}\right) y_k \tag{16}$$

$$y_{k+1} = \left(1 - \frac{1}{\alpha}\frac{E_{\alpha,\alpha}(-{t_k}^{\alpha})}{E_{\alpha}(-{t_k}^{\alpha})}\left[(t_{k+1})^{\alpha} - (t_k)^{\alpha}\right]\right) y_k \tag{17}$$

$$y_{k+1} = y_k - \frac{1}{2}\left(\frac{1}{\alpha}\frac{E_{\alpha,\alpha}(-{t_k}^{\alpha})}{E_{\alpha}(-{t_k}^{\alpha})}\left[(t_{k+1})^{\alpha} - (t_k)^{\alpha}\right]\right)\left[y_k + y^{*}_{k+1}\right] \tag{18}$$

where y^{*}_{k+1} is y_{k+1} of Eqn. (17);

$$y_{k+1} = \left(1 - \left[1 - \frac{\mathrm{E}_{\alpha}(-(t_{k+1})^{\alpha})}{\mathrm{E}_{\alpha}(-{t_k}^{\alpha})}\right]\right) y_k = \frac{\mathrm{E}_{\alpha}(-(t_{k+1})^{\alpha})}{\mathrm{E}_{\alpha}(-{t_k}^{\alpha})} y_k \tag{19}$$

$$y_{k+1} = y_k - \frac{1}{2}\left[1 - \frac{\mathrm{E}_{\alpha}(-(t_{k+1})^{\alpha})}{\mathrm{E}_{\alpha}(-{t_k}^{\alpha})}\right]\left[y_k + y^{*}_{k+1}\right],$$

where y^{*}_{k+1} is y_{k+1} of Eqn. (19) (20)

With the OMFDM (15) viewed as an improved version of the FEFDM (16), comparisons against the analytic solutions are presented in Figure 1 and Figure 2 below using discrete representations obtained from the FEFDM, MFEFDM (17), and FNSFD (19), as well as their improved versions (respectively, OMFDM, IMFEFDM (18), and IFNSFD (20))

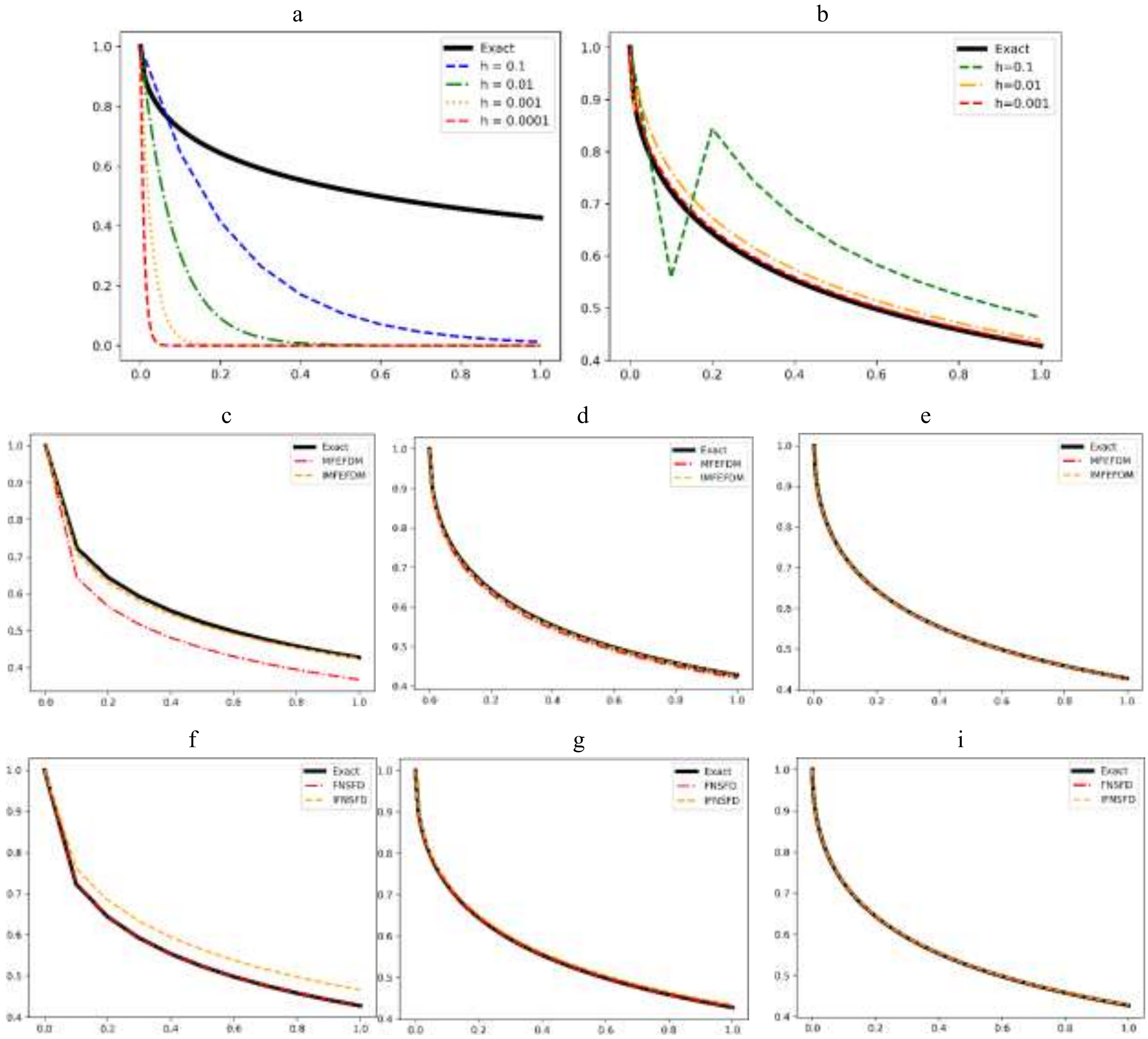

Fig. 1: Solution profiles for Example 1 when $\alpha = 0.5$ (a. FEFDM, b. OMFDM; (c–e) MFEFDM and IFEFDM (c. h=0.1, d. h=0.01, e. h=0.001); (f–i) NSFD and INSFD (f. h=0.1, g. h=0.01, i. h=0.001)

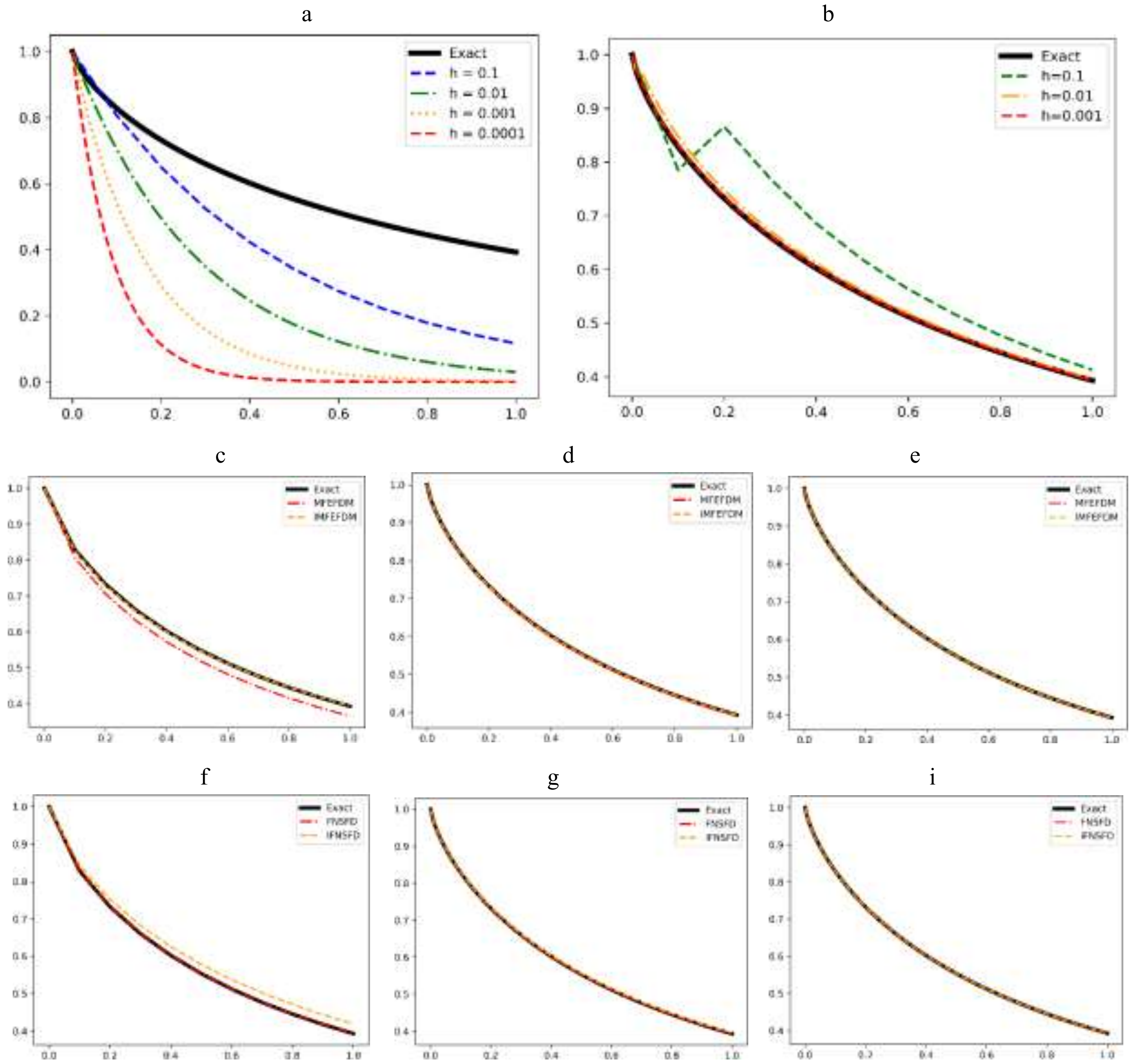

Fig. 2: Solution profiles for Example 1 when $\alpha = 0.75$. (a. FEFDM, b. OMFDM; (c–e) MFEFDM and IFEFDM (c. h=0.1, d. h=0.01, e. h=0.001); (f–i) NSFD and INSFD (f. h=0.1, g. h=0.01, i. h=0.001)

It is clear from Figure 1 and Figure 2 that the FEFDM does not perform well for all step sizes; this inferior performance persists for all values of $\alpha \in (0,1)$. While it performs better than the FEFDM, the OMFDM is seen to under-perform both the MFEFDM and the FNSFD as well as their improved versions for all step sizes. The absolute and percentage errors for $\alpha = 0.5$ and various step sizes are presented in Table 1 below to further quantify these performance differences for all the six considered method.

Table 1. Error-values for Example 1 when $\alpha = 0.5$

h	error	FEFDM	OMFM	MFEFDM	IMFEFDM	FNSFD	IFNSFD
0.1	Abs E	0.415469	0.054394	0.060024	0.002517	0	0.038582
	% E	97.16684	12.72124	14.03785	0.588624	0	9.023249
0.01	Abs E	0.427577	0.011239	0.007778	0.000703	0	0.006339
	% E	99.99852	2.62848	1.819015	0.164483	0	1.482554
0.001	Abs E	0.427584	0.003192	0.000998	9.15E-05	0	0.00088
	% E	100	0.746423	0.233314	0.021411	0	0.205787

Example 2

$D_t^{\alpha}y(t) = 1 - y(t)\,, y(0) = 0,\ \ 0 \le t \le 1;\ 0 < \alpha \le 1.$

Example 2 is a slight extension of Example 1. Comparisons against the analytic solutions for Example 2 are presented in graphical form in Figure 3 and Figure 4 below using discrete representations obtained from the Caputo fractional Euler, modified fractional Euler, and ESDDFD-based NSFD Euler methods (respectively, FEFDM, MFEFDM, and FNSFD), and their improved versions (respectively, OMFM, IMFEFDM, and IFNSFD).

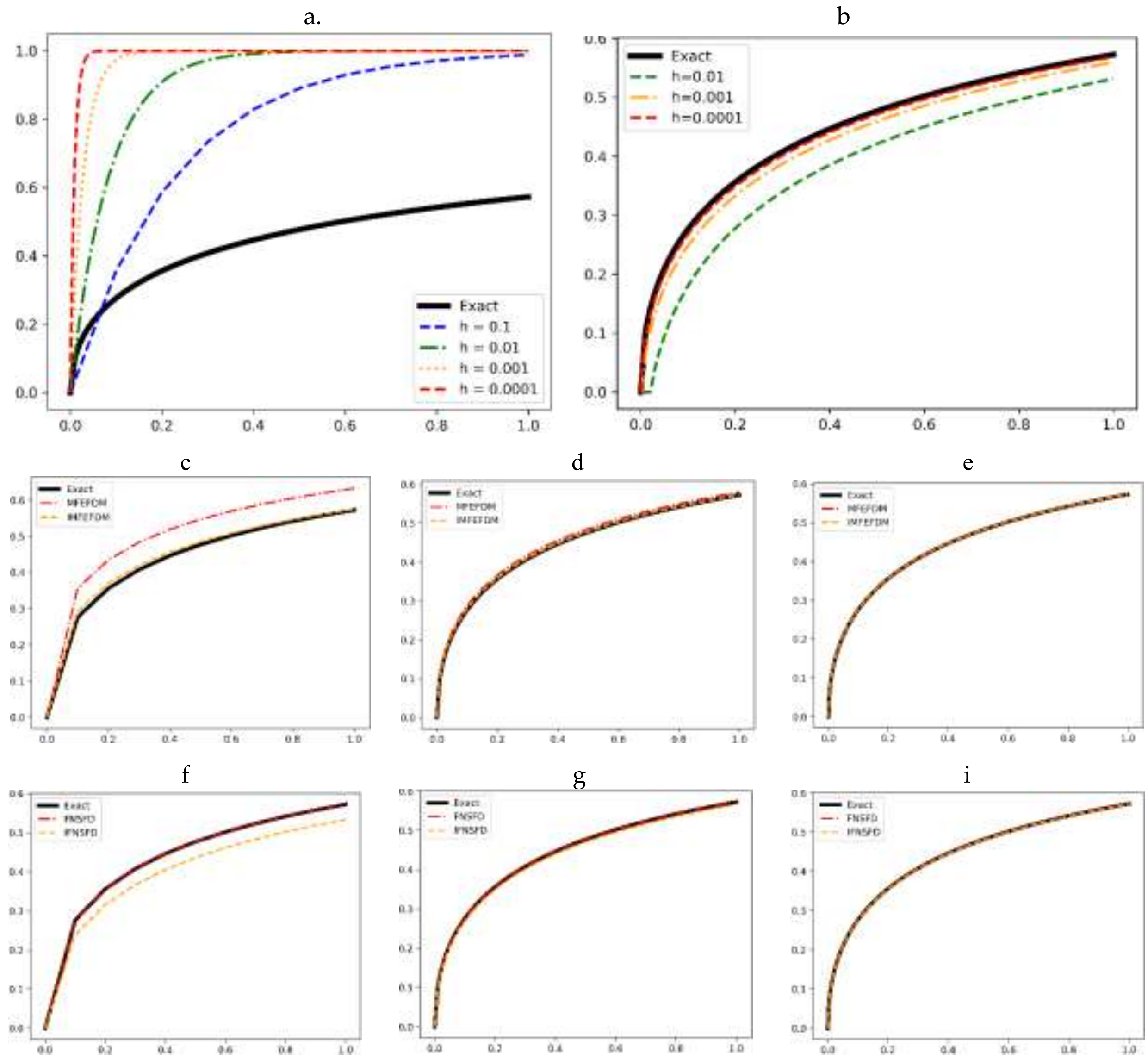

Fig. 3: Solution profiles for Example 2 when $\alpha = 0.5$. (a. FEFDM, b. OMFDM; (c–e) MFEFDM and IFEFDM (c. h=0.1, d. h=0.01, e. h=0.001); (f–i) NSFD and INSFD (f. h=0.1, g. h=0.01, i. h=0.001)

Consistent with the results of Example 1, it is seen from Figure 3 above and Figure 4 below that the FEFDM under-performs all the other methods while the MFEFDM, the FNSFD, and their improved versions all outperform the OMFDM.

The absolute and percentage errors from Example 2 for $\alpha = 0.5$ and various step sizes are presented in Table 2 below to further quantify the performance differences for all the six considered methods.

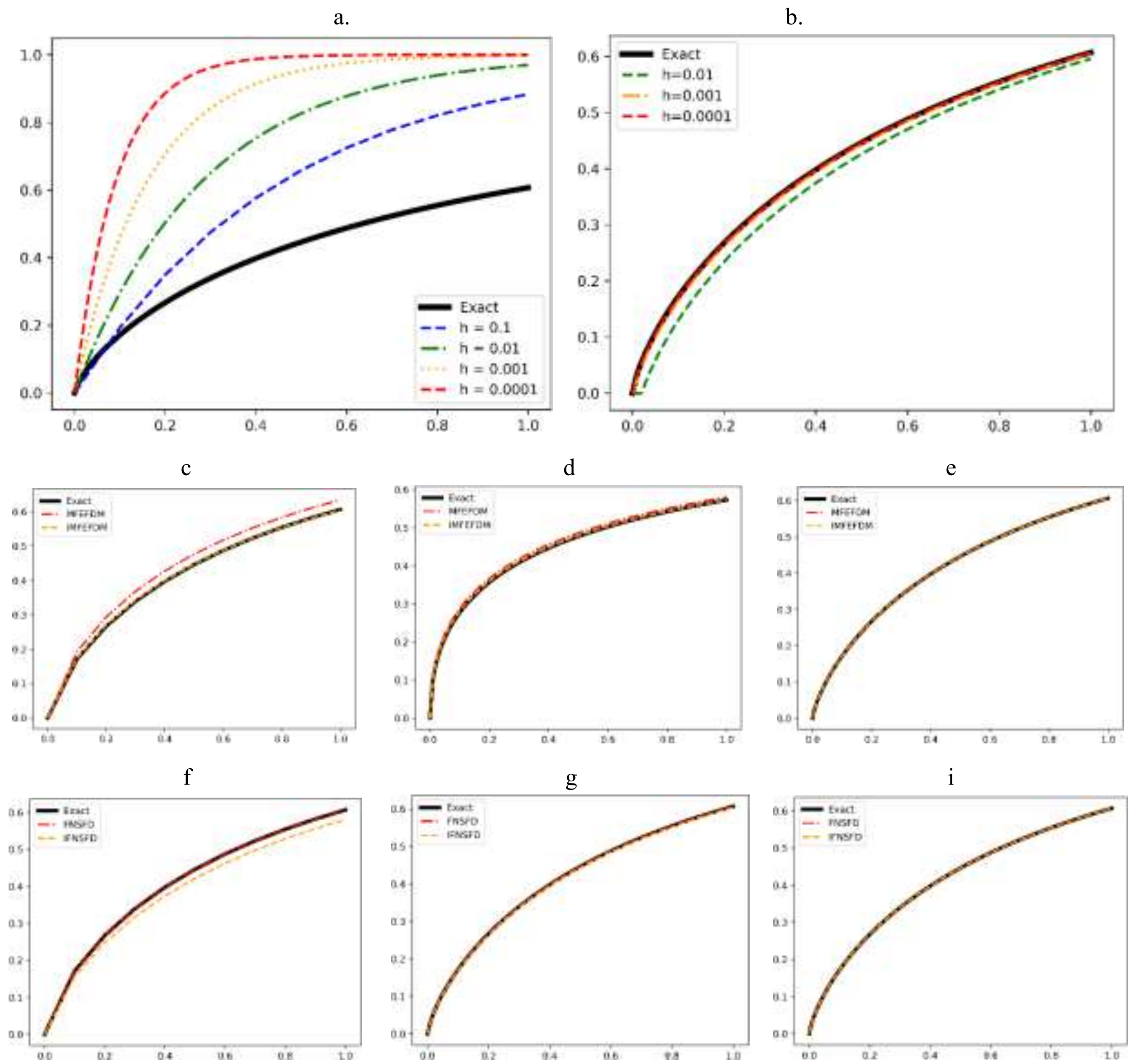

Fig. 4: Solution profiles for Example 2 when $\alpha = 0.75$. (a. FEFDM, b. OMFDM; (c–e) MFEFDM and IFEFDM (c. h=0.1, d. h=0.01, e. h=0.001); (f–i) NSFD and INSFD (f. h=0.1, g. h=0.01, i. h=0.001)

Table 2. Error-values for Example 2 when $\alpha = 0.5$

h	error	FEFDM	OMFM	MFEFDM	IMFEFDM	FNSFD	IFNSFD
0.1	Abs E	0.415469	0.132849	0.060024	0.002517	2.11E-15	0.038582
	% E	72.58168	23.20846	10.486	0.439691	3.66E-13	6.740186
0.01	Abs E	0.427577	0.041251	0.007778	0.000703	2.11E-15	0.006339
	% E	74.69689	7.206459	1.358768	0.122865	3.66E-13	1.107438
0.001	Abs E	0.427584	0.013144	0.000998	9.15E-05	1.55E-15	0.00088
	% E	74.69799	2.296192	0.174281	0.015993	2.66E-13	0.153718

6 Conclusion

A discretization method for the fractional initial value problem, with the Caputo fractional derivative, has been considered that extends the integer Euler method and is termed the generalized fractional Euler's method in recent literature; its justification using a fractional series expansion is recalled. It has been shown that the method is valid only for $\alpha = 1$. A modified generalized fractional Euler's method and a corresponding nonstandard method are proposed along with their improved Euler counterparts. Numerical experiments are presented comparing the FEFDM with the Odibat-Momani algorithm derived from the FEFDM and the four suggested alternative fractional Euler and improved fractional Euler methods. Graphical evidence and tabulation of absolute and percentage errors show that the FEFDM has very large errors and that the proposed methods outperform both the FEFDM and the Odibat-Momani algorithm. The proposed methods have the potential to improve the numerical simulation of models of the form (2) in direct applications (such as in, [10], [11], [12], [13]), as well as in developing other methods (such as in, [14], [15], [16], [17], [18]). As a next step, the authors intend to apply these methods to the numerical simulation and analysis of various fractional disease models.

References:

[1] I. Podlubny, *Fractional Differential Equations,* Academic Press, 1999.

[2] M. Dalir and M. Bashour, Applications of Fractional Calculus *Applied Mathematical Sciences*, 4(21), 2010, pp.1021–1032

[3] M. Caputo, Linear Models of Dissipation whose Q is almost Frequency Independent-II. *Geophys. J. R. ustr. SOC.*, 13, 1967, pp.529–539

[4] K. Diethelm, N.J. Ford, A.D. Freed, and Yu. Luchko, Algorithms for the fractional calculus: a selection of numerical methods. *Comput. Meth. Appl. Mech. Engrg.*, 194, 2005, pp.743–773.

[5] Z. M.Odibat and S. Momani, An Algorithm for the numerical solution of differential equations of fractional order, *Journal of applied mathematics & informatics*, 26 (12), 2008, pp.15–27

[6] C. Li and F. Zeng, Finite Difference Methods For Fractional Differential Equations. *International Journal of Bifurcation and Chaos*, 22(4), 2012, 1230014 (28 pages)

[7] D. Baleanu, K. Diethelm, E. Scalas, and J.J. Trujillo, *Fractional Calculus: Models and numerical methods*, World Scientific, 2012.

[8] G-h.Gao, Z-z. Sun, and H-w. Zhang, A new fractional numerical differentiation formula to approximate the Caputo fractional derivative and its applications, *Journal of Computational Physics*, 259, 2014, pp.33–50

[9] R. Garrappa, Numerical Solution of Fractional Differential Equations: A Survey and a Software Tutorial. *Mathematics*. 6(2), 2018:16. https://doi.org/10.3390/math6020016

[10] M.M. Khadera and M. Adel, Numerical treatment of the fractional SIRC model and influenza A using generalized Euler method, *J. of Modern Methods in Numerical Math* 6:1, 2015, pp.44–56

[11] D. Yaro, S.K. Omari-Sasu, P. Harvim, A.W. Saviour, and B.A. Obeng, Generalized Euler Method for Modeling Measles with Fractional Differential Equations, *International Journal of Innovative Research & Development*, 4(4), 2015, pp.358–366

[12] D. Vivek, K. Kanagarajan, and S. Harikrishnan, Numerical solution of fractional-order logistic equations by fractional Euler's method, *International Journal for Research in Applied Science & Engineering Technology* (IJRASET), 2016, 4(III), pp.775–780

[13] Y.Y.Y. Noupoue, Y. Tandoğdu, and M. Awadalla, On numerical techniques for solving the fractional logistic differential equation, *Advances in Difference Equations*, 108 (2019), 2019, pp.1–13. https://doi.org/10.1186/s13662-019-2055-y

[14] C. Li and F. Zeng, The Finite Difference Methods for Fractional Ordinary Differential Equations, *Numerical Functional Analysis and Optimization*, 34(2), 2013, pp.149–179, DOI: 10.1080/01630563.2012.706673: https://doi.org/10.1080/01630563.2012.706673

[15] N.A. Khan, O.A. Razzaq, and F. Riaz, Numerical Simulations For Solving Fuzzy Fractional Differential Equations By Max-Min Improved Euler Methods, *JACSM*, 7(1), 2015, pp.53–83

[16] F.S. Fadhel and H.H. Khayoon, Numerical Solution of Ordinary Differential Equations of Fractional Order Using Variable Step Size Method, Special Issue: 1st Scientific International Conference, College of Science, Al-Nahrain University, 21-22/11/2017, Part I, pp.143–149, [Online], https://www.iasj.net/iasj/download/875d174d6dd06236 (Accessed Date: October 17, 2023)

[17] H.F. Ahmed, Fractional Euler Method; An Effective Tool For Solving Fractional Differential Equations, *Journal of the Egyptian Mathematical Society*, 26(1), 2018, pp.38–43. DOI: 10.21608/JOEMS.2018.9460

[18] M.S. Mechee and S.H. Aidib, Generalized Euler and Runge-Kutta methods for solving classes of fractional ordinary differential equations, *Int. J. Nonlinear Anal. Appl.* 13(1), 2022, pp.1737–1745 http://dx.doi.org/10.22075/ijnaa.2022.5788

[19] Z. Odibat and N. Shawagfeh, Generalized Taylor's formula. *Mathematics and Computation*, 186(1), 2007, 286–293. DOI: 10.1016/j.amc.2006.07.102

[20] D.P. Clemence-Mkhope, Spectral Non-integer Derivative Representations and the Exact Spectral Derivative Discretization Finite Difference Method for the Fokker-Planck Equation. 2021, [Online], https://ui.adsabs.harvard.edu/link_gateway/2021arXiv210602586C/arxiv:2106.02586 (Accessed Date: October 17, 2023)

[21] R.E. Mickens, *Nonstandard Finite Difference Schemes: Methodology And Applications*, World Scientific Publishing Company, 2020.

[22] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh. A new definition of fractional derivative. *J. Comput. Appl. Math.* 2014, *264*, pp.65–70.

[23] D.P. Clemence-Mkhope and B.G.B. Clemence-Mkhope, The Limited Validity of the Conformable Euler Finite Difference Method and an Alternate Definition of the Conformable Fractional Derivative to Justify Modification of the Method. *Mathematical and Computational Applications*. 2021; 26(4):66. https://doi.org/10.3390/mca26040066

[24] J. Peng and K. Li, A note on property of the Mittag-Leffler function. *Journal of Mathematical Analysis and Applications*, 370(2), 2010, pp.635–638

[25] F. Mainardi, A Note on the Equivalence of Fractional Relaxation Equations to Differential Equations with Varying Coefficients. Reprinted from: *Mathematics* 2018, 6,8, DOI: 10.3390/math6010008

[26] S.S. Zeid, M. Yousefi, and A.V. Kamyad, Approximate Solutions for a Class of Fractional-Order Model of HIV Infection via Linear Programming Problem. *American Journal of Computational Mathematics*, 6, 2016, pp.141–152. DOI: 10.4236/ajcm.2016.62015.

[27] Y. Ding and H. Ye, A fractional-order differential equation model of HIV infection of CD4+ T-cells, Mathematical and Computer Modelling, 50 (3–4), 2009, pp.386–392, https://doi.org/10.1016/j.mcm.2009.04.019.

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Conflict of Interest

The authors have no conflicts of interest to declare.