Varieties of Systems of DEF Generated by Isomorphic Transformations

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Abstract: - Despite more than a century of origin and development, the theory of discrete exponential function (DEF) systems continues to attract the attention of mathematicians and application specialists in various fields of science and technology. One of the most successful applications of the DEF systems is the spectral processing of discrete signals based on fast Fourier transform (FFT) algorithms in the DEF bases. The construction of structural schemes of FFT algorithms is preceded, as a rule, by the factorization of the DEF matrices. The main problem encountered when factorizing DEF matrices is that the elements of such matrices are the degrees of phase multipliers, which are complex-valued quantities. In this connection, the computational complexity of factorization of DEF matrices may be too large, especially when the number of components of the matrix order decomposition is large. In this paper, we propose a relatively simple method of mutually unambiguous transition from complex-valued DEF matrices to matrices whose elements are natural numbers equal to the degree indices of phase multipliers in the canonical DEF matrices. Through this bijective transformation, the factorization of DEF matrices becomes significantly more manageable, streamlining the overall process of factorization.

Key-Words: - discrete exponential functions, mother and daughter systems of DEF, isomorphic transformations, factorization of DEF matrices, synthesis of DEF systems, interrelation of DEF systems.

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1 Introduction

Discrete exponential function (DEF) systems are a fundamental concept that plays a crucial role in signal and data processing and analysis. Various scientific and technological domains, including telecommunications, medical diagnostics, radio communications, geophysics, environmental research, and many others, widely use DEF systems.

The genuinely revolutionary role of the DEF systems played in the formation and development of the theory and practical applications of the discrete Fourier transform (DFT), which had a significant influence on the formation of algorithmic and software of modern information processing tools. The initial steps in developing the DEF algorithms were at the dawn of the XIX century. The first mentions of the DFT systems appeared in the, [1], published in 1801. In his dissertation, Gauss outlines a method for computing the DFT, although he did not use that name. Another important work was a paper by, [2], published in 1815. This paper developed a method similar to the DFT and was used to solve problems involving probability. Undoubtedly, the author made an invaluable contribution to the formation of the Fourier transform theory of the theory himself (20s of the XIX century). However, he concentrated his attention on the continuous transformation.

The first references to DEF systems appeared in the works, [3], [4]. In the modern sense, the DEF algorithm was formulated and optimized to the Fast Fourier Transform (FFT) jointly by, [5]. This work remains one of the key publications in FFT and eventually became known as the Cooley-Tukey Algorithm. Among the numerous areas of applications of the FFT algorithms, let us highlight spectral signal processing, [6], [7], [8], [9], computer vision, [10], data transmission in telecommunication systems, [11], medical information processing, [12], time series analysis, [13], quantum computing, [14], geophysics and geodesy, [15], and many others.

Note that the FFT algorithm applies only to such discrete sequences of samples of a continuous signal whose sampling volume, denoted by N, is a composite number. The simplicity of the FFT tree becomes particularly pronounced when N is a power of two, denoted as $N = 2^n$, where n is a natural number. Building an FFT tree in the DEF basis for a composite N not equal to degree two can be computationally challenging, especially for large dimensions N. One effective strategy to solve this challenge is to pre-factorize the DEF matrices.

However, because the elements of the DEF matrices are complex-valued quantities, the positive effect of the factorization may not be so impressive.

The primary purpose of this study is to develop a method of mutually unambiguous transition from complex-valued representations of the DEF matrices to matrices whose elements are natural numbers. Achievement of this goal will considerably simplify both the construction of FFT trees in the DEF bases and the factorization of the DEF matrices of large orders.

2 Basic Relations

The non-canonical system of the DEF of N-order is a matrix

$$\boldsymbol{E}_N = \{e(k,t)\} = \{\Psi^{kt}\}, \quad k,t = \overline{0, N-1}, \qquad (1)$$

in which $e(k,t)$ are basis functions of the k-order of the discrete argument (time) t, and

$$\Psi = \exp\left(-j\frac{2\pi}{N}\right) \qquad (2)$$

— phase multiplier (PM).

Taking into account the periodicity of FM (2), you can reduce the matrix (1) to the canonical form

$$\boldsymbol{E}_N = \{e(k,t)\} = \{\Psi^{(kt)_N}\}, \quad k,t = \overline{0, N-1}, \qquad (3)$$

where $(x)_m$ is a modular arithmetic function equal to the value of the number x modulo m.

Using the relation (3), let us compose, as an example, the six-order canonical DEF matrix

$$\boldsymbol{E}_6 = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & t \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ k \end{matrix} & \begin{bmatrix} \Psi^0 & \Psi^0 & \Psi^0 & \Psi^0 & \Psi^0 & \Psi^0 \\ \Psi^0 & \Psi^1 & \Psi^2 & \Psi^3 & \Psi^4 & \Psi^5 \\ \Psi^0 & \Psi^2 & \Psi^4 & \Psi^0 & \Psi^2 & \Psi^4 \\ \Psi^0 & \Psi^3 & \Psi^0 & \Psi^3 & \Psi^0 & \Psi^3 \\ \Psi^0 & \Psi^4 & \Psi^2 & \Psi^0 & \Psi^4 & \Psi^2 \\ \Psi^0 & \Psi^5 & \Psi^4 & \Psi^3 & \Psi^2 & \Psi^1 \end{bmatrix} \end{array} . \qquad (4)$$

We will number the rows and columns of the matrices (and the numbering of the elements of basic functions) by natural numbers starting from zero.

From the complex-valued matrix (4), we easily pass to a matrix with integer non-negative elements. For this purpose, it is enough to keep in its rows only the values of the degree indices of the phase multipliers. As a result of the proposed reduction, we arrive at the matrix

$$\boldsymbol{E}_6^{(1)} = \{e(k,t)\} = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & t \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ k \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 4 & 0 & 2 & 4 \\ 0 & 3 & 0 & 3 & 0 & 3 \\ 0 & 4 & 2 & 0 & 4 & 2 \\ 0 & 5 & 4 & 3 & 2 & 1 \end{bmatrix} \end{array} , \qquad (5)$$

which is isomorphic to the matrix (4). The isomorphism arises from the bijective mapping $\Psi^q \leftrightarrow q$, where q is a natural number. Elementary analysis of the matrix (5) leads to the relation

$$\{e(k,t)\} = \{(kt)_N\}, \quad k,t = \overline{0, N-1}, \qquad (6)$$

which we will call the *generalized basis of the* DEF in the isomorphic image space.

3 Synthesis of Symmetric DEF Systems

The synthesis of symmetric DEF systems represents an essential aspect of the theory and practice of signal processing and control systems. Discrete exponential functions are widely used to analyze and model dynamic systems. Symmetric DEF systems have specific mathematical properties that make them particularly attractive for several applications. Symmetry in the context of DEF systems means that their characteristics are preserved for particular operations, such as reflection or rotation. This property facilitates the analysis and control of such systems, making them more predictable and stable.

You can obtain symmetric DEF systems through specific arrangements of matrix row permutations (6), which we will hereafter refer to as the *mother* matrices. Let us take into account that you cannot rearrange the upper row of the mother matrix (the zero-order basis function) to any other row to any other row since it leads to the loss of symmetry of the matrix. Consequently, there are (N-1)! different ways of permutations of basic functions $e(k,t)$, some of which (let us denote their number by L_N) lead to the formation of the symmetric DEF systems.

Let us introduce some simple terminological definitions applicable to further presentation of the material.

Definition 1. A *primitive is called such a basis function* $e(k,t)$ *of an N-order DEF matrix in the isomorphic image space that contains a complete system of deductions modulo N, i.e., a set of non-negative integers from zero to* N-1.

For example, the basic functions $e(1,t)$ $e(5,t)$ are primitive in the matrix (5).

Definition 2. The *basis function* $e(k,t)$, *located in the first line of the DEF system, will be called the forming* (*generating*) *function of the system.*

The constitutive function of the DEF mother system is the function $e(1,t)$.

Definition 3. *All symmetric DEF systems that are not mother systems will be called daughter DEF systems.*

The definitions formulated above form the basis of the following fundamental proposition.

Statement 1. *Only primitive basis functions can serve as the forming functions of the symmetric systems DEF.*

Proof. Consider the mother system of the fourth-order DEF, whose matrix in the image space has the form

$$\begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \quad t \\ \boldsymbol{E}_4^{(1)} = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ k \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 1 \end{bmatrix} \end{array} . \qquad (7)$$

Let us illustrate the outcomes obtained when we try to apply as a forming basis function the only one of the three non-zero functions of the matrix (7) that is not primitive, i.e., the function $e(2,t) = (0,2,0,2)$. Due to the inherent symmetry of matrices, where the elements in columns align with those in the corresponding rows, we can deduce the following intermediate result

$$\begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \quad t \\ \begin{array}{c} 0 \\ 1 \\ 2 \rightarrow \\ 3 \\ k \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & & \\ 0 & 2 & & \end{bmatrix} \end{array} .$$

The line highlighted by the arrow does not correspond to any of the basic functions of the matrix (7). Any basis function of N-order that is not primitive leads to a similar situation (two zeros in any row of the matrix), which completes the proof of Statement 1.

We give the following statement (quite obvious and not requiring proof) to support the systematization of the presentation of the material.

Statement 2: *Any primitive basis function of the DEF system placed in the first row of the synthesizable matrix, uniquely determines the order of the basic functions placed in all other rows of the DEF matrix.*

Let us illustrate the application of Statement 2 by using the synthesis of the six-order daughter matrix of the DEF. Let us choose for this purpose from the mother system (5) the primitive basis function $e(5,t) = (0,5,4,3,2,1)$. Placing in the second row and the second column of the matrix (5) instead of the elements of the function $e(1,t)$ the elements of the function $e(5,t)$, we obtain such an incomplete matrix

$$\begin{array}{cc} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \quad t \\ \boldsymbol{E}_6^{(2)} = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ k \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 4 & 3 & 2 & 1 \\ 0 & 4 & & & & \\ 0 & 3 & & & & \\ 0 & 2 & & & & \\ 0 & 1 & & & & \end{bmatrix} \end{array} . \qquad (8)$$

Filling the empty cells of the matrix $\boldsymbol{E}_6^{(2)}$ in (8) becomes a trivial problem, allowing two solution variants. In the first of them, you replace the underdetermined rows of the daughter matrices with the corresponding rows of the mother system. The second variant assumes the direct calculation of missing elements $a_{k,t}$ in the rows of the daughter matrix by the formula

$$a_{k,t} = (kt)_N, \quad t = \overline{2, N-1},$$

where k is the value of the right element of the row of the daughter matrix.

And to conclude the paragraph, we will also conclude with an unproven statement

Statement 3. *The number L_N of the symmetric DEF systems coincides with the value of the Euler function from the order N of the DEF matrices, i.e*.,

$$L_N = \varphi(N).$$

4 Interrelation of DEF Systems

The totality of mother and daughter symmetric systems of the DEF of *N*-order are structurally interconnected. That is, if the structural form of at least one DEF system (matrix) is known, we calculate the structural forms of the other systems uniquely.

Definition 4. *By the structural form of the DEF systems, we will understand the sequence of order basis functions in the DEF matrices*.

A simple relation defines the relationship between the mother system of the DEF and the daughter system. Let a be the value of the first element of the first-order basis function of the DEF daughter system and let $e(k,t)$ and $d(k,t)$ be the basis functions of the mother and daughter systems, respectively. Then, as empirically established,

$$d(k,t) = e((ak)_N, t), \quad k,t = \overline{0, N-1},$$

and the coefficient $a > 1$ is mutually simple with *N*.

In total, there are only three groups of the DEF systems consisting of a mother and one daughter matrix. These are the systems whose order *N* is 3, 4, and 6, and the numbers a correspond to the values 2, 3, and 5 respectively, i.e., $a = N-1$.

If the order *N* of the systems DEF is a prime number, then *N*-1 symmetric DEF matrices correspond to each. In particular, Fig. 1 shows the transition algorithm between the DEF systems of simple seven-order

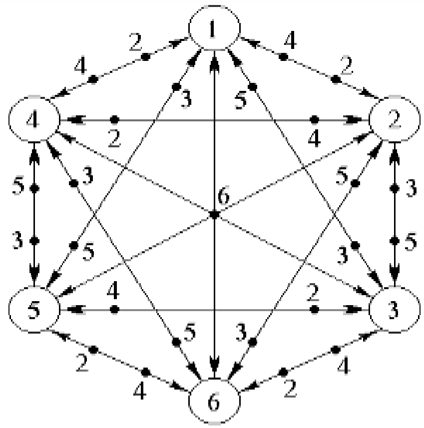

Fig. 1: Interconnection graph of the seven-order DEF systems

Fig. 2 shows the interconnection of composite-order DEF systems $N = 9$.

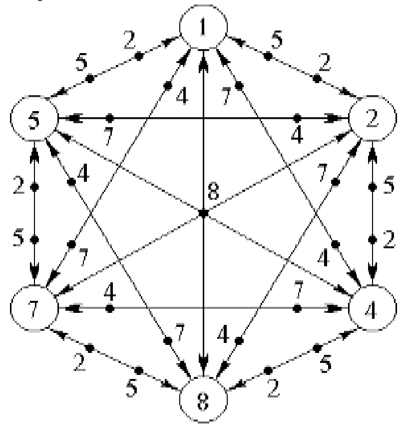

Fig. 2: Interconnection graph of the ninth-order DEF systems

Similarly, you can establish the interconnection of the DEF systems of arbitrary order.

5 Factorization of the DEF Matrices

The necessity to perform the procedure of factorization of the DEF matrices is related to the problem of constructing algorithms (structural schemes) of FFT on a given basis. Only matrices whose order *N* is a composite number can be factorized, i.e., provided that

$$N = \prod_{i=1}^{k} n_i^{l_i}, \tag{9}$$

where n_i are prime numbers, k is the number of different prime factors forming the composite number *N*, and the degree l_i are natural numbers that determine the multiplicity of the prime numbers n_i.

The representation (9) corresponds to the factorized DEF matrix of *N*-order

$$\boldsymbol{E}_N = \prod_{i=1}^{k} \boldsymbol{F}_i \ . \tag{10}$$

Matrices $\boldsymbol{F}_i$ by analogy with simple numbers we will call *simple matrices*.

A remarkable feature of $\boldsymbol{F}_i$ matrices is that they include many zero elements. Such matrices are called strongly sparse matrices.

Definition 5. *To strongly sparse matrices* $\boldsymbol{F}_i$, *generated by factorization of the DEF matrices in isomorphic space and corresponding to prime numbers* n_i, *we will refer to matrices, each row, and each column, which contain* n_i *non-zero elements*.

Later in this paragraph, we will clarify how the term "non-zero element" should be understood.

Let $\boldsymbol{x}$ be the vector column of the input signal values, $\boldsymbol{E}_N$ be the transformation matrix, and $\boldsymbol{y}$ be the vector column of output signals, i.e.,

$$\boldsymbol{y} = \boldsymbol{E}_N \boldsymbol{x} \ . \tag{11}$$

Regarding spectral analysis, we can expect that the spectrum of the discrete signal $\boldsymbol{x}$ is on the basis $\boldsymbol{E}_N$. Taking into account factorization (10), let us represent the spectrum (11) in the form

$$\boldsymbol{y} = \left(\prod_{i=1}^{k} \boldsymbol{F}_i \right) \boldsymbol{x} \ . \tag{12}$$

Suppose that we organize the procedure of matrix transformations (12) to exclude operations of calculating products on zero elements of matrices $\boldsymbol{F}_i$. In this case, the determination of the vector $\boldsymbol{y}$ by formula (12) is more efficient than the calculation by formula (11), and the effectiveness increases rapidly with increasing matrix size *N* and the number of its zero elements.

There are many ways of factorization of composite matrices, [16], which lead to different structural schemes of FFTs. We will mainly follow the work of, [17], relying on isomorphic representations of the DEF matrices. From now on, we give practical techniques for the factorization of composite DEF matrices considered in the order of increasing their dimensionality *N*.

So, let's turn to the fourth-order DEF mother system.

$$\boldsymbol{E}_4^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 1 \end{bmatrix} . \tag{13}$$

Let us make some clarifications concerning matrix (13). You should consider that the zero elements of this (as well as all subsequent) the DEF matrices are not zero in the usual sense. These zero means that in the place of these elements, there are numbers, corresponding to the values of the phase multipliers Ψ in the zero degree. Naturally, this number is one. When in the factorized matrix, there should be an arithmetic zero in place of an element to eliminate possible ambiguity, we will put a dash in this place.

Let's write the matrix $\boldsymbol{E}_4^{(1)}$ as a product of

$$\boldsymbol{E}_4^{(1)} = \boldsymbol{F}_1 \boldsymbol{F}_2 \,. \tag{14}$$

Matrices $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ correspond to numbers 2, which are elements of the decomposition of order *N* of the matrix (13) equal to four.

Let's choose the elements of the left half of the matrix rows (13) as the elements of the matrix $\boldsymbol{F}_1$. In the matrix $\boldsymbol{F}_1$, we keep the initial position of the selected elements of the mother system $\boldsymbol{E}_4^{(1)}$ for even rows (the numbering of matrix rows is performed from top to bottom by a sequence of numbers from 0 to *N*-1) and shift the elements of odd rows to the right by $N/2$, i.e., by two positions. We make such shifts so that each row and each column of the matrix $\boldsymbol{F}_1$ (according to Definition 5) contains two non-empty elements. We arrive at the matrix

$$\boldsymbol{F}_1 = \begin{bmatrix} 0 & 0 & - & - \\ - & - & 0 & 1 \\ 0 & 2 & - & - \\ - & - & 0 & 3 \end{bmatrix} . \tag{15}$$

The matrix $\boldsymbol{F}_2$ should formed in such a way that the equality (14) holds. As we can see from the upper line of matrix (15), we obtain the zero-order basis function of the system $\boldsymbol{E}_4^{(1)}$ in (13) by selecting these the first two lines of the matrix $\boldsymbol{F}_2$

$$\begin{array}{c} 0 \\ 1 \end{array} \begin{bmatrix} 0 & - & 0 & - \\ - & 0 & - & 0 \end{bmatrix} . \tag{16}$$

Obviously, by multiplying the zero row of the matrix (15) by the matrix (16), and the matrix elements marked with a dash do not participate in the multiplication operation, we obtain the zero-order basis function of the system $\boldsymbol{E}_4^{(1)}$.

Here, it is appropriate to remind you that in the isomorphic mapping, we formally perform the matrix product operation using the same rules as in the usual matrix calculus. Only needs to replace the operation of elementwise product with the operation of elementwise addition modulo *N*. Let us explain the method of such replacement by the following example. Let $N = 4$ and

$$\boldsymbol{M}_1 = \begin{bmatrix} a & b & - & - \\ - & - & 0 & 1 \\ 0 & 2 & - & - \\ - & - & 0 & 3 \end{bmatrix}; \quad \boldsymbol{M}_2 = \begin{bmatrix} c & - & d & - \\ - & e & - & f \end{bmatrix}.$$

Then, the process of forming the upper row of the product of matrices $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ (in isomorphic space) reduces to calculating the four elements using the formulas

$$(a+c)_4;\ (b+e)_4;\ (a+d)_4;\ (b+f)_4.$$

To obtain the first-order basis function of the system $\boldsymbol{E}_4^{(1)}$, you can multiply (according to the scheme described above) the first row of the matrix (13) by the matrix

$$\begin{array}{c} 2 \\ 3 \end{array} \begin{bmatrix} 0 & - & 2 & - \\ - & 0 & - & 2 \end{bmatrix}, \tag{17}$$

containing the second and third rows of the matrix $\boldsymbol{F}_2$.

Combining matrices (16) and (17), we obtain

$$\boldsymbol{F}_2 = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} 0 & - & 0 & - \\ - & 0 & - & 0 \\ 0 & - & 2 & - \\ - & 0 & - & 2 \end{bmatrix}. \tag{18}$$

It is easy to check that the product of the matrices (15) and (18) leads to the matrix (13), which completes the factorization procedure of the fourth-order DEF mother system.

We proceed to factorization of the six-order DEF matrix

$$\boldsymbol{E}_6^{(1)} = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 4 & 0 & 2 & 4 \\ 0 & 3 & 0 & 3 & 0 & 3 \\ 0 & 4 & 2 & 0 & 4 & 2 \\ 0 & 5 & 4 & 3 & 2 & 1 \end{array}\right]. \tag{19}$$

We decompose the order $N = n_1 \cdot n_2 = 6$ of this matrix into two simple factors 2 and 3, and their order in the product determines the type of factored matrices $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$, forming the matrix

$$\boldsymbol{E}_6^{(1)} = \boldsymbol{F}_1 \boldsymbol{F}_2. \tag{20}$$

Assuming $n_1 = 3$ and $n_2 = 2$, then in the matrix $\boldsymbol{F}_1$, each row should contain three consecutive significant elements, and if $n_1 = 2$, then in the matrix $\boldsymbol{F}_1$, each row should contain two consecutive significant elements. The matrix $\boldsymbol{F}_2$ is constructed depending on the form of the matrix $\boldsymbol{F}_1$. Consequently, there are at least two variants of factorization of the six-order DEF matrix. Let us consider both of these variants.

So, let us assume $n_1 = 3$. That means that in each row of the matrix $\boldsymbol{F}_1$, it is necessary to place the first three elements of the corresponding basis functions of the system $\boldsymbol{E}_6^{(1)}$. These elements are in the left part relative to the dashed line in the matrix (19). In addition, recall that the significant elements of odd rows (odd-order basis functions) should shift by three positions to the right. We obtain

$$\boldsymbol{F}_1 = \begin{bmatrix} 0 & 0 & 0 & - & - & - \\ - & - & - & 0 & 1 & 2 \\ 0 & 2 & 4 & - & - & - \\ - & - & - & 0 & 3 & 0 \\ 0 & 4 & 2 & - & - & - \\ - & - & - & 0 & 5 & 4 \end{bmatrix}. \tag{21}$$

To satisfy the equality (20), the matrix $\boldsymbol{F}_2$ should have the form

$$\boldsymbol{F}_2 = \begin{bmatrix} 0 & - & - & 0 & - & - \\ - & 0 & - & - & 0 & - \\ - & - & 0 & - & - & 0 \\ 0 & - & - & 3 & - & - \\ - & 0 & - & - & 3 & - \\ - & - & 0 & - & - & 3 \end{bmatrix}. \tag{22}$$

Multiplying matrices (21) and (22), we obtain the matrix (19), which is evidence of the correctness of the factorization procedure of the six-order DEF matrix $\boldsymbol{E}_6^{(1)}$.

Matrices similar to the matrix (21) are called *row-factorized matrices*. Their distinctive feature is that the significant elements densely fill some parts of the rows of this matrix. Matrices similar to the matrix (22) will be called *diagonal-factorized matrices*. Their distinctive feature is that the significant elements of this matrix densely fill some of its diagonals.

Now, let us turn to the second variant of factorization of the six-order DEF matrix. In this variant $n_1 = 2$ and $n_2 = 3$. This ranking of prime numbers n_i means that in the matrix $\boldsymbol{F}_1$, each row and each column must contain two significant elements, and the elements of the rows are composed of the first two elements of the corresponding basis functions of the system $\boldsymbol{E}_6^{(1)}$. The matrix corresponds to the formulated conditions

$$\boldsymbol{F}_1 = \begin{bmatrix} 0 & 0 & - & - & - & - \\ - & - & 0 & 1 & - & - \\ - & - & - & - & 0 & 2 \\ 0 & 3 & - & - & - & - \\ - & - & 0 & 4 & - & - \\ - & - & - & - & 0 & 5 \end{bmatrix}. \qquad (23)$$

From comparing matrices (23) and (19), we easily arrive at the algorithm for forming a matrix $\boldsymbol{F}_1$. The elements of the rows of the matrix $\boldsymbol{F}_1$ are formed from the elements of the first two columns of the matrix $\boldsymbol{E}_6^{(1)}$ due to their shift to the right by two positions (for the first and fourth rows) and four positions (for the second and fifth rows).

Since the matrix $\boldsymbol{F}_2$ corresponds to a prime number $n_2 = 3$, each row and each column of the six-order matrix $\boldsymbol{F}_2$ must contain three significant elements. As follows from the form of the upper row of the matrix (23), in to form the zero-order basis function of the system $\boldsymbol{E}_6^{(1)}$, the first two rows of the matrix $\boldsymbol{F}_2$ should give the form of

$$\begin{array}{c} 0 \\ 1 \end{array} \begin{bmatrix} 0 & - & 0 & - & 0 & - \\ - & 0 & - & 0 & - & 0 \end{bmatrix}. \qquad (24)$$

Indeed, by multiplying the zero row of the matrix (23) successively by the columns of the matrix (24), we obtain the required zero basis function of the system $\boldsymbol{E}_6^{(1)}$ in (19).

The first row of the matrix (23), which has the form $(-\ -\ 0\ 1\ -\ -)$, can form the first-order basis function of the system $\boldsymbol{E}_6^{(1)}$ only as a result of its multiplication by the columns of the second and third rows of the matrix $\boldsymbol{F}_2$ since only the second and third elements of the first row of the matrix (23) are significant. You can easily see that you should choose these rows (second and third) of the matrix $\boldsymbol{F}_2$ as follows

$$\begin{array}{c} 2 \\ 3 \end{array} \begin{bmatrix} 0 & - & 2 & - & 4 & - \\ - & 0 & - & 2 & - & 4 \end{bmatrix}. \qquad (25)$$

Finally, to form the second-order basis function of the system $\boldsymbol{E}_6^{(1)}$, it suffices to multiply the second row of the matrix (23), i.e., the row $(-\ -\ -\ -\ 0\ 2)$, by the fourth and fifth rows of the matrix $\boldsymbol{F}_2$, which should be of the form

$$\begin{array}{c} 4 \\ 4 \end{array} \begin{bmatrix} 0 & - & 4 & - & 2 & - \\ - & 0 & - & 4 & - & 2 \end{bmatrix}. \qquad (26)$$

Combining matrices (24) – (26), we obtain

$$\boldsymbol{F}_2 = \begin{bmatrix} 0 & - & 0 & - & 0 & - \\ - & 0 & - & 0 & - & 0 \\ 0 & - & 2 & - & 4 & - \\ - & 0 & - & 2 & - & 4 \\ 0 & - & 4 & - & 2 & - \\ - & 0 & - & 4 & - & 2 \end{bmatrix}. \qquad (27)$$

As it is easy to check, the product of matrices (23) and (27) modulo $N = 6$ leads to the matrix (19). Thus, we confirm that the second variant of the factorization of the system $\boldsymbol{E}_6^{(1)}$ is also correct.

The acquired experience of factorization of DEF matrices can transfer to the solution of problems of factorization of matrices of eighth and higher orders. Let us formulate the basic empirical rules of factorization of the DEF matrices of arbitrary order, taking as an example the mother system of DEF of the eighth order, which in the isomorphic representation has the form:

$$\boldsymbol{E}_8^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 2 & 4 & 6 & 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 1 & 4 & 7 & 2 & 5 \\ 0 & 4 & 0 & 4 & 0 & 4 & 0 & 4 \\ 0 & 5 & 2 & 7 & 4 & 1 & 6 & 3 \\ 0 & 6 & 4 & 2 & 0 & 6 & 4 & 2 \\ 0 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{bmatrix}. \qquad (28)$$

STEP 1. *Display the composite number N, determining the order of the DEF matrix, as a product of prime numbers* n_i, *arranging them from left to right in ascending order of indices*

$$N = n_1\, n_2\, \ldots\, n_k,$$

where k is the number of multipliers.

For the eighth order of the matrix (28), we have

$$n_1 = n_2 = n_3 = 2 .$$

STEP 2. *Let us represent the order of the matrix N as a product of*

$$N = n \cdot n_k , \tag{29}$$

where $n = n_1\, n_2 \ldots n_{k-1}$.

The format (29) makes it possible to factorize the matrix $\boldsymbol{E}_8^{(1)}$ in this sequence. According to (29), let us write the number 8 as a product of the factors 4 and 2. To number 4 corresponds to a partially factored matrix, which we denote $\boldsymbol{F}$, and to number 2 corresponds to a matrix, which we denote $\boldsymbol{F}_3$. The product of the matrices $\boldsymbol{F}$ and $\boldsymbol{F}_3$ must be equal to the matrix (28), i.e., $\boldsymbol{E}_8^{(1)} = \boldsymbol{F} \cdot \boldsymbol{F}_3$.

Since the matrix $\boldsymbol{F}$ corresponds to the number $n = 4$, it means that each line of the matrix contains the four first significant elements of the corresponding basis functions of the system (28), and the odd lines should shift four positions to the right, and the vacated elements should fill with dashes. After performing the above operations, we arrive at the matrix

$$\boldsymbol{F} = \begin{vmatrix} 0 & 0 & 0 & 0 & - & - & - & - \\ - & - & - & - & 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 & - & - & - & - \\ - & - & - & - & 0 & 3 & 6 & 1 \\ 0 & 4 & 0 & 4 & - & - & - & - \\ - & - & - & - & 0 & 5 & 2 & 7 \\ 0 & 6 & 4 & 2 & - & - & - & - \\ - & - & - & - & 0 & 7 & 6 & 5 \end{vmatrix} . \tag{30}$$

Concerning the matrices $\boldsymbol{F}$, $\boldsymbol{F}_3$ and $\boldsymbol{E}_8^{(1)}$, we can make the following considerations. First, since the matrix $\boldsymbol{F}_3$ corresponds to the multiplier $n_3 = 2$, it means that all rows and columns of the matrix contain two significant elements each, and you must space these elements diagonally. Second, since the first four elements in the upper row of the matrix $\boldsymbol{F}$ are substantial, the first four rows of the matrix $\boldsymbol{F}_3$, whose form is uniquely determined by the relation

$$\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{vmatrix} 0 & - & - & - & 0 & - & - & - \\ - & 0 & - & - & - & 0 & - & - \\ - & - & 0 & - & - & - & 0 & - \\ - & - & - & 0 & - & - & - & 0 \end{vmatrix} . \tag{31}$$

Finally, the first-order basis function of the system $\boldsymbol{E}_8^{(1)}$ can formed by multiplying the first row of the matrix (30) by the column of the matrix composed of the last four rows of the matrix $\boldsymbol{F}_3$. These rows should look as follows

$$\begin{array}{c} 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{vmatrix} 0 & - & - & - & 4 & - & - & - \\ - & 0 & - & - & - & 4 & - & - \\ - & - & 0 & - & - & - & 4 & - \\ - & - & - & 0 & - & - & - & 4 \end{vmatrix} . \tag{32}$$

The union of rows of the matrices (31) and (32) forms a diagonal factorized matrix

$$\boldsymbol{F}_3 = \begin{vmatrix} 0 & - & - & - & 0 & - & - & - \\ - & 0 & - & - & - & 0 & - & - \\ - & - & 0 & - & - & - & 0 & - \\ - & - & - & 0 & - & - & - & 0 \\ 0 & - & - & - & 4 & - & - & - \\ - & 0 & - & - & - & 4 & - & - \\ - & - & 0 & - & - & - & 4 & - \\ - & - & - & 0 & - & - & - & 4 \end{vmatrix} . \tag{33}$$

It is easy to check that the product of matrices (30) and (33) leads to matrix (28), as it should.

STEP 3. *Since the matrix* $\boldsymbol{F}$ *corresponds to the composite number* $n = 4$, *you can represent in the form of factors*

$$n = n_1 n_2 ,$$

and $n_1 = n_2 = 2$.

Consequently, we can subject the matrix $\boldsymbol{F}$ to deeper factorization and write as a product of

$$\boldsymbol{F} = \boldsymbol{F}_1 \cdot \boldsymbol{F}_2 ,$$

where $\boldsymbol{F}_1$ is a row-factorized matrix and $\boldsymbol{F}_2$ is a diagonal-factorized matrix. Each row and each column of the eighth-order matrices $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ contain two significant elements (since they correspond to prime factors equal to two).

Let us compose the matrix $\boldsymbol{F}_1$, forming it from the row elements of the matrix $\boldsymbol{E}_8^{(1)}$, given by system (28), or the matrix $\boldsymbol{F}$, represented by relation (30). From the significant elements of the rows of the matrix $\boldsymbol{E}_8^{(1)}$ (or $\boldsymbol{F}$) in the matrix $\boldsymbol{F}_1$, we will keep only the first two elements, shifting the remaining ones to the right by two positions relative to the position of the significant elements of the previous row. As a result, we come to the matrix

$$\boldsymbol{F}_1 = \begin{bmatrix} 0 & 0 & - & - & - & - & - & - \\ - & - & 0 & 1 & - & - & - & - \\ - & - & - & - & 0 & 2 & - & - \\ - & - & - & - & - & - & 0 & 3 \\ 0 & 4 & - & - & - & - & - & - \\ - & - & 0 & 5 & - & - & - & - \\ - & - & - & - & 0 & 6 & - & - \\ - & - & - & - & - & - & 0 & 7 \end{bmatrix}. \qquad (34)$$

That is not the only variant of representation of the row-factored matrix $\boldsymbol{F}_1$. For example, we can suggest the following variant

$$\boldsymbol{F}_1^{'} = \begin{bmatrix} 0 & 0 & - & - & - & - & - & - \\ - & - & - & - & 0 & 1 & - & - \\ - & - & 0 & 2 & - & - & - & - \\ - & - & - & - & - & - & 0 & 3 \\ 0 & 4 & - & - & - & - & - & - \\ - & - & - & - & 0 & 5 & - & - \\ - & - & 0 & 6 & - & - & - & - \\ - & - & - & - & - & - & 0 & 7 \end{bmatrix}.$$

Let us take the variant (34) as more regular. You should construct the matrix $\boldsymbol{F}_2$ so that the product of the matrices $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ equals the matrix $\boldsymbol{F}$. It is easy to check that this is the matrix

$$\boldsymbol{F}_2 = \begin{bmatrix} 0 & - & 0 & - & - & - & - & - \\ - & 0 & - & 0 & - & - & - & - \\ - & - & - & - & 0 & - & 2 & - \\ - & - & - & - & - & 0 & - & 2 \\ 0 & - & 4 & - & - & - & - & - \\ - & 0 & - & 4 & - & - & - & - \\ - & - & - & - & 0 & - & 6 & - \\ - & - & - & - & - & 0 & - & 6 \end{bmatrix}. \qquad (35)$$

Each row and each column of matrices (34) and (35) contain two significant elements, and their product is equal to the matrix (30), which completes the procedure of factorization of the system $\boldsymbol{E}_8^{(1)}$.

Let us consider a factorization scheme for an odd ninth-order isomorphic DEF matrix

$$\boldsymbol{E}_9^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 2 & 4 & 6 & 8 & 1 & 3 & 5 & 7 \\ 0 & 3 & 6 & 0 & 3 & 6 & 0 & 3 & 6 \\ 0 & 4 & 8 & 3 & 7 & 2 & 6 & 1 & 5 \\ 0 & 5 & 1 & 6 & 2 & 7 & 3 & 8 & 4 \\ 0 & 6 & 3 & 0 & 6 & 3 & 0 & 6 & 3 \\ 0 & 7 & 5 & 3 & 1 & 8 & 6 & 4 & 2 \\ 0 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{bmatrix}. \qquad (36)$$

Following the above methodology, the factorization of the ninth-order DEF matrix is reduced to the compilation of two sparse matrices corresponding to the factors of the composite number 9, equal to $n_1 = 3$ and $n_2 = 3$. The matrix $\boldsymbol{F}_1$ is row-factorized, i.e.,

$$\boldsymbol{F}_1 = \begin{bmatrix} 0 & 0 & 0 & - & - & - & - & - & - \\ - & - & - & 0 & 1 & 2 & - & - & - \\ - & - & - & - & - & - & 0 & 2 & 4 \\ 0 & 3 & 6 & - & - & - & - & - & - \\ - & - & - & 0 & 4 & 8 & - & - & - \\ - & - & - & - & - & - & 0 & 5 & 1 \\ 0 & 6 & 3 & - & - & - & - & - & - \\ - & - & - & 0 & 7 & 5 & - & - & - \\ - & - & - & - & - & - & 0 & 8 & 7 \end{bmatrix}. \qquad (37)$$

The diagonal-factorized matrix $\boldsymbol{F}_2$ has the form

$$\boldsymbol{F}_2 = \begin{bmatrix} 0 & - & - & 0 & - & - & 0 & - & - \\ - & 0 & - & - & 0 & - & - & 0 & - \\ - & - & 0 & - & - & 0 & - & - & 0 \\ 0 & - & - & 3 & - & - & 6 & - & - \\ - & 0 & - & - & 3 & - & - & 6 & - \\ - & - & 0 & - & - & 3 & - & - & 6 \\ 0 & - & - & 6 & - & - & 3 & - & - \\ - & 0 & - & - & 6 & - & - & 3 & - \\ - & - & 0 & - & - & 6 & - & - & 3 \end{bmatrix}. \qquad (38)$$

The product of matrices (37) and (38) modulo $m = 9$ leads to the system (36), as it should be. Following the above methodology, one can easily solve the factorization problem of the DEF systems of arbitrary order *N*.

6 Further Research

At least we can indicate such directions for further research:

- Development of the FFT algorithms in composite-order DEF bases for handling signals such as audio, images, time series, etc.
- Study the possibility of practical applications of the developed algorithm for factorization of the DEF matrices in natural systems and technologies, such as radio communication, medical diagnostics, image processing, etc.

7 Conclusion

The computation of the vector-matrix product is the basis for many procedures of digital signal processing, a classic example of which is the operations of determining the spectrum of discrete signals in the DEF basis. As the DEF matrices comprise complex-valued elements, the computational cost of multiplying a vector of sampled values from a continuous signal potentially complex itself by a complex-valued matrix is substantial. The factors outlined herein contribute to the computational efficiency realized in this paper. First, due to the isomorphic replacement of the DEF matrices with complex-valued elements by matrices with non-negative integer elements, resulting in a transition to real matrices. Secondly, due to the factorization of real matrices, i.e., their representation is a set of strongly discharged matrices multiplier with consecutive multiplication of the vector of input signal samples by each of the matrices. The reduction in the required computational resources for the realization of vector-matrix operations by the proposed algorithm will increase with the orders of the DEF matrices.

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