Rational solutions to the Burgers' equation from particular polynomials

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Abstract: From particular polynomials, we construct rational solutions to the Burgers' equation as a quotient of a polynomial of degree $n-1$ in x and $n-1-\left|\frac{n}{2}\right|$ in t, by a polynomial of degree n in x and $\left|\frac{n}{2}\right|$ in t, $|n|$ being the greater integer less or equal to n. We call these solutions, solutions of order n.
We construct explicitly these solutions for orders 1 until 20.

Key-Words: Burgers equation, rational solutions, determinants.

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1 Introduction

We consider the Burgers' equation which can be written as

$$u_t + u_{xx} + uu_x = 0 \qquad (1)$$

where the subscripts x and t denote partial derivatives.
In 1915, [1] introduced this equation (1). This equation appears in different contexts in physics as in gas dynamics, [2], acoustics, [3], heat conduction, [4], in soil water, [5], in hydrodynamics turbulence, [6], [7], [8], in shock waves, [9],...
The first solutions has been constructed, [1] in 1915. Other types of methods have been used to solve this equation. We can quote the exp-function method, [13], the tanh-coth method, [14], the groups actions on coset bundles, [15], the Cole-Hopf method, [16, 17, 18], [17], [18], the homotopy perturbation method, [19],...
We can quote some recent results in connection with this study as, [10], [11], [12].
Rational solutions to the Burgers' equation are constructed in this paper. We give solutions as a quotient of a polynomial of degree $n-1$ in x and $n-1-\left[\frac{n}{2}\right]$ in t by a polynomial of degree n in x and $|[frac{n}{2}]$ in t, $|p|]$ being the greater integer less or equal to p.

We explicitly build these solutions for orders between 1 and 20.

2 Rational solutions to the Burger's equation

We consider the following polynomials defined by

$$\begin{aligned} &p_n(x,t) = \sum_{k=0}^{n} \frac{x^k}{k!} \frac{(-t)^{\frac{n-k}{2}}}{\frac{n-k}{2}!} \\ &\left(1-\left(n-k-2\left[\frac{n-k}{2}\right]\right)\right), \\ &\text{for } n \geq 0, \\ &p_n(x,t) = 0 \text{ for } n<0. \end{aligned} \qquad (2)$$

With the choice of these polynomials, we have the following statement
Theorem
The function v_n defined by

$$v_n(x,t) = 2\frac{p_{n-1}(x,t)}{p_n(x,t)}, \qquad (3)$$

where p_n are defined by previous relations (2), is a solution to the Burgers' equation (1)

$$u_t + u_{xx} + uu_x = 0.$$

Remark
In the following, we will call the solution v_n, the solution of order n of the Burgers' equation, (1).

Remark
More explicitly, the previous polynomials can be written as

$$\begin{aligned} &p_{2k}(x,t) = \sum_{l=0}^{n} \frac{x^{2l}}{(2l)!} \frac{(-t)^{k-l}}{k-l!}, \\ &\text{for } k \geq 0, \\ &p_{2k+1}(x,t) = \sum_{l=0}^{n} \frac{x^{2l+1}}{(2l+1)!} \frac{(-t)^{k-l}}{k-l!}, \\ &\text{for } k \geq 0, \\ &p_n(x,t) = 0 \text{ for } n<0. \end{aligned}$$

Proof
The proof is elementary. It is sufficient to evaluate the expression
$A = v_t + v_{2x} + vv_x$.
Taking into account that $(p_n)_x = p_{n-1}$ and $(p_n)_t = -p_{n-2}$, we can write
$A = \left(2\frac{p_{n-1}}{p_n}\right)_t + \left(2\frac{p_{n-1}}{p_n}\right)_{2x} + \left(2\frac{p_{n-1}}{p_n}\right)\left(2\frac{p_{n-1}}{p_n}\right)_x$
$= -\frac{2p_{n-3}p_n + 2p_{n-1}p_{n-2}}{p_n^2}$
$+ \left(\frac{2p_{n-2}p_n - 2p_{n-1}^2}{p_n^2}\right)_x$
$+ 2\frac{p_{n-1}}{p_n}\left(\frac{2p_{n-2}p_n - 2p_{n-1}^2}{p_n^2}\right)$
$= \frac{1}{p_n^3}\left(-2p_{n-3}p_n^2 + 2p_{n-2}p_{n-1}p_n\right.$
$+p_n(2p_{n-3}p_n + 2p_{n-2}p_{n-1}$
$-4p_{n-1}p_{n-2})$
$-2p_{n-1}(2p_{n-2}p_n - 2p_{n-1}^2)$
$\left.+4p_{n-2}p_{n-1}p_n - 4p_{n-1}^3)\right)$
The simplifications then give $A = 0$ and the result.

3 Explicit first order solutions

All these rational solutions are singular. At each order, we see the appearence of curves of singularities. The patterns of singularities are lines, as in figures [1], [3], [5], [7], [9], [11], [13], [15], [17], [19], or horseshoe, as in figures [2], [4], [6], [8], [10], [12], [14], [16], [18], [20], type depending on the order of the solution, .

3.1 First order solutions

Proposition
The function v defined by

$$v_1(x,t) = \frac{2}{x} \tag{4}$$

is a solution to the Burgers' equation (1).

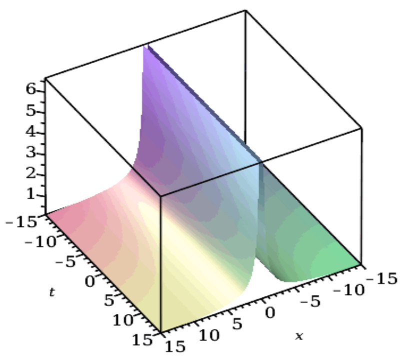

Figure 1. Solution of order 1 to (1).

3.2 Solutions of order two

Proposition
The function v_2 defined by

$$v_2(x,t) = \frac{-4x}{-x^2 + 2\,t}, \tag{5}$$

is a solution to the Burgers' equation (1).

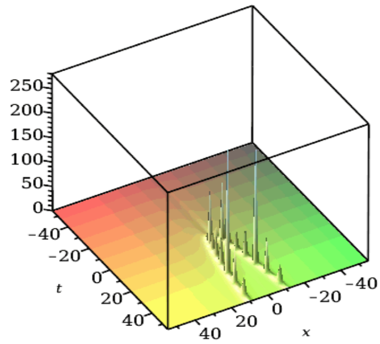

Figure 2. Solution of order 2 to (1).

3.3 Solutions of order three

Proposition
The function v_3 defined by

$$v_3(x,t) = 6\,\frac{-x^2 + 2\,t}{x\,(-x^2 + 6\,t)}, \tag{6}$$

is a solution to the Burgers' equation (1).

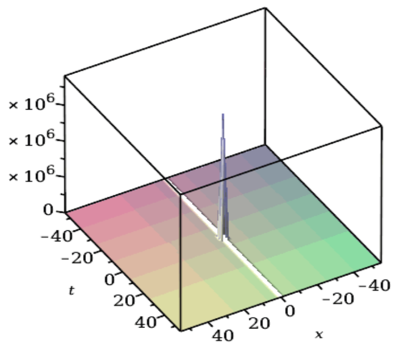

Figure 3. Solution of order 3 to (1).

3.4 Solutions of order four

Proposition
The function v_4 defined by

$$v_4(x,t) = -8\,\frac{x\,(-x^2 + 6\,t)}{x^4 - 12\,x^2t + 12\,t^2}, \tag{7}$$

is a solution to the Burgers' equation (1).

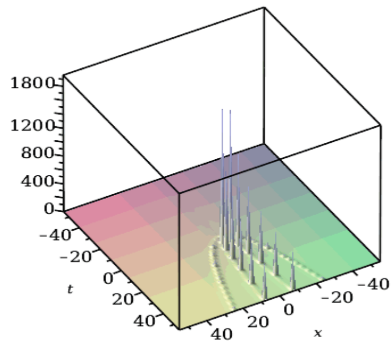

Figure 4. Solution of order 4 to (1).

3.5 Solutions of order five

Proposition

The function v_5 defined by

$$v_5(x,t) = 10\,\frac{x^4 - 12\,x^2 t + 12\,t^2}{x\,(x^4 - 20\,x^2 t + 60\,t^2)}, \tag{8}$$

is a solution to the Burgers' equation (1).

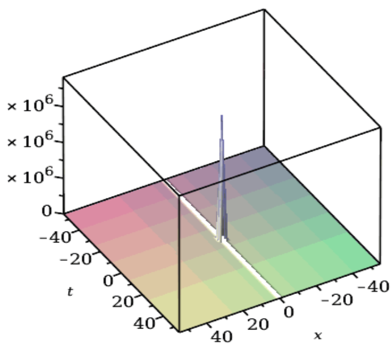

Figure 5. Solution of order 5 to (1).

3.6 Solutions of order six

Proposition

The function v_6 defined by

$$v_6(x,t) = -12\,\frac{x\,(x^4 - 20\,x^2 t + 60\,t^2)}{-x^6 + 30\,x^4 t - 180\,x^2 t^2 + 120\,t^3}, \tag{9}$$

is a solution to the Burgers' equation (1).

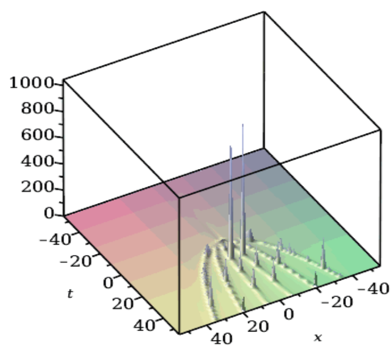

Figure 6. Solution of order 6 to (1).

3.7 Solutions of order seven

Proposition

The function v_7 defined by $v_7(x,t) = 14\frac{n(x,t)}{d(x,t)}$,

$n(x,t) = (-x^6 + 30\,x^4 t - 180\,x^2 t^2 + 120\,t^3)$,

$d(x,t) = x(-x^6 + 42\,x^4 t - 420\,x^2 t^2 + 840\,t^3)$

is a solution to the Burgers' equation (1).

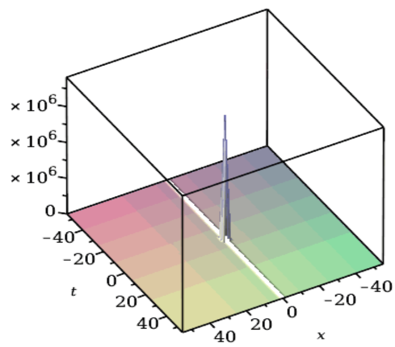

Figure 7. Solution of order 7 to (1).

3.8 Solutions of order eight

Proposition

The function v_8 defined by $v_8(x,t) = -16\frac{n(x,t)}{d(x,t)}$,

$n(x,t) = x(-x^6 + 42\,x^4 t - 420\,x^2 t^2 + 840\,t^3)$,

$d(x,t) = x^8 - 56\,x^6 t + 840\,x^4 t^2 - 3360\,x^2 t^3 + 1680\,t^4$,

is a solution to the Burgers' equation (1).

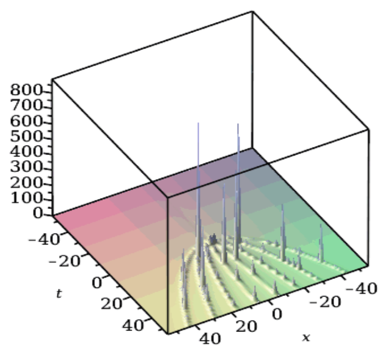

Figure 8. Solution of order 8 to (1).

3.9 Solutions of order nine

Proposition

The function v_9 defined by $v_9(x,t) = 18\frac{n(x,t)}{d(x,t)}$,

$n(x,t) = x^8 - 56\,x^6t + 840\,x^4t^2 - 3360\,x^2t^3 + 1680\,t^4$,

$d(x,t) = x(x^8 - 72\,x^6t + 1512\,x^4t^2 - 10080\,x^2t^3 + 15120\,t^4)$,

is a solution to the Burgers' equation (1).

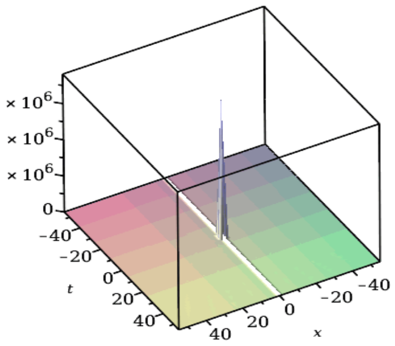

Figure 9. Solution of order 9 to (1).

3.10 Solutions of order ten

Proposition
The function v_{10} defined by $v_{10}(x,t) = 20\frac{n(x,t)}{d(x,t)}$,

$n(x,t) = x(x^8 - 72\,x^6t + 1512\,x^4t^2 - 10080\,x^2t^3 + 15120\,t^4)$,

$d(x,t) = -x^{10} + 90\,x^8t - 2520\,x^6t^2 + 25200\,x^4t^3 - 75600\,x^2t^4 + 30240\,t^5$,

is a solution to the Burgers' equation (1).

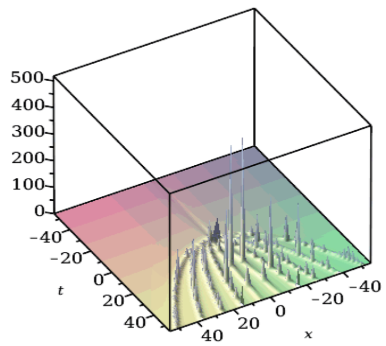

Figure 10. Solution of order 10 to (1).

3.11 Solutions of order eleven

Proposition
The function v_{11} defined by $v_{11}(x,t) = 22\frac{n(x,t)}{d(x,t)}$,

$n(x,t) = -x^{10} + 90\,x^8t - 2520\,x^6t^2 + 25200\,x^4t^3 - 75600\,x^2t^4 + 30240\,t^5$,

$d(x,t) = x(-x^{10} + 110\,x^8t - 3960\,x^6t^2 + 55440\,x^4t^3 - 277200\,x^2t^4 + 332640\,t^5)$,

is a solution to the Burgers' equation (1).

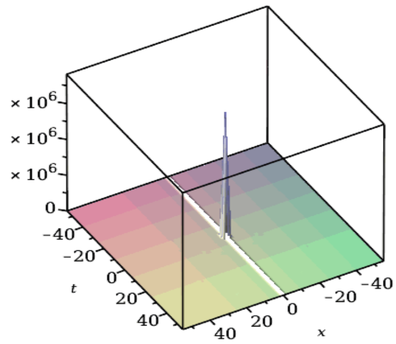

Figure 11. Solution of order 11 to (1).

3.12 Solutions of order twelve

Proposition
The function v_{12} defined by $v_{12}(x,t) = \frac{n(x,t)}{d(x,t)}$,

$n(x,t) = -24x(-x^{10} + 110\,x^8t - 3960\,x^6t^2 + 55440\,x^4t^3 - 277200\,x^2t^4 + 332640\,t^5)$,

$d(x,t) = x^{12} - 132\,tx^{10} + 5940\,t^2x^8 - 110880\,t^3x^6 + 831600\,t^4x^4 - 1995840\,t^5x^2 + 665280\,t^6$

is a solution to the Burgers' equation (1).

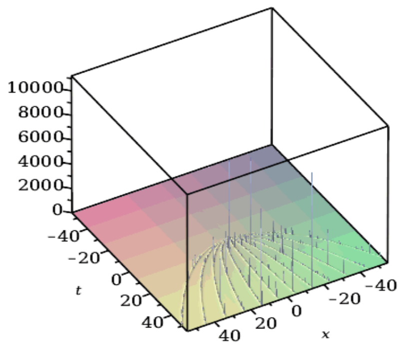

Figure 12. Solution of order 12 to (1).

3.13 Solutions of order thirteen

Proposition
The function v_{13} defined by $v_{13}(x,t) = \frac{n(x,t)}{d(x,t)}$,

$n(x,t) = 26(x^{12} - 132\,tx^{10} + 5940\,t^2x^8 - 110880\,t^3x^6 + 831600\,t^4x^4 - 1995840\,t^5x^2 + 665280\,t^6)$,

$d(x,t) = x(x^{12} - 156\,tx^{10} + 8580\,t^2x^8 - 205920\,t^3x^6 + 2162160\,t^4x^4 - 8648640\,t^5x^2 + 8648640\,t^6)$

is a solution to the Burgers' equation (1).

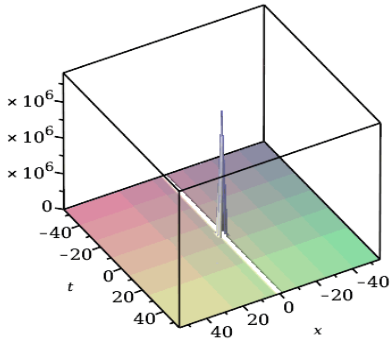

Figure 13. Solution of order 13 to (1).

3.14 Solutions of order fourteen

Proposition

The function v_{14} defined by $v_{14}(x,t) = \frac{n(x,t)}{d(x,t)}$,

$n(x,t) = -28x(x^{12} - 156\,tx^{10} + 8580\,t^2x^8 - 205920\,t^3x^6 + 2162160\,t^4x^4 - 8648640\,t^5x^2 + 8648640\,t^6)$,

$d(x,t) = -x^{14} + 182\,tx^{12} - 12012\,t^2x^{10} + 360360\,t^3x^8 - 5045040\,t^4x^6 + 30270240\,t^5x^4 - 60540480\,t^6x^2 + 17297280\,t^7$

is a solution to the Burgers' equation (1).

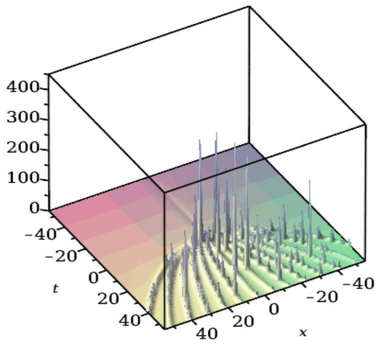

Figure 14. Solution of order 14 to (1).

3.15 Solutions of order fifthteen

Proposition

The function v_{15} defined by $v_{15}(x,t) = \frac{n(x,t)}{d(x,t)}$,

$n(x,t) = 30(-x^{14} + 182\,tx^{12} - 12012\,t^2x^{10} + 360360\,t^3x^8 - 5045040\,t^4x^6 + 30270240\,t^5x^4 - 60540480\,t^6x^2 + 17297280\,t^7)$,

$d(x,t) = x(-x^{14} + 210\,tx^{12} - 16380\,t^2x^{10} + 600600\,t^3x^8 - 10810800\,t^4x^6 + 90810720\,t^5x^4 - 302702400\,t^6x^2 + 259459200\,t^7)$

is a solution to the Burgers' equation (1).

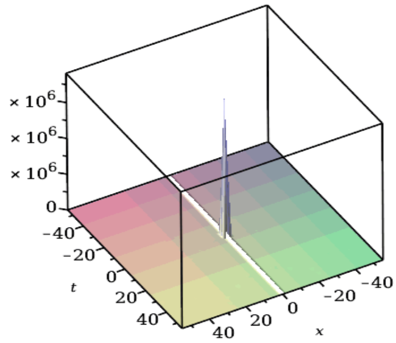

Figure 15. Solution of order 15 to (1).

3.16 Solutions of order sixteen

Proposition

The function v_{16} defined by $v_{16}(x,t) = \frac{n(x,t)}{d(x,t)}$,

$n(x,t) = -32x(-x^{14} + 210\,tx^{12} - 16380\,t^2x^{10} + 600600\,t^3x^8 - 10810800\,t^4x^6 + 90810720\,t^5x^4 - 302702400\,t^6x^2 + 259459200\,t^7)$,

$d(x,t) = x^{16} - 240\,tx^{14} + 21840\,t^2x^{12} - 960960\,t^3x^{10} + 21621600\,t^4x^8 - 242161920\,t^5x^6 + 1210809600\,t^6x^4 - 2075673600\,t^7x^2 + 518918400\,t^8$

is a solution to the Burgers' equation (1).

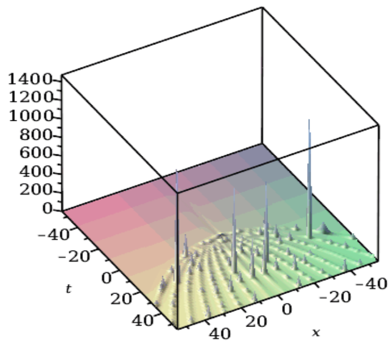

Figure 16. Solution of order 15 to (1).

3.17 Solutions of order seventeen

Proposition

The function v_{17} defined by $v_{17}(x,t) = \frac{n(x,t)}{d(x,t)}$,

$n(x,t) = 34(x^{16} - 240\,tx^{14} + 21840\,t^2x^{12} - 960960\,t^3x^{10} + 21621600\,t^4x^8 - 242161920\,t^5x^6 + 1210809600\,t^6x^4 - 2075673600\,t^7x^2 + 518918400\,t^8)$,

$d(x,t) = x(x^{16} - 272\,tx^{14} + 28560\,t^2x^{12} - 1485120\,t^3x^{10} + 40840800\,t^4x^8 - 588107520\,t^5x^6 + 4116752640\,t^6x^4 - 11762150400\,t^7x^2 + 8821612800\,t^8)$

is a solution to the Burgers' equation (1).

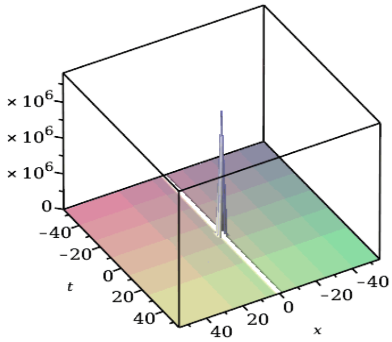

Figure 17. Solution of order 17 to (1).

3.18 Solutions of order eighteen

Proposition

The function v_{18} defined by $v_{18}(x,t) = \frac{n(x,t)}{d(x,t)}$,

$n(x,t) = -36x(x^{16} - 272\,tx^{14} + 28560\,t^2x^{12} - 1485120\,t^3x^{10} + 40840800\,t^4x^8 - 588107520\,t^5x^6 + 4116752640\,t^6x^4 - 11762150400\,t^7x^2 + 8821612800\,t^8)$,

$d(x,t) = -x^{18} + 306\,tx^{16} - 36720\,t^2x^{14} + 2227680\,t^3x^{12} - 73513440\,t^4x^{10} + 1323241920\,t^5x^8 - 12350257920\,t^6x^6 + 52929676800\,t^7x^4 - 79394515200\,t^8x^2 + 17643225600\,t^9$

is a solution to the Burgers' equation (1).

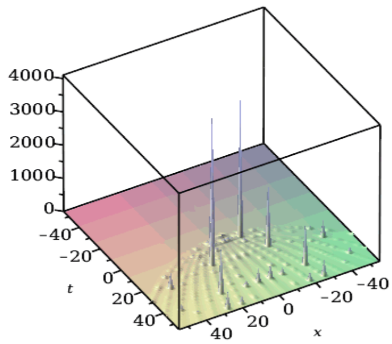

Figure 18. Solution of order 18 to (1).

3.19 Solutions of order nineteen

Proposition

The function v_{19} defined by $v_{19}(x,t) = \frac{n(x,t)}{d(x,t)}$,

$n(x,t) = 38(-x^{18} + 306\,tx^{16} - 36720\,t^2x^{14} + 2227680\,t^3x^{12} - 73513440\,t^4x^{10} + 1323241920\,t^5x^8 - 12350257920\,t^6x^6 + 52929676800\,t^7x^4 - 79394515200\,t^8x^2 + 17643225600\,t^9)$,

$d(x,t) = x(-x^{18} + 342\,tx^{16} - 46512\,t^2x^{14} + 3255840\,t^3x^{12} - 126977760\,t^4x^{10} + 2793510720\,t^5x^8 - 33522128640\,t^6x^6 + 201132771840\,t^7x^4 - 502831929600\,t^8x^2 + 335221286400\,t^9)$

is a solution to the Burgers' equation (1).

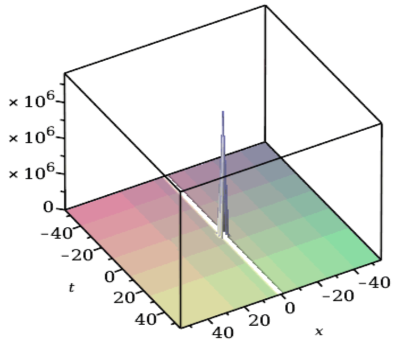

Figure 19. Solution of order 19 to (1).

3.20 Solutions of order twenty

Proposition

The function v_{20} defined by $v_{20}(x,t) = \frac{n(x,t)}{d(x,t)}$,

$n(x,t) = -40x(-x^{18} + 342\,tx^{16} - 46512\,t^2x^{14} + 3255840\,t^3x^{12} - 126977760\,t^4x^{10} + 2793510720\,t^5x^8 - 33522128640\,t^6x^6 + 201132771840\,t^7x^4 - 502831929600\,t^8x^2 + 335221286400\,t^9)$,

$d(x,t) = x^{20} - 380\,tx^{18} + 58140\,t^2x^{16} - 4651200\,t^3x^{14} + 211629600\,t^4x^{12} - 5587021440\,t^5x^{10} + 83805321600\,t^6x^8 - 670442572800\,t^7x^6 + 2514159648000\,t^8x^4 - 3352212864000\,t^9x^2 + 670442572800\,t^{10}$

is a solution to the Burgers' equation (1).

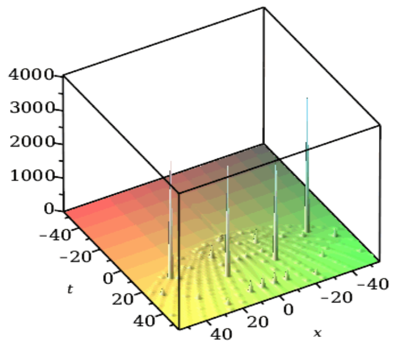

Figure 20. Solution of order 20 to (1).

4 Conclusion

We have given an expression of rational solutions to the Burgers' equation involving particular polynomials.

In particular, we have constructed explicit solutions to the Burgers' equation for the orders $n = 1$ until $n = 20$.

All these solutions are singular. We can classify them by the pattern of their singulatities.

The singularities of these solutions depend on the orders of the solutions. When we consider odd order solutions we have always the line $x = 0$ of singularities. In the case of even order solutions $n = 2p$, the singularities form horseshoe patterns with p branches. It will be interesting to construct solutions of this equation depending on some real parameters.

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