On the Existence of One-Point Time on an Oriented Set

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Abstract: - The oriented set notion is the elementary fundamental concept of the theory of changeable sets. In turn, the changeable set theory is closely related to Hilbert's sixth problem. From the formal point of view, any oriented set is a simple relational system with a single reflexive binary relation. Such mathematical structure is the simplest construction, within the framework of which it is possible to give a mathematically strict definition of the time concept. In this regard, the problem of the existence of time with given properties on an oriented set is very interesting. In the present paper, we establish the necessary and sufficient condition for the existence of one-point time on an oriented set. From the intuitive point of view, any one-point time is the time related to the evolution of a system, which consists of a single object (for example, from a single material point). The main result of the paper provides that the one-point time exists on the oriented set if and only if this oriented set is a quasi-chain. Also, using the obtained result, we solve the problem of describing all possible images of linearly ordered sets, which naturally arises in the theory of ordered sets.

Key-Words: - Binary relations, reflexive relations, oriented sets, changeable sets, time, ordered sets, quasi-ordered sets.

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1 Introductory Remarks

The subject of this article is closely related to the theory of changeable sets. In turn, this theory is connected with the famous sixth Hilbert problem, that is the problem of mathematically strict formulation for the fundamentals of theoretical physics. The last problem was posed in 1900, but it remains very relevant today, [1], [2], [3], [4], [5], [6], [7]. From the intuitive point of view, changeable sets can be interpreted as sets of objects, which can be in the process of permanent transformations. Namely, these objects can change their properties, appear or disappear, break down into several parts or, conversely, unite into a single unit. Moreover, the evolution picture of a changeable set may depend of the area of observation or reference frame. The problem of the creation the mathematical theory of changeable sets (that is the “sets” possessing the properties listed above) emerged in particular in the papers [8], [9], [10], [11], [12], [13]. In the papers of the author of this article the theory of changeable sets was developed on the mathematically strict level. The most complete and systematic presentation of this theory can be found in the preprint, [14]. For more information about scientific papers in peer-reviewed journals, where the foundations of the changeable set theory were first published, see the reference list in the end of preprint.

The notion of oriented set is the basic most elementary concept of the theory of changeable sets. Oriented sets were introduced in [15], as the most simple abstract models of the collections of evolving objects in the framework of one (fixed) reference frame ([14], Section 1). Moreover, in the aforementioned papers it was introduced the concept of time on oriented sets. As well in the article [15], (in Theorem 4.1) the sufficient condition of existence of one-point time for oriented sets is established ([14], Theorem 1.3.1). Note that from the intuitive point of view, one-point time should be understood as the time associated with the evolution of a system consisting of only one object (for example, from one material point). Emphasize that Theorem 4.1 from [15], gives only sufficient conditions for the existence of one-point time. That is why in the paper [14], (Problem 1.3.1) the problem of detection of necessary and sufficient conditions for existence of one-point time on an oriented set is posed. Below in this paper, the solution of the above problem is presented.

Namely, we specify the properties for the oriented set to be able to define the one-point time on it. Using the obtained result, we solve the problem of describing all possible images of linearly ordered sets. Such a problem naturally arises in the theory of ordered sets.

2 On Oriented Sets and One-point Time

Definition 1. *Let, M be any nonempty set ($M \neq \emptyset$).*

*An arbitrary reflexive binary relation $\lhd$ on M (that is a relation satisfying $\forall x \in M \ x \lhd x$) we name an **orientation**, and the pair $\mathcal{M} = (M, \lhd)$ we call an **oriented set**. In this case the set M is named the **basic** set or the set of all **elementary states** of the oriented set $\mathcal{M}$ and it is denoted by $\mathfrak{Bs}(\mathcal{M})$. The relation $\lhd$ we name the **directing relation of changes (transformations)** of $\mathcal{M}$, and denote it by $\underset{\mathcal{M}}{\leftarrow}$.*

In the case where the oriented set $\mathcal{M}$ is known in advance we use the notation $\leftarrow$ instead of $\underset{\mathcal{M}}{\leftarrow}$. For the elements $x, y \in \mathfrak{Bs}(\mathcal{M})$ the record $y \leftarrow x$ should be understood as "*the elementary state y is the result of transformations (or the transformation offspring) of the elementary state x*".

Let $\mathcal{M}$ be an oriented set.

Definition 2. *The nonempty subset $N \subseteq \mathfrak{Bs}(\mathcal{M})$ is referred to as **transitive** in $\mathcal{M}$ if for any $x, y, z \in N$ such, that $z \leftarrow y$ and $y \leftarrow x$ we have $z \leftarrow x$.*

*The transitive subset $L \subseteq \mathfrak{Bs}(\mathcal{M})$ is referred to as **chain** in $\mathcal{M}$ if for any $x, y \in L$ at least one of the relations $y \leftarrow x$ or $x \leftarrow y$ is true.*

*Oriented set $\mathcal{M}$ is called a **chain oriented set** if the set $\mathfrak{Bs}(\mathcal{M})$ is the chain of $\mathcal{M}$, that is if the relation $\leftarrow$ if transitive on $\mathfrak{Bs}(\mathcal{M})$ and for any $x, y \in \mathfrak{Bs}(\mathcal{M})$ at least one of the conditions $x \leftarrow y$ or $y \leftarrow x$ is satisfied (note that this is the case, where the oriented set $\mathcal{M}$ is a linearly quasi-ordered set).*

Recall that a (partially) ordered set is an ordered pair of kind $\mathbb{T} = (\boldsymbol{T}, \leq)$ with reflexive, asymmetric and transitive binary relation $\leq$ on $\boldsymbol{T}$. The pair $\mathbb{T}$ is called an linearly ordered set if the following additional condition holds:

(LnO) for every $t, \tau \in \boldsymbol{T}$ it is performed at least one of the correlations $t \leq \tau$ or $\tau \leq t$.

Definition 3. *Let $\mathcal{M}$ be an oriented set and $\mathbb{T} = (\boldsymbol{T}, \leq)$ be a linearly ordered set. A mapping $\psi: \boldsymbol{T} \to 2^{\mathfrak{Bs}(\mathcal{M})}$ is referred to as **time** on $\mathcal{M}$ if the following conditions are satisfied:*

1. For any elementary state $x \in \mathfrak{Bs}(\mathcal{M})$ there exists an element $t \in \boldsymbol{T}$ such that $x \in \psi(t)$.

2. If $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$, $x_2 \leftarrow x_1$ and $x_1 \neq x_2$, then there exist elements $t_1, t_2 \in \boldsymbol{T}$ such that $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ and $t_1 < t_2$ (this means that there is a temporal separateness of successive unequal elementary states).

In this case:

- *The elements $t \in \boldsymbol{T}$ we call the moments of time*
- *The pair $\mathcal{H} = (\mathbb{T}, \psi) = ((\boldsymbol{T}, \leq), \psi)$ we name by **chronologization** of $\mathcal{M}$.*

We say that an oriented set $\mathcal{M}$ ***can be chronologized*** if there exists at least one chronologization of $\mathcal{M}$. It turns out that any oriented set can be chronologized. To make sure this we may consider any linearly ordered set $\mathbb{T} = (\boldsymbol{T}, \leq)$, which contains at least two elements and put:

$$\psi(t) := \mathfrak{Bs}(\mathcal{M}), \quad t \in \boldsymbol{T}.$$

It is easy to verify that the conditions of Definition 3 for this function $\psi(\cdot)$ are satisfied. More nontrivial methods to chronologize an oriented set were considered, in particular, in [15].

Definition 4. *Let $\mathcal{M}$ be an oriented set and $\mathbb{T} = (\boldsymbol{T}, \leq)$ be a linearly ordered set.*

*1. The time $\psi: \boldsymbol{T} \to 2^{\mathfrak{Bs}(\mathcal{M})}$ is called **quasi one-point** if for every $t \in \boldsymbol{T}$ the set $\psi(t)$ is a singleton.*

*2. The time ψ is called **one-point** if the following conditions are satisfied:*

(a) The time ψ is quasi one-point;

(b) for any $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ the conditions $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ and $t_1 \leq t_2$, lead to $x_2 \leftarrow x_1$.

We say that an oriented set $\mathcal{M}$ can be chronologized quasi one-point / one-point if there exists at least one chronologization $\mathcal{H} = ((\boldsymbol{T}, \leq), \psi)$ of $\mathcal{M}$ with quasi one-point /one-point time ψ (correspondingly). In this case we name the chronologization $\mathcal{H}$ as quasi one-point /one-point (correspondingly).

Example 1. Let us consider an arbitrary mapping $f: \mathcal{I} \to \mathbb{R}^d$ ($d \in \mathbb{N}$), where $\mathcal{I} \subseteq \mathbb{R}$ is some connected subset of Real axis $\mathbb{R}$. This mapping can be interpreted as equation of motion of a single material point in the space $\mathbb{R}^d$. This mapping f generates the oriented set $\mathcal{M}_f = \left(\mathfrak{Bs}(\mathcal{M}_f), \underset{\mathcal{M}_f}{\leftarrow}\right)$, where

$\mathfrak{Bs}(\mathcal{M}_f) = \mathfrak{R}(f) = \{f(t) \mid t \in \mathcal{I}\} \subseteq \mathbb{R}^d$ and for $x, y \in \mathfrak{Bs}(\mathcal{M}_f)$, the correlation $y \underset{\mathcal{M}_f}{\leftarrow} x$ is valid if and only if there exist $t_1, t_2 \in \mathcal{I}$ such, that $x = f(t_1)$, $y = f(t_2)$ and $t_1 \leq t_2$. Consider the following set-valued mapping:

$$\psi_f(t) = \{f(t)\} \subseteq \mathfrak{Bs}(\mathcal{M}), \quad t \in \mathcal{I}.$$

It is easy to verify, that the mapping $\psi_f(\cdot)$ satisfies the conditions of Definition 3 and Definition 4 (item 2). Consequently $\psi_f(\cdot)$ is an one-point time on $\mathcal{M}_f$.

Example 1 makes clear the definition of one-point time. It is evident, that *any one-point time is quasi one-point*. Examples contained in the paper [15], show that the inverse statement is not true in the general case ([14], Example 1.3.2).

Theorem 1. (ZF+LO, [15]**).** *Any oriented set can be quasi one-point chronologized*.

Note that proof of Theorem 1 can be found also in [14] (see Theorem 1.3.2).

Remark 1. Proof of Theorem 1 uses the Linear Ordering principle (LO) in addition to Zermelo–Fraenkel axiomatic system (ZF). This principle asserts that any set can be linearly ordered. It is evident that the above principle follows from the famous well-ordering Zermelo's theorem, and therefore, from the axiom of choice (AC). But it is known that LO-principle also follows from Ultrafilter theorem of Tarski (UFT) and, moreover, it is logically weaker than this theorem and therefore than the axiom of choice, [16]. On the relationship between LO and AC see, also, [17].

Theorem 2 (ZF+LO, [15]**).** *Any chain oriented set can be one-point chronologized*.

Note that the proof of Theorem 2 can be also found in [14]. It turns out that Theorem 2 is not reversible. And the next example demonstrates the existence of non-chain oriented sets, which can be one-point chronologized.

Example 2. Consider the function $\mathrm{f}_0\colon [0,2\pi] \to \mathbb{R}^2$, defined by the formula:

$$\boldsymbol{f}_0(t) = (\cos t, \sin t) \qquad (t \in [0,2\pi]).$$

According to Example 1, using this function, we may construct the oriented set $\mathcal{M}_{\boldsymbol{f}_0}$. This oriented set can be one-point chronologized by means of the time:

$$\psi_{\boldsymbol{f}_0}(t) = \{\boldsymbol{f}_0(t)\} \qquad (t \in [0,2\pi]).$$

At the same time, this oriented set is not a chain, because the binary relation $\underset{\mathcal{M}_{f_0}}{\longleftarrow}$ is not transitive on $\mathfrak{Bs}(\mathcal{M}_{f_0})$. Indeed, consider the points: $x_1 := (0,-1) = \boldsymbol{f}_0\left(\frac{3}{2}\pi\right)$, $x_2 := (1,0) = \boldsymbol{f}_0(0) = \boldsymbol{f}_0(2\pi)$, $x_3 := (0,1) = \boldsymbol{f}_0\left(\frac{\pi}{2}\right)$. For these points we have: $x_1, x_2, x_3 \in \mathfrak{R}(\boldsymbol{f}_0) = \mathfrak{Bs}(\mathcal{M}_{f_0})$ and $x_2 \underset{\mathcal{M}_{f_0}}{\longleftarrow} x_1$, $x_3 \underset{\mathcal{M}_{f_0}}{\longleftarrow} x_2$ but the correlation $x_3 \underset{\mathcal{M}_{f_0}}{\longleftarrow} x_1$ is false.

The above facts generate the following problem:

Problem 1. *Find necessary and sufficient conditions of existence of one-point chronologization for oriented set*.

Note that Problem 1 was also posed in [14], (Problem 1.3.1). The main aim of the present paper is to give the solution of Problem 1.

3 Quasi-chain Oriented Sets and Formulation of Main Theorem

Notation 1. *On any oriented set $\mathcal{M}$ we introduce the following additional binary relation:*

■ *For every $x, y \in \mathfrak{Bs}(\mathcal{M})$ we note $y \overset{+}{\underset{\mathcal{M}}{\leftarrow}} x$ if and only if*:

$$y \underset{\mathcal{M}}{\leftarrow} x \quad \text{and} \quad x \underset{\mathcal{M}}{\not\leftarrow} y.$$

■ *In the cases where it does not lead to misunderstanding we use the notation $y \overset{+}{\leftarrow} x$ instead of the record $y \overset{+}{\underset{\mathcal{M}}{\leftarrow}} x$.*

Notation 2. *Let M be an arbitrary set and $R_1, R_2, \ldots, R_n \subseteq M^2$ ($n \in \mathbb{N}$) be any binary relations on M. Further for $x_0, \ldots, x_n \in M$ we use the abbreviated notation:*

$$x_0 R_1 x_1 R_2 x_2 \ldots x_{n-1} R_n x_n$$

for indication the fact that:

$$(x_0 R_1 x_1) \& (x_1 R_2 x_2) \& \ldots \& (x_{n-1} R_n x_n).$$

Assertion 1. *Let $\mathcal{M}$ be an oriented set, $\mathbb{T} = (\boldsymbol{T}, \leq)$ be a linearly ordered set and $\psi\colon \boldsymbol{T} \to 2^{\mathfrak{Bs}(\mathcal{M})}$ be an one-point time on $\mathcal{M}$. Then for any $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ the conditions:*

$$x_1 \in \psi(t_1), \; x_2 \in \psi(t_2) \quad \text{and} \quad x_2 \overset{+}{\leftarrow} x_1$$

lead to the inequality:

$$t_1 < t_2.$$

Proof. Indeed, suppose that $\mathcal{M}$ is an oriented set, $\mathbb{T} = (\boldsymbol{T}, \leq)$ is a linearly ordered set and $\psi\colon \boldsymbol{T} \to 2^{\mathfrak{Bs}(\mathcal{M})}$ is an one-point time on $\mathcal{M}$. Let the elements $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ be such that $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ and $x_2 \overset{+}{\leftarrow} x_1$. Assume the contrary: $t_2 \leq t_1$. Then, according to Definition 4 (item 2), from the conditions $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ and $t_2 \leq t_1$ it follows that $x_1 \leftarrow x_2$. But the last correlation is in contradiction to the condition $x_2 \overset{+}{\leftarrow} x_1$. Hence the assumption about $t_2 \leq t_1$ is false. Therefore we have $t_1 < t_2$. ■

Definition 5. *The oriented set $\mathcal{M}$ is called* ***quasi-chain*** *if and only if the following conditions are satisfied:*

(QL1) *For any $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ it holds at least one from the correlations $x_2 \leftarrow x_1$ or $x_1 \leftarrow x_2$.*

(QL2) *For every $x_0, x_1, x_2, x_3 \in \mathfrak{Bs}(\mathcal{M})$ the condition $x_3 \overset{+}{\leftarrow} x_2 \leftarrow x_1 \overset{+}{\leftarrow} x_0$ ensures the correlation $x_3 \overset{+}{\leftarrow} x_0$ (quasi-transitivity).*

Remark 2. It is easy to prove that the transitivity of the binary relation $\leftarrow$ on the oriented set $\mathcal{M}$ implies its quasi-transitivity. It turns out that the inverse statement in general is not valid. Example 2 shows that there exist the oriented set $\mathcal{M} = \mathcal{M}_{\mathrm{f}_0}$ such that the relation $\underset{\mathcal{M}}{\leftarrow}$ is quasi-transitive but not transitive. So quasi-chain oriented set must not be chain.

The main result of this paper is the following theorem.

Theorem 3 (ZF+UFT). *An oriented $\mathcal{M}$ set can be one-point chronologized if and only if it is a quasi-chain.*

Remark 3. We emphasize that proof of the necessity for Theorem 3 does not require the Ultrafilter Tarski theorem (UFT). This theorem is needed only for the proof of sufficiency of the condition, pointed out in Theorem 3.

The proof of Theorem 3 is divided into two main lemmas. Lemma 1 in the next section assures the necessity for Theorem 3, whereas Lemma 2 (see below) provides the sufficiency.

4 Proof of Necessity for Theorem 3

Lemma 1. *If the oriented set $\mathcal{M}$ can be one-point chronologized then it is a quasi-chain.*

Proof. Let $\mathbb{T} = (\boldsymbol{T}, \leq)$ be a linearly ordered set and $\psi : \boldsymbol{T} \to 2^{\mathfrak{Bs}(\mathcal{M})}$ be an one-point time on the oriented set $\mathcal{M}$.

$\equiv>$ **1**. First we will validate the condition **(QL1)**. Chose any $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$. By Definition 3 the time points $t_1, t_2 \in \boldsymbol{T}$ must exist such, that $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$. Since $\mathbb{T} = (\boldsymbol{T}, \leq)$ is a linearly ordered set then for $t_1, t_2 \in \boldsymbol{T}$ at least one of the inequalities must be fulfilled $t_1 \leq t_2$ or $t_2 \leq t_1$. In Accordance with Definition 4, in the case $t_1 \leq t_2$ we obtain $x_2 \leftarrow x_1$. Similarly in the case $t_2 \leq t_1$ we deduce $x_1 \leftarrow x_2$.

$\equiv>$ **2**. Now we validate the condition **(QL2)**. Consider any elements $x_0, x_1, x_2, x_3 \in \mathfrak{Bs}(\mathcal{M})$ such, that $x_3 \overset{+}{\leftarrow} x_2 \leftarrow x_1 \overset{+}{\leftarrow} x_0$. Consider any $t_0, t_3 \in \boldsymbol{T}$ such, that $x_0 \in \psi(t_0)$, $x_3 \in \psi(t_3)$ (by Definition 3 such t_0, t_3 exist). Since $x_2 \leftarrow x_1$, then, according to Definition 3 the time points $t_1, t_2 \in \boldsymbol{T}$ must exist such that $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ and $t_1 \leq t_2$. Taking into account the correlations $x_0 \in \psi(t_0)$, $x_1 \in \psi(t_1)$ and $x_1 \overset{+}{\leftarrow} x_0$, as well as Assertion 1, we obtain, $t_0 < t_1$. Similarly from the correlations $x_2 \in \psi(t_2)$, $x_3 \in \psi(t_3)$ and $x_3 \overset{+}{\leftarrow} x_2$ we deduce $t_2 < t_3$. Therefore the following inequalities are performed:

$$t_0 < t_1 \leq t_2 < t_3.$$

That is why $t_0 < t_3$. Thus we have:

$$\forall\, t_0, t_3 \in \boldsymbol{T} \; \Big(\big(x_0 \in \psi(t_0)\big) \& \big(x_3 \in \psi(t_3)\big) \Rightarrow (t_0 < t_3) \Big). \tag{1}$$

In accordance with the statement, proven in the item 1, at least one from the correlations $x_0 \leftarrow x_3$ or $x_3 \leftarrow x_0$ must hold. Assume, that $x_0 \leftarrow x_3$. Then, by Definition 3 the elements $\tilde{t}_0, \tilde{t}_3 \in \boldsymbol{T}$ must exist such that $x_0 \in \psi(\tilde{t}_0)$, $x_3 \in \psi(\tilde{t}_3)$ and $\tilde{t}_3 \leq \tilde{t}_0$. But the last inequality is in a contradiction to (1). Hence, the correlation $x_0 \leftarrow x_3$ is impossible. Thus the only possible one it remains the correlation $x_3 \overset{+}{\leftarrow} x_0$, that it was necessary to prove. ■

The proof of the sufficiency for Theorem 3 is much more complicated. First of all we need to work out some auxiliary technical results for this purpose. This work will be done in the next section.

5 Some Auxiliary Technical Results

5.1 Some Additional Properties of Quasi-chain Oriented Sets

Assertion 2. *Let, $\mathcal{M}$ be a quasi-chain oriented set and $x_0, x_1, x_2, x_3 \in \mathfrak{Bs}(\mathcal{M})$ be arbitrary elementary* *states of $\mathcal{M}$. Then the following properties are performed:*

(QL3) *If* $x_3 \leftarrow x_2 \overset{+}{\leftarrow} x_1 \leftarrow x_0$ *then* $x_3 \leftarrow x_0$.

(QL4) *If* $x_3 \overset{+}{\leftarrow} x_2 \leftarrow x_1$ *then* $x_3 \leftarrow x_1$.

(QL5) *If* $x_3 \leftarrow x_2 \overset{+}{\leftarrow} x_1$ *then* $x_3 \leftarrow x_1$.

(QL6) *If* $x_3 \overset{+}{\leftarrow} x_2 \overset{+}{\leftarrow} x_1$ *then* $x_3 \overset{+}{\leftarrow} x_1$.

Proof. The proofs of the properties (QL3)–(QL6) are listed below.

≡> **(QL3)**. Let $x_0, x_1, x_2, x_3 \in \mathfrak{Bs}(\mathcal{M})$ and $x_3 \leftarrow x_2 \overset{+}{\leftarrow} x_1 \leftarrow x_0$. Assume that the correlation $x_3 \leftarrow x_0$ is false (IE $x_3 \not\leftarrow x_0$). Then, taking into account the fact that the oriented set $\mathcal{M}$ is quasi-chain, we get $x_0 \overset{+}{\leftarrow} x_3$. Thus, we have, $x_0 \overset{+}{\leftarrow} x_3 \leftarrow x_2 \overset{+}{\leftarrow} x_1$. Hence, by Definition 5 (condition (QL2)) we get, $x_0 \overset{+}{\leftarrow} x_1$, which is in a contradiction to the correlation $x_1 \leftarrow x_0$. Therefore assumption about $x_3 \not\leftarrow x_0$ is false. So we have $x_3 \leftarrow x_0$.

≡> **(QL4)**. Suppose that $x_1, x_2, x_3 \in \mathfrak{Bs}(\mathcal{M})$ and $x_3 \overset{+}{\leftarrow} x_2 \leftarrow x_1$. Then, by Definition 1, we have, $x_3 \leftarrow x_3 \overset{+}{\leftarrow} x_2 \leftarrow x_1$. Thence, using Property (QL3), we obtain $x_3 \leftarrow x_1$.

≡> **(QL5)**. If we assume that $x_3 \leftarrow x_2 \overset{+}{\leftarrow} x_1$, then we will have $x_3 \leftarrow x_2 \overset{+}{\leftarrow} x_1 \leftarrow x_1$. Thence, applying Property (QL3), we obtain $x_3 \leftarrow x_1$.

≡> **(QL6)**. If we suppose that $x_3 \overset{+}{\leftarrow} x_2 \overset{+}{\leftarrow} x_1$, then we will deliver $x_3 \overset{+}{\leftarrow} x_2 \leftarrow x_2 \overset{+}{\leftarrow} x_1$. Thence, by Definition 5 (condition (QL2)), we deduce $x_3 \overset{+}{\leftarrow} x_1$. ■

5.2 Finite-repeating Time on Oriented Sets

Definition 6. *Let* $\mathbb{T} = (\boldsymbol{T}, \leq)$ *be a linearly ordered set and* $\mathcal{M}$ *be an oriented set.*

• *The time* $\psi: \boldsymbol{T} \to 2^{\mathfrak{Bs}(\mathcal{M})}$ *will be named as* ***finite-repeating*** *if and only if for every* $x \in \mathfrak{Bs}(\mathcal{M})$ *the following condition is fulfilled:*

$$\mathbf{card}(\{t \in \boldsymbol{T} \mid x \in \psi(t)\}) < \aleph_0$$

(where $\mathbf{card}(\boldsymbol{M})$ *is the cardinality of a set* $\boldsymbol{M}$*). Moreover, the number:*

$$\mathrm{Rp}_x(\psi) = \mathbf{card}(\{t \in \boldsymbol{T} \mid x \in \psi(t)\})$$

will be refereed to as ***repeatability*** *of the time* ψ *relatively the element* $x \in \mathfrak{Bs}(\mathcal{M})$.

• *Let* $n \in \mathbb{N}$. *The time* ψ *is named as* n ***-repeating*** *if and only if the time* ψ *is finite-repeating and*

$$\forall x \in \mathfrak{Bs}(\mathcal{M}) \; (\mathrm{Rp}_x(\psi) = n).$$

Notation 3. *Let* $\psi: \boldsymbol{T} \to 2^{\mathfrak{Bs}(\mathcal{M})}$ *be a* ***finite-repeating*** *time on the oriented set* $\mathcal{M}$. *For every* $x \in \mathfrak{Bs}(\mathcal{M})$ *we note:*

$$\hat{\psi}^+(x) := \max(\{t \in \boldsymbol{T} \mid x \in \psi(t)\});$$
$$\hat{\psi}^-(x) := \min(\{t \in \boldsymbol{T} \mid x \in \psi(t)\}),$$

where maximum and minimum should be understood it the sense of the linearly ordered set $\mathbb{T} = (\boldsymbol{T}, \leq)$.

Assertion 3. *Let* $\mathbb{T} = (\boldsymbol{T}, \leq)$ *be a linearly ordered set and* $\psi: \boldsymbol{T} \to 2^{\mathfrak{Bs}(\mathcal{M})}$ *be a finite-repeating one-point time on the oriented set* $\mathcal{M}$. *Then for any* $x, x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ *the following properties are holding:*

(FR1) $\hat{\psi}^-(x) \leq \hat{\psi}^+(x)$. *If, in addition,* $\mathrm{Rp}_x(\psi) \geq 2$ *then* $\hat{\psi}^-(x) < \hat{\psi}^+(x)$.

(FR2) *The correlation* $x_2 \leftarrow x_1$ *is true if and only if* $\hat{\psi}^-(x_1) \leq \hat{\psi}^+(x_2)$. *If, in addition,* $x_1 \neq x_2$ *then* $x_2 \leftarrow x_1$ *if and only if* $\hat{\psi}^-(x_1) < \hat{\psi}^+(x_2)$.

(FR3) $x_2 \overset{+}{\leftarrow} x_1$ *if and only if* $\hat{\psi}^+(x_1) < \hat{\psi}^-(x_2)$.

(FR4) *If, in addition, the time* ψ *is* n*-repeating with* $n \geq 2$ *then* $x_2 \leftarrow x_1$ *if and only if* $\hat{\psi}^-(x_1) < \hat{\psi}^+(x_2)$.

Proof. **(FR1)**: Let $x \in \mathfrak{Bs}(\mathcal{M})$. Then according to Notation 3, we have $\hat{\psi}^-(x) = \min(\{t \in \boldsymbol{T} \mid x \in \psi(t)\}) \leq \max(\{t \in \boldsymbol{T} \mid x \in \psi(t)\}) = \hat{\psi}^+(x)$. If, in addition, $\mathrm{Rp}_x(\psi) \geq 2$ then the set $\{t \in \boldsymbol{T} \mid x \in \psi(t)\}$ contains at least two elements. So minimum of this set is less then maximum.

(FR2): **First we suppose that** $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ **and** $x_2 \leftarrow x_1$**.**

Then in the case $x_1 = x_2$ we have the inequality $\hat{\psi}^-(x_1) \leq \hat{\psi}^+(x_2)$ according to Property (FR1). Hence we will consider that $x_1 \neq x_2$. Since $x_2 \leftarrow x_1$ and $x_1 \neq x_2$, then, by Definition 3, the time points $t_1, t_2 \in \boldsymbol{T}$ exist such that $x_1 \in \psi(t_1)$, $x_2 \in$

$\psi(t_2)$ and $t_1 < t_2$. Therefore:

$$\hat{\psi}^-(x_1) = \min(\{t \in \boldsymbol{T} \mid x_1 \in \psi(t)\}) \leq \leq t_1 < t_2 \leq \max(\{t \in \boldsymbol{T} \mid x_2 \in \psi(t)\}) = = \hat{\psi}^+(x_2).$$

So, for every $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ it is performed the following implication:

$$\left(x_2 \leftarrow x_1\right) \& (x_1 \neq x_2) \Rightarrow \Rightarrow \left(\hat{\psi}^-(x_1) < \hat{\psi}^+(x_2)\right). \tag{2}$$

Thus, in the both cases for any $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ we have the implication:

$$\left(x_2 \leftarrow x_1\right) \Rightarrow \left(\hat{\psi}^-(x_1) \leq \hat{\psi}^+(x_2)\right). \tag{3}$$

Conversely, suppose that $\hat{\psi}^-(x_1) \leq \hat{\psi}^+(x_2)$**.** Put:

$$\hat{t}_1 := \hat{\psi}^-(x_1), \qquad \hat{t}_2 := \hat{\psi}^+(x_2).$$

Then in accordance with Notation 3, we have, $x_1 \in \psi(\hat{t}_1)$, $x_2 \in \psi(\hat{t}_2)$ and $\hat{t}_1 \leq \hat{t}_2$. Hence, by Definition 4, we deduce $x_2 \leftarrow x_1$. Thus for every $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ we have the implication:

$$\left(\hat{\psi}^-(x_1) \leq \hat{\psi}^+(x_2)\right) \Rightarrow \left(x_2 \leftarrow x_1\right) \tag{4}$$

The implications (3) and (4) assure the desired equivalence:

$$(x_2 \leftarrow x_1) \Leftrightarrow \left(\hat{\psi}^-(x_1) \leq \hat{\psi}^+(x_2)\right).$$

If we assume that, in addition, $x_1 \neq x_2$ then from (2) and (4) we deliver the equivalence:

$$(x_2 \leftarrow x_1) \Leftrightarrow \left(\hat{\psi}^-(x_1) < \hat{\psi}^+(x_2)\right).$$

(FR3): Let $x_2 \overset{+}{\leftarrow} x_1$. Assume that $\hat{\psi}^-(x_2) \leq \hat{\psi}^+(x_1)$. Then according to Property (FR2), we obtain the correlation $x_1 \leftarrow x_2$, which contradicts to $x_2 \overset{+}{\leftarrow} x_1$. Therefore, $\hat{\psi}^+(x_1) < \hat{\psi}^-(x_2)$.

Conversely, suppose that $\hat{\psi}^+(x_1) < \hat{\psi}^-(x_2)$. Then, applying Property (FR1), we deliver $\hat{\psi}^-(x_1) \leq \hat{\psi}^+(x_1) < \hat{\psi}^-(x_2) \leq \hat{\psi}^+(x_2)$. Hence, according to Property (FR2), we obtain $x_2 \leftarrow x_1$. Assume that the condition $x_1 \leftarrow x_2$ also is performed. Then by Property (FR2), we get the inequality $\hat{\psi}^-(x_2) \leq \hat{\psi}^+(x_1)$, which contradicts to the inequality $\hat{\psi}^+(x_1) < \hat{\psi}^-(x_2)$. That is the assumption about $x_1 \leftarrow x_2$ is wrong. That is why we have $x_2 \overset{+}{\leftarrow} x_1$.

(FR4): In the case $x_1 \neq x_2$ Property (FR4) follows from Property (FR2). In the case $x_1 = x_2$ this property follows from Property (FR1). ■

Remarks on the idea of proof the sufficiency of Theorem 3. It turns out that it is technically easier to prove the existence of 2-repeating one-point time on the quasi-chain oriented set $\mathcal{M}$. Taking into account this situation, we can take the set $\boldsymbol{T} = \mathfrak{Bs}(\mathcal{M}) \times \{0,1\}$ as the set of time points and consider the mapping:

$$\psi(t) = \psi((x, \alpha)) = \{x\}, \quad t = (x, \alpha) \in \boldsymbol{T} \quad \left(x \in \mathfrak{Bs}(\mathcal{M}),\ \alpha \in \{0,1\}\right). \tag{5}$$

Then for the proof of desired result it is sufficient to find the linear order relation $\leq$ on $\boldsymbol{T}$, which turns the mapping (5) into a one-point time. Further we will consider that the desired order $\leq$ satisfies the following natural additional condition:

$$(x, 0) \leq (x, 1) \qquad \left(\forall x \in \mathfrak{Bs}(\mathcal{M})\right) \tag{6}$$

Assume that the mapping (5) is an one-point time. Taking into account convention (6), for every $x \in \mathfrak{Bs}(\mathcal{M})$ we obtain the equalities:

$$\hat{\psi}^-(x) = (x, 0), \qquad \hat{\psi}^+(x) = (x, 1). \tag{7}$$

From the equalities (7) and properties **(FR2)**, **(FR3)** (see. Assertion 3) it follows that the desired order $\leq$ on $\boldsymbol{T} = \mathfrak{Bs}(\mathcal{M}) \times \{0,1\}$ must have the following properties:

- If $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ and $x_2 \leftarrow x_1$ then $(x_1, 0) \leq (x_2, 1)$.
- If $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$ and $x_2 \overset{+}{\leftarrow} x_1$ then $(x_1, 1) \leq (x_2, 0)$.

6 Proof of Sufficiency for Theorem 3

For proving the main result of this section we need the following auxiliary assertion:

Assertion 4 ([16]**).** *Let* $(\boldsymbol{T}, \preceq)$ *be a partially ordered set. Then the linear order* $\leq$ *on the set* $\boldsymbol{T}$ *exists such that* $\preceq\ \subseteq\ \leq$.

Emphasize that the inclusion of binary relations in Assertion 4 should be understood in set-theoretic sense, that is the record "$\preceq\ \subseteq\ \leq$" means that for any $t_1, t_2 \in \boldsymbol{T}$ the correlation $t_1 \preceq t_2$ leads to $t_1 \leq t_2$.

It is known that Assertion 4 is a consequence of Ultrafilter Tarski theorem (UFT). In turn UFT follows from the axiom of choice (AC), moreover UFT is logically weaker than AC. So Assertion 4 also is logically weaker than AC. But, from the other hand, it is known that this Assertion can not be obtained from Zermelo–Fraenkel axiomatic system without AC (for details see [16], Theorem 2.18 and Proposition 4.39).

The next lemma ensures the sufficiency for Theorem 3.

Lemma 2. *If the oriented set $\mathcal{M}$ is a quasi-chain then it can be one-point chronologized. Moreover there exists the chronologization $\mathcal{H}=\big((\boldsymbol{T},\le),\psi\big)$ of $\mathcal{M}$ with 2-repeating one-point time ψ.*

Proof. Let $\mathcal{M}$ be quasi-chain oriented set. Denote:

$$\boldsymbol{T}:=\mathfrak{Bs}(\mathcal{M})\times\{0,1\}.$$

First we introduce the binary relation $\trianglelefteq$ on the set $\boldsymbol{T}$ by the following rule:

$\vartriangleright$ For any $t_1=(x_1,\alpha_1)\in\boldsymbol{T}$, $t_2=(x_2,\alpha_2)\in\boldsymbol{T}$ we write $t_1\trianglelefteq t_2$ if and only if at least one of the following conditions is satisfied:

(Preo1) $t_1=t_2$ (that is $x_1=x_2$ and $\alpha_1=\alpha_2$);

(Preo2) $x_2\leftarrow x_1,\ \alpha_1=0,\ \alpha_2=1$;

(Preo3) $x_2\overset{+}{\leftarrow}x_1,\ \alpha_1=1,\ \alpha_2=0$.

Also, we note $t_1\vartriangleleft t_2$ if and only if $t_1\trianglelefteq t_2$ and $t_1\ne t_2$.

The introduced relation $\trianglelefteq$ is obviously reflexive (ie $t\trianglelefteq t$ $(\forall t\in\boldsymbol{T})$). Moreover, we are going to prove that this relation has the following property of the "weak" transitivity:

(WT) If $t_0,t_1,t_2,t_3\in\boldsymbol{T}$ and $t_0\vartriangleleft t_1\vartriangleleft t_2\vartriangleleft t_3$ then $t_0\vartriangleleft t_3$.

Indeed, let $t_0,t_1,t_2,t_3\in\boldsymbol{T}$ and $t_0\vartriangleleft t_1\vartriangleleft t_2\vartriangleleft t_3$, where $t_i=(x_i,\alpha_i)$, $x_i\in\mathfrak{Bs}(\mathcal{M})$, $\alpha_i\in\{0,1\}$ $\big(i\in\overline{0,3}=\{0,1,2,3\}\big)$. Since $t_0\vartriangleleft t_1$, Condition **(Preo1)** can not be performed for the elements t_0 and t_1. Hence one of the conditions **(Preo2)**, **(Preo3)** must be fulfilled. If Condition **(Preo2)** is fulfilled, we have, $x_1\leftarrow x_0,\ \alpha_0=0,\ \alpha_1=1$. Next, since $t_1\vartriangleleft t_2$, Condition **(Preo1)** can not be performed for the elements t_1 and t_2. Condition **(Preo2)** also can not be performed for the elements t_1 and t_2, because $\alpha_1=1\ne 0$. Therefore Condition **(Preo3)** is fulfilled, that is $x_2\overset{+}{\leftarrow}x_1$, $\alpha_2=0$. Similarly we verify that $x_3\leftarrow x_2,\ \alpha_3=1$. Hence, we have $x_3\leftarrow x_2\overset{+}{\leftarrow}x_1\leftarrow x_0$. And, applying Property **(QL3)** (see Assertion 2), we deliver $x_3\leftarrow x_0$. And, taking into account that $\alpha_0=0,\ \alpha_3=1$, we see that Condition **(Preo2)** is performed for t_0 and t_3. That is why, $t_0\trianglelefteq t_3$. But, since $\alpha_0=0\ne1=\alpha_3$, then $t_0\ne t_3$. Thus, $t_0\vartriangleleft t_3$. Similarly, in the case where **(Preo3)** is fulfilled for t_0 and t_1, we successively obtain:

1) $x_1\overset{+}{\leftarrow}x_0,\ \alpha_0=1,\ \alpha_1=0$; **2)** $x_2\leftarrow x_1,\ \alpha_2=1$; **3)** $x_3\overset{+}{\leftarrow}x_2,\ \alpha_3=0$.

Thence, by Definition 5 (item **(QL2)**), we deduce, $x_3\overset{+}{\leftarrow}x_0$. So, taking into account that $\alpha_0=1,\ \alpha_3=0$ and $\alpha_0\ne\alpha_3$, we obtain $t_0\vartriangleleft t_3$.

Let us prove that the relation $\trianglelefteq$ is asymmetric, i.e.:

(AS) If $t_0,t_1\in\boldsymbol{T}$, $t_0\trianglelefteq t_1$ and $t_1\trianglelefteq t_0$ then $t_0=t_1$.

Indeed, suppose that $t_0\trianglelefteq t_1$ and $t_1\trianglelefteq t_0$, where $t_i=(x_i,\alpha_i)$, $x_i\in\mathfrak{Bs}(\mathcal{M})$, $\alpha_i\in\{0,1\}$ $\big(i\in\overline{0,1}\big)$. Assume that $t_0\ne t_1$. Then Condition **(Preo1)** can not be performed for the elements t_0 and t_1. So, by conditions **(Preo2)**, **(Preo3)**, at least on of the following two cases must hold:

[**case 1**] $x_1\leftarrow x_0$ and $x_0\overset{+}{\leftarrow}x_1$ or

[**case 2**] $x_1\overset{+}{\leftarrow}x_0$ and $x_0\leftarrow x_1$.

But really each of these cases is impossible (by definition of relation $\overset{+}{\leftarrow}$). The contradiction obtained above proves that $t_0=t_1$.

Now, using the properties **(AS)** and **(WT)**, we will prove that the relation $\trianglelefteq$ has the following property of "stronged" asymmetry:

(AS(n)) If $n\in\mathbb{N}$, $t_0,\dots,t_n\in\boldsymbol{T}$ and $t_0\trianglelefteq t_1\trianglelefteq\cdots\trianglelefteq t_n\trianglelefteq t_0$ then $t_0=t_1=\cdots=t_n$.

Indeed, let $n\in\mathbb{N}$, $t_0,\dots,t_n\in\boldsymbol{T}$ and $t_0\trianglelefteq t_1\trianglelefteq\cdots\trianglelefteq t_n\trianglelefteq t_0$. In the case $n=1$ the desired result follows from Property **(AS)**. In the case $n=2$ we have $t_0\trianglelefteq t_1\trianglelefteq t_2\trianglelefteq t_0$. If we assume that $t_0=t_1$, we obtain $t_0=t_1\trianglelefteq t_2\trianglelefteq t_1$. Thence, using Property **(AS)** we obtain $t_0=t_1=t_2$. Similarly we get the equality $t_0=t_1=t_2$ in the cases $t_1=t_2$ and $t_2=t_0$. If we assume that $t_0\ne t_1\ne t_2\ne t_0$ then we obtain $t_0\vartriangleleft t_1\vartriangleleft t_2\vartriangleleft t_0$. And, by Property **(WT)**, we deduce the impossible correlation $t_0\vartriangleleft t_0$. Hence, the case $t_0\ne t_1\ne t_2\ne t_0$ is impossible. And in all possible cases we obtain the desired result for $n=2$.

Now we consider any number $n\in\mathbb{N}$ such that $n>2$. Our inductive assumption is that Property

(AS(k)) holds for all $k \in \mathbb{N}$ such that $k < n$. Let $t_0, \dots, t_n \in \boldsymbol{T}$ and $t_0 \trianglelefteq t_1 \trianglelefteq t_2 \trianglelefteq \cdots \trianglelefteq t_n \trianglelefteq t_0$. First we assume that $t_0 \neq t_1 \neq t_2 \neq t_3$. Then we obtain $t_0 \triangleleft t_1 \triangleleft t_2 \triangleleft t_3 \trianglelefteq \cdots \trianglelefteq t_0$. Thence by Property **(WT)**, we deduce $t_0 \trianglelefteq t_3 \trianglelefteq \cdots \trianglelefteq t_0$. So, by inductive assumption, we have $t_0 = t_3 = \cdots = t_n$. The equality $t_0 = t_3$ together with the correlation, obtained before leads to the correlation $t_0 \triangleleft t_1 \triangleleft t_2 \triangleleft t_0$, which, by Property **(WT)** leads to the contradiction $t_0 \triangleleft t_0$. Thus, the assumption $t_0 \neq t_1 \neq t_2 \neq t_3$ is false. In the case where $t_0 = t_1$ or $t_1 = t_2$ or $t_2 = t_3$ Property **(AS(n))** can be reduced to Property **(AS(n-1))**, which is valid, according to inductive assumption. Therefore, the inductive transition is well-founded. That is why, Property **(AS(n))** holds for each $n \in \mathbb{N}$.

Let $\preceq$ be a transitive closure (transitive hull) of the relation $\trianglelefteq$ in the sense of [18] (see page 337) or [19], (see page 69), that is binary relation on $\boldsymbol{T}$ satisfying the following condition:

(PO) For $t, \tau \in \boldsymbol{T}$ the correlation $t \preceq \tau$ is valid if and only if there exist $n \in \mathbb{N}$ and $t_0, \dots, t_n \in \boldsymbol{T}$ such, that $t_0 = t$, $t_n = \tau$ and $t_0 \trianglelefteq \cdots \trianglelefteq t_n$.

The following inclusion holds:

$$\trianglelefteq \quad \subseteq \quad \preceq. \tag{8}$$

Indeed, if assume that $t, \tau \in \boldsymbol{T}$ and $t \trianglelefteq \tau$, then we can put $n := 1$, $t_0 := t$, $t_1 := \tau$. And, according to **(PO)**, we obtain $t \preceq \tau$.

It follows from the reflexivity of the relation $\trianglelefteq$ and the inclusion (8) that the relation $\preceq$ is also reflexive. According to Property **(AS(n))** that the relation $\preceq$ is asymmetric (that is, if $t \preceq \tau$ and $\tau \preceq t$ then $t = \tau$). Being a transitive closure of the relation $\trianglelefteq$, the relation $\preceq$ is transitive (according to [18] (see Theorem and Definition 28.18) or [19], (see Theorem 5.7)). So if $t \preceq \tau$ and $\tau \preceq u$ then $t \preceq u$. Note that the transitivity of the relation $\preceq$ is not difficult to check also by the direct verification method. Thus, the relation $\preceq$ is a partial order on $\boldsymbol{T}$. Therefore, by Assertion 4, *there exists a linear order relation* $\leq$ *on* T *such that* $\preceq \subseteq \leq$. Then, using (8), we get the inclusion:

$$\trianglelefteq \quad \subseteq \quad \leq. \tag{9}$$

Denote: $\mathbb{T} := (\boldsymbol{T}, \leq)$. Also we define the mapping $\psi: \boldsymbol{T} \to 2^{\mathfrak{Bs}(\mathcal{M})}$ by formula (5). That is for an arbitrary $t = (x, \alpha) \in \boldsymbol{T}$ we put $\psi(t) = \psi\big((x, \alpha)\big) := \{x\}$. We are going to prove that the mapping ψ is an one-point time on the oriented set $\mathcal{M}$.

1. According to formula (5), for any $x \in \mathfrak{Bs}(\mathcal{M})$ we obtain:

$$x \in \{x\} = \psi\big((x, 0)\big) = \psi(t_x), \qquad \text{where}$$
$$t_x = (x, 0) \in \boldsymbol{T}.$$

Hence, the first condition of Definition 3 is satisfied.

2. Let $x_1, x_2 \in \mathfrak{Bs}(\mathcal{M})$, $x_2 \leftarrow x_1$ and $x_1 \neq x_2$. Denote, $t_1 := (x_1, 0)$, $t_2 := (x_2, 1)$. Then for elements t_1 and t_2 it is performed Condition **(Preo2)**. Therefore $t_1 \trianglelefteq t_2$. Thence, by inclusion (9), we deduce the inequality $t_1 \leq t_2$. And, since $t_1 \neq t_2$, we have $t_1 < t_2$. Moreover, by formula (5), we have $\psi(t_1) = x_1$, $\psi(t_2) = x_2$. Hence, the second condition of Definition 3 also is satisfied.

Thus, in accordance with Definition 3, *the mapping* ψ *is a time on the oriented set* $\mathcal{M}$.

3. Let's prove that the time ψ is one-point.
3.1. According to formula (5) the set $\psi(t)$ consists of one element. Hence, by Definition 4, the time ψ is quasi one-point.
3.2 Suppose that $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ and $t_1 \leq t_2$. Then it follows from the quasi-one-pointness of time ψ that $\psi(t_1) = \{x_1\}$, $\psi(t_2) = \{x_2\}$. In accordance with the formula (5), the last two equalities are possible only if there exist numbers $\alpha_1, \alpha_2 \in \{0,1\}$ such that $t_1 = (x_1, \alpha_1)$, $t_2 = (x_2, \alpha_2)$. Denote, $t_1' := (x_1, 0)$, $t_2' := (x_2, 1)$. Then, according to the conditions **(Preo1)**, **(Preo2)**, taking into account the reflexivity of the relation $\leftarrow$, we obtain $t_1' \trianglelefteq t_1$ and $t_2 \trianglelefteq t_2'$. Hence, taking into account the inclusion (9), we have $t_1' \leq t_1$ and $t_2 \leq t_2'$ (where $t_1 \leq t_2$, according to the above). That is why:

$$t_1' \leq t_2'. \tag{10}$$

Also, by formula (5), we obtain $\psi(t_1') = \{x_1\}$, $\psi(t_2') = \{x_2\}$. Now we are going to prove that $x_2 \leftarrow x_1$. Assume the the contrary, $x_2 \not\leftarrow x_1$. Then, since the oriented set $\mathcal{M}$ is a quasi-chain, we deduce $x_1 \overset{+}{\leftarrow} x_2$. Consequently, according to condition **(Preo3)**, we obtain $t_2' \trianglelefteq t_1'$ and therefore $t_2' \leq t_1'$. The last inequality together with (10) ensures $t_1' = t_2'$, which is impossible, because $t_1' := (x_1, 0)$, $t_2' := (x_2, 1)$. The obtained contradiction proves that $x_2 \leftarrow x_1$.

Thus the both conditions of Definition 4 are satisfied. Therefore the time ψ is one-point.
4. Now we are going to prove that the time ψ is

2-repeating. Using formula (5) for each $x \in \mathfrak{Bs}(\mathcal{M})$ we obtain:

$$\mathrm{Rp}_x(\psi) = \mathbf{card}\left(\{t \in \boldsymbol{T} \mid x \in \psi(t)\}\right) = \\ = \mathbf{card}\left(\left\{t \in \boldsymbol{T} \mid \psi(t) = \{x\}\right\}\right) = \\ = \mathbf{card}(\{(x,0),(x,1)\}) = 2.$$

So, by Definition 6, the time ψ is 2-repeating.
The lemma is completely proven. ■

Now Theorem 3 follows from Lemma 1 and Lemma 2.

In fact, applying Lemma 1 and Lemma 2 we can readily deduce the following, more powerful theorem.

Theorem 4. *For an oriented set $\mathcal{M}$ the following statements are equivalent:*

(1) $\mathcal{M}$ *is a quasi-chain*;

(2) *admits an one-point time*;

(3) $\mathcal{M}$ *admits* 2*-repeating one-point time*.

7 On Images of Linearly Ordered Sets

In this short section we deduce one interesting corollary from Theorem 3 in the theory of ordered sets. Namely it will be obtained the description of all oriented sets, which can be represented as images of linearly ordered sets. First of all we formulate the definition of image of linearly ordered set.

Let $\mathcal{M}$ be an oriented set and $\boldsymbol{U}: \mathfrak{Bs}(\mathcal{M}) \to \mathcal{X}$ be a mapping from $\mathfrak{Bs}(\mathcal{M})$ to $\mathcal{X}$. Then we can introduce the binary relation $\leftarrow_{(1)}$ on the set $M_1 = \boldsymbol{U}[\mathfrak{Bs}(\mathcal{M})] = \mathfrak{R}(\boldsymbol{U})$ by the following rule:

⇛ For $\tilde{x}, \tilde{y} \in M_1$ we note $\tilde{y} \leftarrow_{(1)} \tilde{x}$ if and only if there exist $x, y \in \mathfrak{Bs}(\mathcal{M})$ such, that $\tilde{x} = \boldsymbol{U}(x)$, $\tilde{y} = \boldsymbol{U}(y)$ and $y \leftarrow x$.

It is not difficult to verify that the ordered pair $\mathcal{M}_1 = \left(M_1, \leftarrow_{(1)}\right)$ is an oriented set, moreover $\mathfrak{Bs}(\mathcal{M}_1) = M_1$ and $\underset{\mathcal{M}_1}{\leftarrow} = \leftarrow_{(1)}$.

Definition 7. *An oriented set $\mathcal{M}_1$ is referred to as* ***image*** *of the oriented set $\mathcal{M}$ under the mapping $U: \mathfrak{Bs}(\mathcal{M}) \to \mathcal{X}$ if and only if:*

1. $\mathfrak{Bs}(\mathcal{M}_1) = \boldsymbol{U}[\mathfrak{Bs}(\mathcal{M})] = \mathfrak{R}(\boldsymbol{U})$.

2. For $\tilde{x}, \tilde{y} \in \mathfrak{Bs}(\mathcal{M}_1)$ the correlation $\tilde{y} \underset{\mathcal{M}_1}{\leftarrow} \tilde{x}$ holds if and only if there exist $x, y \in \mathfrak{Bs}(\mathcal{M})$ such, that $\tilde{x} = \boldsymbol{U}(x)$, $\tilde{y} = \boldsymbol{U}(y)$ and $y \underset{\mathcal{M}}{\leftarrow} x$.

It is apparently that for each mapping $\boldsymbol{U}: \mathfrak{Bs}(\mathcal{M}) \to \mathcal{X}$ there exists an unique image under the mapping $\boldsymbol{U}$. We will use the notation $\boldsymbol{U}\left[[\mathcal{M}]\right]$ for the image of the oriented set $\mathcal{M}$ under the mapping $\boldsymbol{U}: \mathfrak{Bs}(\mathcal{M}) \to \mathcal{X}$.

It is evidently that every linearly ordered set $\mathbb{T} = (\boldsymbol{T}, \leq)$ is an oriented set with:

$$\mathfrak{Bs}(\mathbb{T}) = \boldsymbol{T}, \qquad \underset{\mathbb{T}}{\leftarrow} = \leq.$$

Therefore, it is meaningful to consider the image of the linearly ordered set $\mathbb{T} = (\boldsymbol{T}, \leq)$ under some mapping of kind $\boldsymbol{U}: \boldsymbol{T} \to \mathcal{X}$. And the image of the linearly ordered set $\mathbb{T}$ is the oriented set $\boldsymbol{U}\left[[\mathbb{T}]\right]$. That is why the following problem naturally arises:

Problem 2. *Can an arbitrary oriented set be represented as the image $\boldsymbol{U}\left[[\mathbb{T}]\right]$ of some linearly ordered set $\mathbb{T}$? If it can not, describe all oriented sets that can be represented as an image of some linearly ordered set.*

The key for solution of Problem 2 gives the following Assertion.

Assertion 5. *An oriented set $\mathcal{M}$ can be represented as image of some linearly ordered set if and only if $\mathcal{M}$ can be one-point chronologized.*

Proof. Indeed, suppose that the ordered set $\mathcal{M}$ can be represented in the form $\mathcal{M} = \boldsymbol{U}\left[[\mathbb{T}]\right]$, where $\mathbb{T} = (\boldsymbol{T}, \leq)$ is a linearly ordered set. So, $\boldsymbol{U}$ is the mapping of kind $\boldsymbol{U}: \boldsymbol{T} \to \mathfrak{Bs}(\mathcal{M})$ with $\mathfrak{R}(\boldsymbol{U}) = \mathfrak{Bs}(\mathcal{M})$. Here we denote by $\geq$ the binary relation, inverse to $\leq$ (ie for $x, y \in \boldsymbol{T}$ the condition $y \geq x$ holds if and only if $x \leq y$). According to Duality Principle (see [20], page 14), the ordered pair:

$$\mathbb{T}_{\geq} = (\boldsymbol{T}, \geq) \tag{11}$$

is the linearly ordered set as well. It is not difficult to verify that the mapping:

$$\boldsymbol{T} \ni t \mapsto \psi(t) = \{\boldsymbol{U}(t)\} \subseteq \mathfrak{Bs}(\mathcal{M})$$

is an one-point time on $\mathcal{M}$ (relatively the linearly ordered set $\mathbb{T}_{\geq}$). Conversely, let $\mathbb{T} = (\boldsymbol{T}, \leq)$ be a linearly ordered set and $\psi: \boldsymbol{T} \to 2^{\mathfrak{Bs}(\mathcal{M})}$ be one-point time on the oriented set $\mathcal{M}$. Then, by Definition 4, for every time point $t \in \boldsymbol{T}$ the element $\boldsymbol{x}_{(t)} \in \mathfrak{Bs}(\mathcal{M})$ exists such, that $\psi(t) = \left\{\boldsymbol{x}_{(t)}\right\}$. Consider the mapping:

$$\boldsymbol{T} \ni t \mapsto \boldsymbol{U}(t) = \boldsymbol{x}_{(t)} \in \mathfrak{Bs}(\mathcal{M}).$$

It is easy to verify that for this mapping $\boldsymbol{U}$ it is performed the equality $\mathcal{M} = \boldsymbol{U}\left[[\mathbb{T}_{\geq}]\right]$, where the linearly ordered set $\mathbb{T}_{\geq}$ is determined by the formula (11). ■

Assertion 5 together with Theorem 3 stipulate the following corollary.

Corollary 1. *An oriented set $\mathcal{M}$ can be* *represented as image of some linearly ordered set if and only if it is a quasi-chain.*

References:

[1] A.N. Gorban, "Hilberts sixth problem: the endless road to rigour" , *Phil Trans R Soc A*, 376(2118), 2018, p. 20170238, http://dx.doi.org/10.1098/rsta.2017.0238 .

[2] L. Corry, "Hilbert's sixth problem: between the foundations of geometry and the axiomatization of physics" , *Phil Trans R Soc A*, 376(2118), 2018, https://doi.org/10.1098/rsta.2017.0221 .

[3] L. Accardi, "Quantum probability and Hilbert's sixth problem" , *Phil Trans R Soc A*, 376(2118), 2018, https://doi.org/10.1098/rsta.2018.0030 .

[4] S. Majid, "On the emergence of the structure of physics" , *Phil Trans R Soc A*, 376(2118), 2018, https://doi.org/10.1098/rsta.2017.0231 .

[5] M. Slemrod, "Hilbert's sixth problem and the failure of the Boltzmann to Euler limit" , *Phil Trans R Soc A*, 376(2118), 2018, https://doi.org/10.1098/rsta.2017.0222 .

[6] G. M. D'Ariano, "The solution of the sixth Hilbert problem: the ultimate Galilean revolution" , *Phil Trans R Soc A*, 376(2118), 2018, https://doi.org/10.1098/rsta.2017.0224 .

[7] A. N. Gorban and I. Y. Tyukin, "Blessing of dimensionality: mathematical foundations of the statistical physics of data" , *Phil Trans R Soc A*, 376(2118), 2018, https://doi.org/10.1098/rsta.2017.0237 .

[8] A.P. Levich, "Methodological difficulties in the way to understanding the phenomenon of time" , in *Time of the end of time: Proceedings of Moscow-Petersburg Philosophical Club*, 2009, pp. 66–88, [Online]. http://www.chronos.msu.ru/old/RREPORTS/levich_trudnosti.pdf (Accessed Date: February 24, 2024).

[9] A.P. Levich, "Time as variability of natural systems: ways of quantitative description of changes and creation of changes by substantial flows" , in On the Way to Understanding the Time Phenomenon: the Constructions of Time in Natural Science. Part 1. Interdisciplinary Time Studies, *World Scientific*, chap. 5, 1995, pp. 149–192, [Online]. http://www.chronos.msu.ru/old/EREPORTS/levich1.pdf (Accessed Date: February 24, 2024).

[10] A.P. Levich, "Modeling of "dynamic sets"" , in *Irreversible processes in nature and technique*, MSTU named after N.E. Bauman, pp. 3–46.

[11] Michael Barr, Colin Mclarty and Charles Wells, "Variable Set Theory", 1986, 12 pages, [Online]. http://www.math.mcgill.ca/barr/papers/vst.pdf (Accessed Date: February 24, 2024).

[12] John L. Bell., "Abstract and Variable Sets in Category Theory" , in *What is Category Theory?*, Polimetrica International Scientific Publisher, 2006, pp. 9–16.

[13] Lawvere FW, Rosebrugh R. *Sets for Mathematics*. Cambridge University Press, 2003.

[14] Ya.I. Grushka, "Draft introduction to abstract kinematics. (Version 2.0)" , Preprint: ResearchGate, 2017, pp. 1–208, https://doi.org/10.13140/RG.2.2.28964.27521 .

[15] Ya.I. Grushka, "Primitive changeable sets and their properties" , *Mathematical Bulletin of Taras Shevchenko Scientific Society*, 9, 2012, pp. 52–80.

[16] Herrlich Horst, *Axiom of Choice*, Lecture Notes in Mathematics, Springer-Verlag, 2006.

[17] David Pincus, "The dense linear ordering principle" , *Journal of Symbolic Logic*, 62(2), 1997, pp. 438–456.

[18] Rudolf Lidl and Gnter Pilz, *Applied Abstract Algebra*, Undergraduate Texts in Mathematics, Springer New York, NY, 2nd edn., 1998, https://doi.org/10.1007/978-1-4757-2941-2 .

[19] T.O. Karnaukh and A.B. Stavrovsky, *Introduction to discrete mathematics*, Kyiv University, 2006.

[20] Garrett Birkhoff, *Lattice theory*, Third edition. American Mathematical Society Colloquium Publications, Vol. XXV, American Mathematical Society, Providence, R.I., New York, 1967.

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