Generalized almost Contractions on Extended Quasi-Cone B-Metric Spaces

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Abstract: - Fixed Point Theory is among the most valued research topics nowadays. Over the years, it has been developed in three directions: by generalizing the metric space, by establishing new contractive conditions, and by applying its results to various fields such as Differential Equations, Integral Equations, Economics, etc. In this paper, we define a new class of cone metric spaces called the class of extended quasi-cone b-metric spaces. Extended quasi-cone b-metric spaces generalize cone metric spaces and quasi-cone b-metric spaces. We have studied topological issues, such as the right and left topologies, right (left) Cauchy, and convergent sequences. Furthermore, there are determined generalized τ-almost contractions, which extend the almost contractions. The highlight of this study is the investigation of the existence and uniqueness of a fixed point for some types of generalized τ-almost contractions in extended quasi-cone b-metric space. We prove some corollaries and theorems for known contractions in extended quasi-cone b-metric spaces. Our results generalize some known theorems given in literature due to the new cone metric spaces and contractions. Concrete examples illustrate theoretical outcomes. In addition, we show an application of the main results to Integral Equations, which provides the applicative side of them.

Key-Words: - Extended quasi cone b-metric space, fixed point, Cauchy sequence, generalized almost contraction, Convergent sequence, Right (left) open balls, Right (left) topology.

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1 Introduction

The mathematician in, [1], defined the cone metric by generalizing the metric by replacing the set of nonnegative real numbers with a cone. Later, in, [2], authors extended this concept to quasi-cone metric and proved some great fixed-point results that generalized Banach's contraction in quasi-cone metric space. Initiating by, [3], in which were studied the concept of b-metric space by modifying the triangle inequality of a metric, the authors in, [4], established cone b-metric space, investigated its topological properties, and obtained some approximating fixed-point results in these spaces.
Recently, in, [5], there is determined a new space called extended b-metric space. Many authors have surveyed fixed points on these spaces, [6], [7], [8], [9], [10], [11].

Encouraged by their results, in this paper, we study the extended quasi-cone b-metric space. We have set out left and right Cauchy sequences, convergent sequences, and some topological points in this space. Also, we prove several fixed points results for generalized almost contractions. In addition, we give an application in Integral Equations of our main theorem.

2 Preliminaries

Definition 2.1. [1], Let P be a subset of E, where E is an ordered Banach space. The set P is said to be a cone if it satisfies the following conditions:

1. $P \neq \{0\}$,
2. $ax + by \in P$ for each $a, b \in R$ and $x, y \in P$,
3. if $x \in P$ then $-x \in P$.

The cone P is called normal if for every $x, y \in P$, $x \leq y$ then $\|x\| \leq K\|y\|$, where $K > 0$. K is called the normality constant of P.

The authors in, [1], have defined a partial ordering relation in cone P as follows.

For each $x, y \in P, x \leq y$ if $y - x \in P$ and $x < y$ if $x \leq y$ and $x \neq y$; for every $x, y \in P, x \ll y$ only if $y - x \in intP$.

Definition 2.2. [1], Let P be a cone and X a non–empty set. The map $d: X \times X \to P$ is called a cone metric if it satisfies the following conditions:

1. $d(x, y) = 0$ if and only if $x = y$, for every $x, y \in X$,
2. $d(x, y) = d(y, x)$ for every $x, y \in X$,
3. $d(x, z) \leq d(x, y) + d(y, z)$ for each $x, y, z \in X$.

The ordered couple (X, d) is called a cone metric space.

The authors in, [2], generalized the cone metric space to quasi-cone metric space as follows:

Definition 2.3. [2], Let P be a cone and X a nonempty set. The map $d: X \times X \to P$ is called a quasi-cone metric if it satisfies the following conditions:

1. $d(x, y) = 0$ if and only if $x = y$, for every $x, y \in X$,
2. $d(x, z) \leq d(x, y) + d(y, z)$ for each $x, y, z \in X$.

The ordered couple (X, d) is called a quasi-cone metric space.

In, [5], there were extended b-metric spaces to extended b-metric spaces by replacing the third condition of the metric.

Definition 2.4. [5], Let X be a nonempty set and $\tau: X \times X \to [1, +\infty)$ be a function. The mapping $d_\tau: X \times X \to [0, +\infty)$ is called extended b-metric space if it satisfies the following conditions:

1. $d_\tau(x, y) = 0$ if and only if $x = y$, for every $x, y \in X$;
2. $d_\tau(x, y) = d_\tau(y, x)$ for every $x, y \in X$,
3. $d_\tau(x, z) \leq \tau(x, z)(d_\tau(x, y) + d_\tau(y, z))$ for each $x, y, z \in X$.

The pair (X, d_τ) is called an extended b-metric space.

Inspired by authors in, [2], and, [5], we define a new generalization of quasi-cone space as below:

Definition 2.5. Let P be a cone, $\tau: X \times X \to [1, +\infty)$ be a function, and X a nonempty set. The mapping $q_\tau: X \times X \to P$ is called extended quasi-cone b-metric if it satisfies the following conditions:

1. $d_\tau(x, y) = 0$ if and only if $x = y$, for every $x, y \in X$,
2. $d_\tau(x, z) \leq \tau(x, z)(d_\tau(x, y) + d_\tau(y, z))$ for each $x, y, z \in X$.

The pair (X, q_τ) is called an extended cone b-metric space.

Example 2.6. Let $(E = R, \|\ \ \|)$ be an ordered Banach space where $\|\ \ \|$is the Euclidian norm and $P = [0, +\infty)$ is a cone, $X = \{0,1,2\}$. Define $\tau: X \times X \to [1, +\infty)$, $\tau(x, y) = x + y + 1$ and $q_\tau: X \times X \to P$, $q_\tau(0,0) = q_\tau(1,1) = q_\tau(2,2) = 0$, $q_\tau(0,1) = \frac{8}{9}$, $q_\tau(1,0) = \frac{4}{5}$, $q_\tau(1,2) = \frac{7}{9}$, $q_\tau(2,1) = \frac{2}{5}$, $q_\tau(0,2) = \frac{1}{9}$, and $q_\tau(2,0) = \frac{3}{5}$.

We see that the function satisfies both condition 1 and 2 of extended quasi-cone b-metric space. As a result, q_τ is an extended quasi-cone b-metric, and the couple (X, q_τ) is an extended quasi-cone b-metric space, but it is not a quasi-cone metric space.

Remark 2.7. If $\tau: X \times X \to [1, +\infty)$, $\tau(x, y) = s$ for each $x, y \in X$, where $s \in [1, +\infty)$ is constant, then the pair (X, q_τ) is a quasi-cone b-metric space.

Below, we present some topological aspects of extended quasi-cone b-metric spaces.

Let E be an ordered Banach space, $P \subset E$ a regular cone, and normal in E with normality constant $K \geq 1$ and (X, q_τ) an extended quasi-cone b-metric space.

We take $a \in X$, $c \gg 0$ where $c \in P$.

Definition 2.8. The set $B_r(a,c) = \{x \in X, q_\tau(a,x) \ll c\}$, $(B_l(a,c) = \{x \in X: q_\tau(a,x) \ll c\}$) is called the right (left) open ball with center a and radius c.

Definition 2.9. The set $B_r^{cl}(a,c) = \{x \in X: q_\tau(x,a) \leq c\}$ $(B_l^{cl}(a,c) = \{x \in X: q_\tau(a,x) \leq c\})$ is called the right (left) closed ball with center a and radius c.

Theorem 2.10. Let (X, q_τ) be an extended quasi-cone b-metric space. The family $\mathfrak{I}^r = \{\varphi, X, G \subset X, \text{for each } a \in G, \text{there exists } B_r(a,c) \subset \text{G}\}$ is a topology in X.

The topology $\mathfrak{I}^r$is is called right topology induced by the extended quasi-cone b-metric q_τ.

Similarly, we can define the left topology induced by the extended quasi-cone b-metric q_τ.

We establish the following results for the right topology in (X, q_τ).

Definition 2.11. The set $A \subset X$ is called open if $A \in \mathfrak{I}^r$.

Definition 2.12. The set $V \subset X$ is called the right neighborhood of $a \in X$ if there exists an open ball centered in a such that $B_r(a,c) \subset V$.

Remark 2.13. The topology $\mathfrak{I}^r$ in (X, q_τ) satisfies the First Axiom of Countability.

Theorem 2.14. The set $A \subset X$ is right open if and only if, for each point $a \in A$, there exists $B_r(a,c)$ such that $B_r(a,c) \subset A$.

Definition 2.15. The set $A \subset X$ is right closed if its complement in X is right to open.

Theorem 2.16. The space $(X, \mathfrak{I}^r)$ is T_1.

Proof. We have to show that for each $x \in X$, the set $\{x\}$ is right closed, or the set $X - \{x\}$ is right open. Taking $a \in X - \{x\}$, then $a \neq x$ and $q_p(x,a) \gg 0$. Denote $c = q_\tau(x,a)$ and since $c \gg 0$, there exists $n \in N$ such that $0 \ll \frac{c}{2n} \ll c$. We prove that $B(a, \frac{c}{2n}) \subset X - \{x\}$. Considering $z \in B(a, \frac{c}{2n})$, then $q_\tau(z,a) \ll \frac{c}{2n} \ll c = q_\tau(x,a)$. As a result, $z \neq x$, and $z \in X - \{x\}$. Consequently, $\mathfrak{I}^r$ is T_1.

Definition 2.17. Let (X, q_τ) be an extended quasi-cone b-metric space and (x_n) a sequence in X.

1. The sequence (x_n) is called right (left) convergent to $x \in X$ if, for every $c \gg 0$, there exists $n_0 \in N$, such that for each $n \geq n_0$, it implies $q_\tau(x_n, x) \ll c$ $(q_\tau(x, x_n) \ll c)$ or $\lim_{n \to +\infty} q_\tau(x_n, x) = 0$ $(\lim_{n \to +\infty} q_\tau(x, x_n) = 0)$.
2. The sequence (x_n) is called bi-convergent to $x \in X$ if it is right and left convergent to $x \in X$.
3. The sequence (x_n) is called right (left) Cauchy in X if, for every $c \gg 0$, there exists $n_0 \in N$, such that for each $n > m > n_0$ it yields $q_\tau(x_n, x_m) \ll c$ $(q_\tau(x_m, x_n) \ll c)$ or $\lim_{n,m \to +\infty} q_\tau(x_n, x_m) = 0$ $(\lim_{n,m \to +\infty} q_\tau(x_m, x_n) = 0)$.
4. The sequence (x_n) is called bi-Cauchy in X if it is right and left Cauchy in X.
5. The extended quasi-cone b-metric space (X, q_τ) is called complete if every bi-Cauchy sequence in X is bi-convergent.

Remark 2.18. There exist sequences that are left Cauchy or right Cauchy but not bi-Cauchy.

Example 2.19. Taking $X = [0,1], E = R^2, P = \{(c_1, c_2): c_1, c_2 \geq 0\}$ and considering $q_\tau: X \times X \to P$

$$q_\tau(x,y) = \begin{cases} ((x-y)^2, p^2(x-y)^2), & x \geq y \\ (p^2, 1), & x < y \end{cases}$$

for $0 < p < 1$ and $\tau: X \times X \to [1, +\infty), \tau(x,y) = e^{(x-y)^2}$, the couple $(X,\ q_\tau)$ is an extended quasi-cone b-metric space.

Let $x_n = \frac{1}{n}$ be a sequence in $X = [0,1]$. For $n < m$, we have $q_\tau(x_m, x_n) = q_\tau\left(\frac{1}{m}, \frac{1}{n}\right) = (p^2, 1)$, which shows that the sequence $x_n = \frac{1}{n}$ is not right Cauchy.

For $n > m$, we have $q_\tau(x_m, x_n) = q_\tau\left(\frac{1}{m}, \frac{1}{n}\right) = ((x-y)^2, p^2(x-y)^2)$ which proves that the sequence $(\frac{1}{n})$ is left Cauchy.

Authors in, [12], presented a new class of weak contractions called almost contraction:

Let (X, d) be a metric space and $T: X \to X$ a function. T is called almost contraction or (h, L)-contraction if it satisfies the following inequality:

$$d(Tx, Ty) \leq hd(x, y) + Ld(x, Ty)$$

for each $(x, y) \in X \times X$, $h \in (0,1)$ and $L \geq 0$.

In, [13], extended this contraction to generalized almost contraction.

Let (X, d) be a metric space and $T: X \to X$ a function. T is called generalized almost contraction if it completes the following inequality:

$$d(Tx, Ty) \leq h \max\{d(x, y), d(Tx, x), d(Ty, y), \frac{d(Tx, y) + d(x, Ty)}{2}\} + L \min\{d(x, y), d(Tx, x), d(Ty, y), d(Tx, y), d(x, Ty)\}$$

for each $(x, y) \in X \times X$, $h \in (0,1)$ and $L \geq 0$.

He proved the existence and the uniqueness of fixed points on the respective contractions in metric space.

Definition 2.20. [14], The function $\varphi: P \to P$ is called a comparison function if it satisfies the following conditions:

1. $\varphi(t) < t$ for each $t \in P$,
2. $\lim_{n\to+\infty} \|\varphi^n(t)\| = 0$, for each $t \in P$.

Initiating from above, we determine a new class of generalized almost contractions called τ-almost contraction, as follows:

Denote

$M(x, y) = \max\{q_\tau(x, y), q_\tau(Tx, x), q_\tau(Ty, y), \frac{q_\tau(Tx, y) + q_\tau(x, Ty)}{2\tau(x, y)}\}$ and $m(x, y) = \min\{q_\tau(x, y), q_\tau(Tx, x), q_\tau(Ty, y), q_\tau(Tx, y), q_\tau(x, Ty)\}$.

The function T is called a generalized τ-almost contraction if it satisfies the following inequality:

$$q_\tau(Tx, Ty) \leq \varphi\big(M(x, y)\big) + Lm(x, y) \quad (1)$$

for every $x, y \in X$ where $\varphi: P \to P$ is a comparison function, and $L \geq 0$.

3 Main Results

3.1 Fixed Point Results

In this section, we show some fixed-point results for generalized τ-almost contractions in extended quasi-cone b-metric space.

Lemma 3.1.1. Let (X, q_τ) be a complete and Hausdorff extended quasi-cone b-metric space, with the normality constant of cone K, and $T: X \to X$ a generalized τ-almost contraction. Let $x_0 \in X$ be a point in X such that $O(x_0)$is bounded. Then, for each $n \geq 1$, the inequalities $q_\tau(T^{n+1}x_0, T^n x_0) \leq \varphi^n(c)$ (2) and $q_\tau(T^n x_0, T^{n+1}x_0) \leq \varphi^n(c)$ (3) hold, where $c \in P$.

Proof. Since the orbit $O(x_0)$ is bounded, there exits $c \in P$ such that $\delta\big(O(x_0)\big) \leq c$. To prove that $q_\tau(T^{n+1}x_0, T^n x_0) \leq \varphi^n(c)$, for each $n \geq 1$, we use the mathematical induction method. For $n = 1$, we have

$$q_\tau(T^2x_0, Tx_0) = q_\tau(T(Tx_0), Tx_0)$$

$$\leq \varphi\left(\begin{matrix}\max\{q_\tau(Tx_0, x_0), q_\tau(T^2x_0, Tx_0), \\ q_\tau(Tx_0, x_0), \frac{q_\tau(T^2x_0, x_0) + q_\tau(Tx_0, Tx_0)}{2\tau(Tx_0, x_0)}\}\end{matrix}\right)$$

$$+L \min\{q_\tau(Tx_0, x_0), q_\tau(T^2x_0, Tx_0), q_\tau(Tx_0, x_0), q_\tau(T^2x_0, x_0), q_\tau(Tx_0, Tx_0)\}$$

$$= \varphi(\max\{q_\tau(Tx_0, x_0), q_\tau(T^2x_0, Tx_0)\})$$

$$= \varphi(\max\{q_\tau(Tx_0, x_0), q_\tau(T^2x_0, Tx_0)\}).$$

Since the orbit of a point x_0 is bounded, $\delta\big(O(x_0)\big) \leq c$, we have

$\max\{ q_\tau(Tx_0, x_0), q_\tau(T^2x_0, Tx_0)\} \leq c$.

As a result, the inequality $q_\tau(T^2x_0, Tx_0) \leq \varphi(c)$ holds.

Suppose that for $k < n, q_\tau\left(T^{k+1}x_0, T^kx_0\right) \leq \varphi^k(c)$ is true.

Let's prove the inequality for $n > k$.

We see that the extended quasi-cone metric q_τ satisfies the following inequality:

$$q_\tau\left(T(T^nx_0), T(T^{n-1}x_0)\right)$$

$$\leq \varphi\left(\begin{matrix}\max\{ q_\tau(T^nx_0, T^{n-1}x_0), q_\tau(T^{n+1}x_0, T^nx_0), \\ q_\tau(T^nx_0, T^{n-1}x_0), \\ \frac{q_\tau(T^{n+1}x_0, T^{n-1}x_0) + q_\tau(T^nx_0, T^nx_0)}{2\tau(T^nx_0, T^{n-1}x_0)}\}\end{matrix}\right)$$

$$+L\min\{ q_\tau(T^nx_0, T^{n-1}x_0), q_\tau(T^{n+1}x_0, T^nx_0), \\ q_\tau(T^nx_0, T^{n-1}x_0), q_\tau(T^{n+1}x_0, T^{n-1}x_0), \\ q_\tau(T^nx_0, T^nx_0)\}$$

$$\leq \varphi\left(\begin{matrix}\max\{ q_\tau(T^nx_0, T^{n-1}x_0), q_\tau(T^{n+1}x_0, T^nx_0), \\ \frac{q_\tau(T^nx_0, T^{n-1}x_0) + q_\tau(T^{n+1}x_0, T^nx_0)}{2}\}\end{matrix}\right)$$

$$= \varphi(\max\{ q_\tau(T^nx_0, T^{n-1}x_0), q_\tau(T^{n+1}x_0, T^nx_0)\}).$$

Considering the values of $\max\{ q_\tau(T^nx_0, T^{n-1}x_0), q_\tau(T^{n+1}x_0, T^nx_0)\}$, we have the following cases:

If $\max\{ q_\tau(T^nx_0, T^{n-1}x_0), q_\tau(T^{n+1}x_0, T^nx_0)\} = q_\tau(T^nx_0, T^{n-1}x_0)$, then we have:

$q_\tau(T^{n+1}x_0,\ T^nx_0) \leq \varphi\left(q_\tau(T^nx_0, T^{n-1}x_0)\right) \leq \varphi\left(\varphi^{n-1}(c)\right) = \varphi^n(c)$, we get $q_\tau(T^{n+1}x_0,\ T^nx_0) \leq \varphi^n(c)$

If $q_\tau(T^{n+1}x_0,\ T^nx_0) \leq \varphi\left(q_\tau(T^{n+1}x_0, T^nx_0)\right)$ then $q_\tau(T^{n+1}x_0, T^nx_0) = 0 \leq \varphi^n(c)$.

Consequently, for every $n = 1,2, \ldots$ the inequality $q_\tau(T^{n+1}x_0,\ T^nx_0) \leq \varphi^n(c)$ holds.

In addition, q_τ completes the following inequality:

$$q_\tau\left(T(T^{n-1}x_0), T(T^nx_0)\right)$$

$$\leq \varphi\left(\begin{matrix}\max\{ q_\tau(T^{n-1}x_0, T^nx_0), q_\tau(T^nx_0, T^{n-1}x_0), \\ q_\tau(T^{n+1}x_0, T^nx_0)\}, \\ \frac{q_\tau(T^nx_0, T^nx_0) + q_\tau(T^{n-1}x_0, T^{n+1}x_0)}{2\tau(T^{n-1}x_0, T^nx_0)}\end{matrix}\right)$$

$$+L\min\{ q_\tau(T^{n-1}x_0, T^nx_0), q_\tau(T^nx_0, T^{n-1}x_0), \\ q_\tau(T^{n+1}x_0, T^nx_0), q_\tau(T^nx_0, T^nx_0), \\ q_\tau(T^{n-1}x_0, T^{n+1}x_0)\}$$

$$= \varphi\left(\max\{ q_\tau(T^{n-1}x_0, T^nx_0), q_\tau(T^nx_0, T^{n-1}x_0)\right)$$

$$\leq \varphi^n(c).$$

So, the inequality $q_\tau(T^nx_0,\ T^{n+1}x_0) \leq \varphi^n(c)$ is true for each $n = 1,2, \ldots$.

Theorem 3.1.2. Let (X, q_τ) be a complete and Hausdorff extended quasi cone b-metric space, with the normality constant of cone K, and $T: X \to X$ is generalized τ-almost contraction. Let $x_0 \in X$ be a point in X such that $O(x_0)$ is bounded, $\delta\left(O(x_0)\right) \leq c$ and $\lim_{n,m\to+\infty} \frac{\|\varphi^{m+1}(c)\|}{\|\varphi^m(c)\|}\tau(x_n, x_m) < 1$. Then, the function T has a unique fixed point in X.

Proof. Defining the sequence $\{x_n = T^nx_0\}$, we consider the following cases.

If there exists $n = 1,2, \ldots,\ x_n = x_{n+1}$, then the function T has a fixed point x_n.

Suppose that $x_n \neq x_{n+1}$, for each $n = 1,2, \ldots$. Using Lemma 3.1.1, we have that for all $n = 1,2, \ldots$,

$$q_\tau(x_{n+1}, x_n) = q_\tau(T^{n+1}x_0,\ T^nx_0) \leq \varphi^n(c).$$

Firstly, we prove that the sequence $(x_n = T^nx_0)$ is bi-Cauchy (right and left Cauchy).

Taking $n > m$, and using step-by-step the third condition of extended quasi-cone metric, yields

$$q_\tau(x_n, x_m) \leq \tau(x_n, x_m)\big(q_\tau(x_n, x_{m+1}) \\ + q_\tau(x_{m+1}, x_m)\big)$$

$$\leq\ \tau(x_n, x_m)q_\tau(x_n, x_{m+1}) \\ + \tau(x_n, x_m)q_\tau(x_{m+1}, x_m)$$

$$\leq \tau(x_n, x_m)\tau(x_n, x_{m+1})\big(q_\tau(x_n, x_{m+2}) + q_\tau(x_{m+2}, x_{m+1})\big)$$

$$+\tau(x_n, x_m)q_\tau(x_{m+1}, x_m) \leq \cdots$$

$$\leq \tau(x_n, x_m)\tau(x_n, x_{m+1}) \ldots \tau(x_n, x_{n-1})q_\tau(x_n, x_{n-1})$$

$$+\cdots+$$

$$\tau(x_n, x_m)\tau(x_n, x_{m+1})q_\tau(x_{m+2}, x_{m+1})$$

$$+\tau(x_n, x_m)q_\tau(x_{m+1}, x_m)$$

$$\leq \tau(x_n, x_1)\tau(x_n, x_2) \ldots \tau(x_n, x_m)\tau(x_n, x_{m+1}) \ldots$$

$$\tau(x_n, x_{n-1})\varphi^{n-1}(c) + \cdots +$$

$$\tau(x_n, x_1)\tau(x_n, x_2) \ldots \tau(x_n, x_m)$$

$$\tau(x_n, x_{m+1})\varphi^{m+1}(c) +$$

$$\tau(x_n, x_1)\tau(x_n, x_2) \ldots \tau(x_n, x_m)\varphi^m(c).$$

Taking the norm of both sides and using the normality property of the cone, we obtain

$$\|q_\tau(x_n, x_m)\| \leq$$

$$K\left(\left\|\begin{matrix}\tau(x_n, x_1)\tau(x_n, x_2) \ldots \tau(x_n, x_m)\tau(x_n, x_{m+1}) \\ \ldots \tau(x_n, x_{n-1})\varphi^{n-1}(c) \\ +\cdots+ \\ \tau(x_n, x_1)\tau(x_n, x_2) \ldots \tau(x_n, x_m) \\ \tau(x_n, x_{m+1})\varphi^{m+1}(c) \\ +(x_n, x_1)\tau(x_n, x_2) \ldots \tau(x_n, x_m)\varphi^m(c)\end{matrix}\right\|\right)$$

$$\leq K\begin{pmatrix}\tau(x_n, x_1)\tau(x_n, x_2) \ldots \tau(x_n, x_m)\tau(x_n, x_{m+1}) \\ \ldots \tau(x_n, x_{n-1})\|\varphi^{n-1}(c)\| \\ +\cdots+ \\ \tau(x_n, x_1)\tau(x_n, x_2) \ldots \tau(x_n, x_m) \\ \tau(x_n, x_{m+1})\|\varphi^{m+1}(c)\| \\ +\tau(x_n, x_1)\tau(x_n, x_2) \ldots \tau(x_n, x_m)\|\varphi^m(c)\|\end{pmatrix}$$

As a result, we have shown that:

$$\|q_\tau(x_n, x_m)\| \leq K(S_{n-1} - S_m),$$

where $\{S_n\}_{n\in N}$ is the sequence of partial sums of series $\sum_{m=1}^{\infty}\|\varphi^m(c)\| \prod_{i=1}^{m} \tau(x_n, x_i)$.

This series is convergent since it satisfies the D'Alembert Criterion.

$$\lim_{n,m\to+\infty} \frac{\|\varphi^{m+1}(c)\| \prod_{i=1}^{m+1} \tau(x_n, x_i)}{\|\varphi^m(c)\| \prod_{i=1}^{m} \tau(x_n, x_i)} =$$

$$\lim_{n,m\to+\infty} \frac{\|\varphi^{m+1}(c)\|}{\|\varphi^m(c)\|}\tau(x_n, x_{m+1}) < 1.$$

In addition $\lim_{n,m\to+\infty}\|q_\tau(x_n, x_m)\| = 0$ and $\lim_{n,m\to+\infty} q_\tau(x_n, x_m) = 0$.

So, we proved that the sequence $\{x_n\}_{n\in N}$ is left Cauchy.

Using the same method, we demonstrate that the sequence $\{x_n\}_{n\in N}$ is right Cauchy and it is Cauchy. Since the space (X, q_τ) is complete, the sequence $\{x_n = T^n x_0\}_{n\in N}$ converges to a point $x^* \in X$, and $\lim_{n\to+\infty} q_\tau(x_n, x^*) = \lim_{n\to+\infty} q_\tau(x^*, x_n) = 0$.
We must show that x^* is a fixed point of mapping T.

$$q_\tau(Tx^*, T^{n+1}x_0) \leq$$

$$\varphi\left(\begin{matrix}\max\{q_\tau(x^*, T^n x_0), q_\tau(Tx^*, x^*), \\ q_\tau(T^{n+1}x_0, T^n x_0)\dfrac{q_\tau(Tx^*, T^n x_0) + q_\tau(x^*, T^{n+1}x_0)}{2\tau(x^*, T^n x_0)}\}\end{matrix}\right)$$

$$+L\ \min\{q_\tau(x^*, T^n x_0), q_\tau(Tx^*, x^*),$$

$$q_\tau(T^{n+1}x_0, T^n x_0), q_\tau(Tx^*, T^n x_0), q_\tau(x^*, T^{n+1}x_0)\}$$

Taking the limits of both sides, we derive to $\lim_{n\to+\infty} q_\tau(Tx^*, T^{n+1}x_0) = 0$.
Similarly, we can show that $\lim_{n\to+\infty} q_\tau(T^{n+1}x_0, Tx^*) = 0$.

Consequently, the sequence $\{x_n = T^n x_0\}_{n\in N}$ converges to the point Tx^*. Since the space is Hausdorff, it yields that $Tx^* = x^*$.

In the end, we prove that x^* is the unique fixed point of T.

Suppose that there exists another fixed point $y^* \in X$, of T, $Ty^* = y^*$.

We see that

$$q_\tau(x^*, y^*) = q_\tau(Tx^*, Ty^*) \leq$$

$$\varphi\left(\begin{matrix}\max\{q_\tau(x^*, y^*), q_\tau(Tx^*, x^*), \\ q_\tau(Ty^*, y^*), \dfrac{q_\tau(Tx^*, y^*) + q_\tau(x^*, Ty^*)}{2\tau(x^*, y^*)}\}\end{matrix}\right\}$$

$$+L\ \min\{q_\tau(x^*, y^*), q_\tau(Tx^*, x^*), q_\tau(Ty^*, y^*),$$

$$q_\tau(Tx^*, y^*), q_\tau(x^*, Ty^*)\} = \varphi(q_\tau(x^*, y^*)).$$

This inequality shows that $q_\tau(x^*, y^*) = 0$, and $x^* = y^*$.

The following example illustrates Theorem 3.1.2.

Example 3.1.3 Considering $X = [0,1], E = R^2$, the cone $P = \{(x, y) \in R^2, x, y \geq 0\}$ with normality constant $K = 1$, $\tau: X \times X \to [1, +\infty)$, $\tau(x, y) = x + y + 1$, and

$$q_\tau: X \times X \to P, q_\tau: (x, y) = \begin{cases} \left(\frac{y}{3}, y\right), & x < y \\ (x, 3x), & x \geq y \end{cases},$$

the couple (X, q) is an extended quasi cone b metric space.

Define $T: X \to X, Tx = \begin{cases} \frac{x^3}{27}, & 0 \leq x < \frac{1}{81} \\ \frac{x}{27}, & \frac{1}{81} \leq x \leq 1 \end{cases}$ and $\varphi: P \to P$, $\varphi(x, y) = (\frac{x}{3}, \frac{y}{3})$ which is a comparison function.

Considering $0 \leq x < \frac{1}{81}, Tx = \frac{x^3}{27}$, we obtain $T^n x_0 = \left(\frac{x_0}{3}\right)^{3^n}, T^m x_0 = \left(\frac{x_0}{3}\right)^{3^m}$.

Furthermore, we achieve

$$\lim_{n,m\to+\infty} \frac{\|\varphi^{m+1}(c)\|}{\|\varphi^m(c)\|} \tau(x_n, x_m) =$$

$$\lim_{n,m\to+\infty} \frac{1}{3}\left((\frac{x_0}{3})^{3^n} + (\frac{x_0}{3})^{3^m} + 1\right) = \frac{1}{3} < 1.$$

In addition, taking into consideration $\frac{1}{81} \leq x \leq 1$, the following equalities hold:

$$Tx = \frac{x}{81}, T^n x_0 = \frac{x_0}{27^n}, T^m x_0 = \frac{x_0}{27^m},$$

and

$$\lim_{n,m\to+\infty} \frac{\|\varphi^{m+1}(c)\|}{\|\varphi^m(c)\|} \tau(x_n, x_m) =$$

$$\lim_{n,m\to+\infty} \frac{1}{3}\left(\frac{x_0}{27^n} + \frac{x_0}{27^m} + 1\right) = \frac{1}{3} < 1.$$

The next step is to demonstrate that function T satisfies the inequality (1) of Lemma 3.1.1 by taking into account the following cases:

If $x, y \in [0, \frac{1}{81}), x < y$, we obtain $q_\tau(Tx, Ty) = q_\tau\left(\frac{x^3}{27}, \frac{y^3}{27}\right) = \left(\frac{y^3}{81}, \frac{y^3}{27}\right) < \left(\frac{y}{9}, \frac{y}{3}\right) \leq \varphi\big(M(x, y)\big) + Lm(x, y)$.

For $x, y \in [0, \frac{1}{81}), x > y$, we get

$$q_\tau(Tx, Ty) = q_\tau\left(\frac{x^3}{27}, \frac{y^3}{27}\right) = \left(\frac{x^3}{27}, \frac{x^3}{9}\right) <$$

$$\varphi\big(M(x, y)\big) + Lm(x, y).$$

Considering $x, y \in [\frac{1}{81}, 1], x < y$, it yields

$$q_\tau(Tx, Ty) = q_\tau\left(\frac{x}{27}, \frac{y}{27}\right) =$$

$$\left(\frac{y}{81}, \frac{y}{27}\right) \leq \varphi\big(M(x, y)\big) + Lm(x, y).$$

Taking $x, y \in \left[\frac{1}{81}, 1\right], x > y$, we have

$$q_\tau(Tx, Ty) = q_\tau\left(\frac{x}{27}, \frac{y}{27}\right) = \left(\frac{x}{27}, \frac{x}{9}\right) \leq$$

$$\varphi\big(M(x, y)\big) + Lm(x, y).$$

For $x \in [0, \frac{1}{81}), y \in [\frac{1}{81}, 1]$, we get the following inequality

$$q_\tau(Tx, Ty) = q_\tau\left(\frac{x^3}{27}, \frac{y}{27}\right) = \left(\frac{y}{81}, \frac{y}{27}\right) \leq$$

$$\varphi\big(M(x, y)\big) + Lm(x, y).$$

If $y \in \left[0, \frac{1}{16}\right), x \in [\frac{1}{16}, 1]$, it comes

$$q_\tau(Tx, Ty) = q_\tau\left(\frac{x}{27}, \frac{y^3}{27}\right) = \left(\frac{x}{27}, \frac{x}{9}\right) \leq$$

$$\varphi\big(M(x, y)\big) + Lm(x, y).$$

As a result, we have that for every $(x, y) \in X \times X$, the inequality $q_\tau(Tx, Ty) \leq \varphi\big(M(x, y)\big) + Lm(x, y)$ holds.

The function T satisfies the conditions of Theorem 3.1.2, and it has a unique fixed point $x = 0$.

Corollary 3.1.4. Let (X, q_τ) be a complete and Hausdorff extended quasi cone b-metric space, with the constant of normality of cone K, and $T: X \to X$a function that satisfies the generalized almost contraction:

$$q_\tau(Tx, Ty) \le hM(x, y) + Lm(x, y) \quad (4)$$

for every $x, y \in X$ where $0 < h < 1$, and $L \ge 0$.

Let $x_0 \in X$ be a point in X such that $O(x_0)$ is bounded $\delta(O(x_0)) \le c$ and $\lim_{n,m\to+\infty} h\tau(x_n, x_m) < 1$. Then, the function T has a unique fixed point in X.

Proof. Taking $\varphi(t) = ht, 0 < h < 1$ in Theorem 3.1.2, we prove Corollary 3.1.4.

Remark 3.1.5. Corollary 3.1.4 extends the result of [14], in extended quasi-cone metric space.

Theorem 3.1.6. Let (X, q_τ) be a complete and Hausdorff extended quasi cone b-metric space, with the constant of normality of cone K, and T: X→X a function that satisfies the nonlinear contraction:

$$q_\tau(Tx, Ty) \le \varphi\big(M(x, y)\big) \quad (5)$$

for every $x, y \in X$ where $\varphi: P \to P$ is a comparison function,

Let $x_0 \in X$ be a point in X such that $O(x_0)$ is bounded $\delta(O(x_0)) \le c$ and $\lim_{n,m\to+\infty} \frac{\|\varphi^{m+1}(c)\|}{\|\varphi^{m}(c)\|}\tau(x_n, x_m) < 1$. Then, the function T has a unique fixed point in X.

Proof. For every $x, y \in X$, we have that

$$q_\tau(Tx, Ty) \le \varphi\big(M(x, y)\big) \le \varphi\big(M(x, y)\big) + Lm(x, y).$$

As a result, the function T satisfies the inequality (1), and it has a unique fixed point in X.

Example 3.1.7. Let X be the segment [0,1], $E = R^2$ and $P = \{(x, y) \in E, x \ge 0, y \ge 0\}$ is a cone. Determine $q_\tau: X^2 \to P$,

$$q_\tau(x, y) = \begin{cases} \left(x, \frac{x}{2}\right), & x > y \\ (0,0), & x = y, \\ \left(\frac{y}{2}, y\right), & y > x \end{cases}$$

where $\tau: X^2 \to [1, +\infty)$, $\tau(x, y) = 2x + y + 5$.

The pair (X, q_τ) is an extended quasi-cone metric space. Let $T: [0,1] \to [0,1]$, $Tx = \frac{x^2+x}{5}$ be a function, and $\varphi: P \to P$, $\varphi(t, s) = \left(\frac{t}{2}, \frac{s}{2}\right)$ is a comparison function.

The function T satisfies the conditions of Theorem 3.1.6. Consequently, it has a unique fixed point in $X, x = 0$.

Corollary 3.1.8. Let (X, q_τ) be a complete and Hausdorff extended quasi cone b-metric space, with a constant of normality of cone K, and $T: X \to X$a function that satisfies the contraction:

$$q_\tau(Tx, Ty) \le hM(x, y) \quad (6)$$

for every $x, y \in X$ where $0 < h < 1$. Let $x_0 \in X$ be a point in X such that $O(x_0)$ is bounded $\delta(O(x_0)) \le c$ and $\lim_{n,m\to+\infty} h\tau(x_n, x_m) < 1$. Then, the function T has a unique fixed point in X.

Remark 3.1.9. Corollary 3.1.8 extends the result in, [11], on extended quasi b-cone metric space.

3.2 An Application to Integral Equation

Fixed Point Theory has a huge application to Integral equations, where it guarantees the existence and uniqueness of the solution. These applications are studied by many authors who have contributed to Fixed Point Theory, [15], [16], [17].

Considering $E = R, P = [0, +\infty), X = C([0,1], \mathbb{R})$, and $q_\tau: X \times X \to P$, given by

$$q_\tau\big(x(t), y(t)\big) = \begin{cases} \sup_{t\in[0,1]} |x(t) - y(t)|, & x(t) < y(t) \\ 2\sup_{t\in[0,1]} |x(t) - y(t)|, & x(t) \ge y(t)' \end{cases}$$

where $\tau\big(x(t), y(t)\big) = |x(t)| + |y(t)| + 2$, the couple (X, q_τ) is a completely extended quasi-cone b-metric space.

Theorem 3.2.1. The integral equation $x(t) = m(t) + \int_0^1 K(t, s) r(s, x(s)) ds$, where $x \in C([0,1], \mathbb{R})$.

and $m: [0,1] \to \mathbb{R}$ is a continuous function, $K: [0,1] \times \mathbb{R} \to [0, +\infty)$ and $r: [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions which satisfy the following conditions:

1. $\int_0^1 (|r(s, x(s)) - r(s, y(s))|) ds \leq$ $\max\{|x(t) - y(t)|, |Tx(t) - x(t)|, |Ty(t) - y(t)|, \frac{|Tx(t)-y(t)|+|x(t)-Ty(t)|}{(|x(t)|+|y(t)|+2)}\}$ for all $t \in [0,1]$,
2. $K(t,s) \leq h < 1, L > 0$;

has a unique solution in $C_{[0,1]}$.

Proof. Define the mapping $T: X \to X$ given by $Tx(t) = \int_0^1 K(t,s) r(s, x(s)) ds$.
Below, we show that the mapping T satisfies the conditions of Corollary 3.1.5.
Firstly, we see that:

$$|Tx(t) - Ty(t)| \leq$$

$$\left| \int_0^1 K(t,s) \left(r(s, x(s)) - r(s, y(s)) \right) ds \right|$$

$$\leq \int_0^1 K(t,s) (|r(s, x(s)) - r(s, y(s))|) ds$$

$$\leq \int_0^1 K(t,s) \max\{|x(s) - y(s)|, |Tx(s) - x(s)|, |Ty(s) - y(s)|, \frac{|Tx(s) - y(s)| + |x(s) - Ty(s)|}{(|x(s)| + |y(s)| + 2)}\} ds \leq$$

$$\leq \left(\int_0^1 K(s,t) ds \right) \max\{q_\tau(x(s), y(s)), q_\tau(Tx(s), x(s)), q_\tau(Ty(s), y(s)), \frac{q_\tau(Tx(s), y(s)) + q_\tau(x(s), Ty(s))}{\tau(x(s), y(s))}\}$$

$$\leq h[\max\{q_\tau(x(s), y(s)), q_\tau(Tx(s), x(s)), q_\tau(Ty(s), y(s)), \frac{q_\tau(Tx(s), y(s)) + q_\tau(x(s), Ty(s))}{\tau(x(s), y(s))}\}]$$

$$\leq hM(x(s), y(s)) + Lm(x(s), y(s)).$$

Hence, we have

$$q_\tau(Tx(s), Ty(s))$$

$$\leq hM(x(s), y(s)) + Lm(x(s), y(s)).$$

Consequently, the function T completes the conditions of Corollary 3.1.5, and it has a unique fixed point. This result leads to proof of the existence and uniqueness of the solution of integral equation $x(t) = h(t) + \int_0^1 K(t,s) r(s, x(s)) ds$.

4 Conclusion

In this paper, we have defined new extended quasi-cone b-metric spaces that generalize quasi-cone b-metric spaces and cone metric spaces. There are proven several topological properties of these spaces. The highlight of this paper is Theorem 3.1.2, which guarantees- the existence and uniqueness of a fixed point for a generalized almost contraction in extended quasi-cone b-metric space. This crucial result extends Theorem 2 in, [5], Theorem 1 in, [12], Theorem 2.1 in, [14], on ordered metric spaces, and Theorem 3 in, [18]. The main theoretical result is associated with an example that shows its applicable side. Theorem 3.2.1 shows an application of Fixed-Point Theory to Integral Equations. It proves that there exists a unique solution for an integral equation. According to this application, this study contributes to Integral Equations

As a further study of this paper, the readers can see with interest the application of the main results section to Differential Equations.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

- Silvana Liftaj has given the idea and proved a significant part of section 3.1.
- Eriola Sila has defined the extended quasi-cone b-metric space, topology, and Example 3.1.3.
- Zamir Selko has found the application of fixed point results to Integral Equations.

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