Applications of Gegenbauer Polynomials to a Certain Subclass of p-Valent Functions

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Abstract: The paper presents a subclass of p-valent functions defined by the means of Gegenbauer Polynomials in the open unit disk $\mathbb{D}$. We investigate the properties of this new class and provide estimations for the modulus of the coefficients a_{p+1} and a_{p+2}, where $p \in \mathbb{N}$, for functions belong to this subclass. Moreover, we examine the classical Fekete-Szegö inequality for functions f belong to the presenting subclass.

Key-Words: Analytic Functions; holomorphic Functions; Univalent Functions; p-Valent Functions; Principle of Subordination; Gegenbauer Polynomials; Chebyshev polynomials; Coefficient estimates; Fekete-Szegö Inequality.

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1 Introduction

Let $\mathcal{H}$ be the class of all functions $f(z)$ that are holomorphic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. An analytic function f in a domain $\mathcal{D} \subset \mathbb{C}$ is called p-valent, if for each $w \in \mathbb{C}$ the equation $f(z) = w$ has at most p roots in $\mathcal{D}$. Therefore, there exists $w_0 \in \mathbb{C}$ such that the equation $f(z) = w_0$ has exactly p roots in $\mathcal{D}$. Let $\mathcal{A}_p$ be the class of all holomorphic functions $f \in \mathcal{H}$ that are given by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad \text{where} \quad z \in \mathbb{D}. \tag{1}$$

Let $\mathcal{S}$ denote the class of all functions f in the class $\mathcal{A} = \mathcal{A}_1$ that are univalent in $\mathbb{D}$. Let $\mathcal{S}_p^*$ be the class of p-valent starlike functions such that we say $f(z) \in \mathcal{A}_p$ in the class $\mathcal{S}_p^*$ if the following condition satisfies for all $z \in \mathbb{D}$:

$$\mathcal{R}\left\{\frac{zf'(z)}{f(z)}\right\} > 0.$$

Also, let S_p^c be the class of p-valent functions such that we say $f(z) \in \mathcal{A}_p$ in the class $\mathcal{S}_p^c$ if the following condition satisfies for all $z \in \mathbb{D}$:

$$\mathcal{R}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0.$$

It is well known (see, for details [1]) and [2]) that if f is analytic in a convex domain $\mathcal{D} \subset \mathbb{C}$ and $\mathcal{R}\left\{e^{i\theta} f'(z)\right\} > 0$ for some real θ and for all $z \in \mathcal{D}$, then $f(z)$ is univalent in $\mathcal{D}$. In [3] the auther extended the previous result, in fact he showed that if $f(z)$ of the form (1) is analytic in a convex domain $\mathcal{D} \subset \mathbb{C}$ and $\mathcal{R}\left\{e^{i\theta} f^{(p)}(z)\right\} > 0$ for some real θ and for all $z \in \mathcal{D}$, then $f(z)$ is at most p-valent in $\mathcal{D}$. Moreover, it can be shown that if $f \in \mathcal{A}_p$ and $\mathcal{R}\left\{f^{(p)}(z)\right\} > 0$ for all $z \in \mathbb{D}$, then $f(z)$ is at most p-valent in $\mathbb{D}$. According to [4] we have if $f \in \mathcal{A}_p$, $p \geq 2$, and $\arg\left\{f^{(p)}(z)\right\} < \frac{3\pi}{4}$ for all $z \in \mathbb{D}$, then $f(z)$ is at most p-valent in $\mathbb{D}$. For more information about p-valent we refer the readers to the articles [5], [6], [7], [8] and the references therein.

Let the functions f and g be analytic in $\mathbb{D}$, we say the function f is subordinate by the function g in $\mathbb{D}$, denoted by $f(z) \prec g(z)$ for all $z \in \mathbb{D}$, if there exists a Schwarz function w, with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{D}$, such that $f(z) = g(w(z))$ for all $z \in \mathbb{D}$. In particular, if the function g is univalent over $\mathbb{D}$ then $f(z) \prec g(z)$ equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D}$. For more information about the Subordination Principle we refer the readers to the monographs [9], [10], [11] and [12].

The research in the geometric function theory has been very active in recent years, the typical problem in this field is studying a functional made up of combinations of the initial coefficients of the functions $f \in \mathcal{A}$. For a function in the class $\mathcal{S}$, it is well-known that $|a_n|$ is bounded by n. Moreover, the coefficient bounds give information about the geometric properties of those functions. For instance, the bound for the second coefficients of functions in the class $\mathcal{S}$ gives the growth and distortion bounds for the class.

Coefficient related investigations of functions belong to the class Σ began around the 1970. It is worth mentioning that, in [13] the author studied the class of bi-univalent functions and derived the bound for $|a_2|$. According to [14] we know that the maximum value of $|a_2|$ is $\frac{4}{3}$ for functions belong to the class Σ. Moreover, in [15] the authors proved that $|a_2| \leq \sqrt{2}$ for functions in the class Σ. Since then, many researchers investigated the coefficient bounds for various subclasses of the bi-univalent function class Σ. However, not much is known about the bounds of the general coefficients $|a_2|$ for $n \geq 4$. In fact, the coefficient estimate problem for the general coefficient $|a_n|$ is still an open problem.

The Fekete-Szegö functional is well known for its rich history in the geometric function theory. Its origin was in [16] when they found the maximum value of $|a_3 - \lambda a_2^2|$, as a function of the real parameter $0 \leq \lambda \leq 1$ for a univalent function f. Since then, maximizing the modulus of the functional $\Psi_\lambda(f) = a_3 - \lambda a_2^2$ for $f \in \mathcal{A}$ with any complex λ is called the Fekete-Szegö problem. There are many researchers investigated the Fekete-Szegö functional and the other coefficient estimates problems, for example see the articles [17], [18], [19], [20], [21], [22], [23], [24] and the references therein.

2 Preliminaries

In this section we present some information that are crucial for the main results of this paper. In [25] the author introduced and studied a subclass $\mathfrak{F}(\gamma)$ of the class $\mathcal{A}$ consisting of functions of the form

$$f(z) = \int_{-1}^{1} K(z,x)\, d\sigma(x), \tag{2}$$

where $K(z,x) = \dfrac{z}{(z^2 - 2tz + 1)^\gamma}$, $\gamma \geq 0$, $-1 \leq x \leq 1$, and σ is the probability measure on $[-1,1]$. Moreover, the function $K(z,x)$ has the following Taylor-Maclaurin series expansion

$$K(z,x) = z + C_1^\gamma(x) z^2 + C_2^\gamma(x) z^3 + C_3^\gamma(x) z^4 + \cdots,$$

where $C_n^\gamma(t)$ denotes the Gegenbauer polynomials of order α. Furthermore, for any real numbers $\gamma, x \in \mathbb{R}$,with $\gamma \geq 0$ and $-1 \leq x \leq 1$, the generating function of Gegenbauer polynomials is given by

$$H_\gamma(z,x) = (z^2 - 2xz + 1)^{-\gamma}, \quad \text{where} \quad z \in \mathbb{D}.$$

Thus, for any fixed x the function $H_\gamma(z,x)$ is analytic on the unit disk $\mathbb{D}$ and its Taylor-Maclaurin series is given by

$$H_\gamma(z,x) = \sum_{n=0}^{\infty} C_n^\gamma(x) z^n.$$

Moreover, Gegenbauer polynomials can be defined in terms of the following recurrence relation:

$$C_n^\gamma(x) = \frac{2x(n+\gamma-1)C_{n-1}^\gamma(x) - (n+2\gamma-2)C_{n-1}^\gamma(x)}{x}, \tag{3}$$

with initial values

$$\begin{aligned} &C_0^\gamma(x) = 1, \quad C_1^\gamma(x) = 2\gamma x, \quad \text{and} \\ &C_2^\gamma(x) = 2\gamma(\gamma+1)x^2 - \gamma. \end{aligned} \tag{4}$$

It is well-known that the Gegenbauer polynomials and their special cases such as Legendre polynomials $L_n(x)$ and the Chebyshev polynomials of the second kind $T_n(x)$ are orthogonal polynomials, where the values of γ are $\gamma = 1/2$ and $\gamma = 1$ respectively, more precisely

$$L_n(x) = C_n^{1/2}(x), \quad \text{and} \quad T_n(x) = C_n^1(x).$$

For more information about the Gegenbauer polynomials and their special cases, we refer the readers to the articles [26], [27], [28], the monographs [29], [30], and the references therein. next, we define our class of p-valent functions which we denote by $\mathcal{A}_p(\alpha, \beta, \gamma)$.

Definition 2.1. *A function $f(z)$ in the family $\mathcal{A}_p$ is said to be in the class $\mathcal{A}_p(\alpha,\beta,\gamma)$ if, for all $z \in \mathbb{D}$, it satisfies the following subordination*

$$\left(\frac{zf'(z) - pf(z)}{\beta p f(z)}\right) + \alpha\left(\frac{f'(z) - pz^{p-1}}{\beta p z^{p-1}}\right) \prec H_\gamma(x,z),$$

where $\gamma > 0$, $x \in [-1,1]$, $0 \leq \alpha \leq 1$, and β is a non-zero complex number.

The following lemma (see, for details [22]) is a well-known fact, but it is crucial for our presented work.

Lemma 2.2. *Let the Schwarz function $w(z)$ be given by:*

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots \quad \textit{where} \quad z \in \mathbb{D},$$

then $|w_1| \leq 1$ and for $t \in \mathbb{C}$

$$|w_2 - t w_1^2| \leq 1 + (|t| - 1)|w_1|^2 \leq \max\{1, |t|\}.$$

The results is sharp for the functions $w(z) = z$ and $w(z) = z^2$.

The primary goal of this article is to investigate a class of p-valent functions in the open unit disk $\mathbb{D}$, which we denote by $\mathcal{A}_p(\alpha,\beta,\gamma)$. For functions in this class, we obtain the estimates for the initial coefficients $|a_{p+1}|$ and $|a_{p+2}|$. Furthermore, we examine the corresponding Fekete-Szegö functional problem for functions in this class.

3 Coefficient bounds of functions in the class $\mathcal{A}_p(\alpha, \beta, \gamma)$

In this section, we provide bounds for the initial coefficients for the functions belong to the class $\mathcal{A}_p(\alpha, \beta, \gamma)$ which are given by equation (1).

Theorem 3.1. *If a function $f \in \mathcal{A}_p$ is given by (1) belong to the class $\mathcal{A}_p(\alpha, \beta, \gamma)$, then*

$$|a_{p+1}| \leq \frac{2\gamma p|x||\beta|}{\alpha p + \alpha + 1}, \tag{5}$$

and

$$|a_{p+2}| \leq \frac{\gamma p|x||\beta|}{\alpha p + 2\alpha + 1} \times max\left\{1, \left|\frac{2(\gamma+1)x^2 - 1}{2x} + \frac{2xp\gamma\beta}{(\alpha p + 1\alpha + 1)^2}\right|\right\}. \tag{6}$$

Proof. Let f be in the class $\mathcal{A}_p(\alpha, \beta, \gamma)$. Then, using Definition 2.1, there exists a holomorphic function ψ on the unit disk $\mathbb{D}$ such that

$$\left(\frac{zf'(z) - pf(z)}{\beta p f(z)}\right) + \alpha\left(\frac{f'(z) - pz^{p-1}}{\beta p z^{p-1}}\right) \prec H_\gamma(x, \psi(z)), \tag{7}$$

where the holomorphic function ψ is given by $\psi(z) = \sum_{n=1}^{\infty} b_n z^n$ such that $\psi(0) = 0$ and $|\psi(z)| < 1$ for all $z \in \mathbb{D}$. Moreover, it is well-known that (see, for details [9]), $|b_j| \leq 1$ for all $j \in \mathbb{N}$.

Now, upon comparing the coefficients of both-sides of equation (7), we obtain the equations

$$\left(\frac{\alpha(p+1)+1}{p\beta}\right) a_{p+1} = C_1^\gamma(x) b_1, \tag{8}$$

and

$$\left(\frac{2\alpha(p+2)+2}{p\beta}\right) a_{p+2} - \left(\frac{1}{p\beta}\right) a_{p+1}^2 = C_1^\gamma(x) b_2 + C_2^\gamma(x) b_1^2. \tag{9}$$

Hence, using equation (8), we get

$$a_{p+1} = \frac{p\beta C_1^\gamma(x) b_1}{\alpha(p+1)+1}. \tag{10}$$

In view of the initial values (4) and the fact $|b_1| \leq 1$, we get

$$|a_{p+1}| \leq \frac{2p\gamma|x||\beta|}{\alpha(p+1)+1},$$

which is the desired estimate of $|a_{p+1}|$.

Secondly, we look for the coefficient estimate of a_{p+2}. Using equation (9), we obtain

$$2(\alpha(p+2)+1)a_{p+2} - a_{p+1}^2 = p\beta[C_1^\gamma(x) b_2 + C_2^\gamma(x) b_1^2]. \tag{11}$$

Using equation (10), the last equation becomes

$$\begin{aligned} a_{p+2} =& \frac{p\beta}{2(\alpha(p+2)+1)} \times \\ & \left(C_1^\gamma(x) b_2 + C_2^\gamma(x) b_1^2 + \frac{p\beta[C_1^\gamma(x)]^2 b_1^2}{(\alpha(p+1)+1)^2}\right) \\ =& \frac{p\beta C_1^\gamma(x)}{2(\alpha(p+2)+1)} \times \\ & \left(b_2 + \frac{C_2^\gamma(x)}{C_1^\gamma(x)} b_1^2 + \frac{p\beta C_1^\gamma(x) b_1^2}{(\alpha(p+1)+1)^2}\right). \end{aligned}$$

Hence, using Lemma 2.2, we obtain

$$|a_{p+2}| \leq \frac{p|\beta|\,|C_1^\gamma(x)|}{2(\alpha(p+2)+1)} \times \max\left\{1, \left|\frac{C_2^\gamma(x)}{C_1^\gamma(x)} + \frac{p\beta C_1^\gamma(x)}{(\alpha(p+1)+1)^2}\right|\right\}.$$

Finally, using the initial values (4), we get the desired estimate of $|a_{p+2}|$. Therefore, this completes the proof of Theorem 3.1. $\square$

The following corollary is just a consequence of Theorem 3.1 when taking $\gamma = 1$. These initial coefficient estimates are related to Chebyshev polynomials of the second kind. The proof is similar to the previous theorem's proof, so we omit the proof's details.

Corollary 3.2. *If a function $f \in \mathcal{A}_p$ is given by (1) belong to the class $\mathcal{A}_p(\alpha, \beta, 1)$, then*

$$|a_{p+1}| \leq \frac{2p|x||\beta|}{\alpha p + \alpha + 1},$$

and

$$|a_{p+2}| \leq \frac{p|x||\beta|}{\alpha p + 2\alpha + 1} \max\left\{1, \left|\frac{4x^2 - 1}{2x} + \frac{2xp\gamma\beta}{(\alpha p + \alpha + 1)^2}\right|\right\}.$$

Remark 1. *The following are special cases of our class of p-valent functions $\mathcal{A}_p(\alpha, \beta, \gamma)$.*

- *Putting, $\alpha = 0$ in Definition 2.1, we get a subclass that satisfies the following subordination:*

$$\frac{1}{\beta}\left(\frac{zf'(z)}{pf(z)} - 1\right) \prec H_\gamma(x, z) \tag{12}$$

- *Putting, $\beta = 1$ in Definition 2.1, we get a subclass that satisfies the following subordination:*

$$\left(\frac{zf'(z)}{pf(z)} - 1\right) + \alpha\left(\frac{f'(z)}{pz^{p-1}} - 1\right) \prec H_\gamma(x, z), \tag{13}$$

Corollary 3.3. *If a function $f \in \mathcal{A}_p$ is given by (1) satisfies the subordination (12), then*

$$|a_{p+1}| \leq 2\gamma p|x||\beta|,$$

and

$$|a_{p+2}| \leq \gamma p|x||\beta| \max\left\{1, \left|\frac{(4p\gamma\beta + 2\gamma + 2)x^2 - 1}{2x}\right|\right\}.$$

Corollary 3.4. *If a function $f \in \mathcal{A}_p$ is given by (1) satisfies the subordination (13), then*

$$|a_{p+1}| \leq \frac{2\gamma p|x|}{\alpha p + \alpha + 1},$$

and

$$|a_{p+2}| \leq \frac{\gamma p|x|}{\alpha p + 2\alpha + 1} \times \max\left\{1, \left|\frac{2(\gamma+1)x^2 - 1}{2x} + \frac{2xp\gamma}{(\alpha p + \alpha + 1)^2}\right|\right\}.$$

4 Fekete-Szegö inequality of the class $\mathcal{A}_p(\alpha, \beta, \gamma)$

In this section, we maximize the modulus of the functional $\Psi_\lambda(f) = a_{p+2} - \lambda a_{p+1}^2$ for real numbers λ and for functions f belong to the class $\mathcal{A}_p(\alpha, \beta, \gamma)$.

Theorem 4.1. *If a function $f \in \mathcal{A}_p$ is given by (1) belong to the class $\mathcal{A}_p(\alpha, \beta, \gamma)$, then for some $\lambda \in \mathbb{R}$ and for $x \in (0, 1]$,*

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \frac{p\gamma x|\beta|}{\alpha p + 2\alpha + 1}, & \text{if } \lambda \in [\lambda_1, \lambda_2] \\ \frac{p\gamma x|\beta||A|}{\alpha p + 2\alpha + 1}, & \text{if } \lambda \notin [\lambda_1, \lambda_2], \end{cases} \tag{14}$$

where

$$\lambda_1 = \frac{(2(\gamma+1)x^2 - 2x - 1)(\alpha p + \alpha + 1)^2 + 4\gamma\beta p x^2}{8\gamma\beta p x^2(\alpha p + 2\alpha + 1)},$$

$$\lambda_2 = \frac{(2(\gamma+1)x^2 + 2x - 1)(\alpha p + \alpha + 1)^2 + 4\gamma\beta p x^2}{8\gamma\beta p x^2(\alpha p + 2\alpha + 1)},$$

and

$$A = \frac{2(\gamma+1)x^2 - 1}{2x} + \frac{2x\gamma\beta(1 - 2\lambda(\alpha p + 2\alpha + 1))}{(\alpha p + \alpha + 1)^2}.$$

Proof. For any real number λ, using equations (11) and (10), we get

$$\begin{aligned} a_{p+2} - \lambda a_{p+1}^2 &= \frac{p\beta[C_1^\gamma(x)b_2 + C_2^\gamma(x)b_1^2]}{2(\alpha p + 2\alpha + 1)} \\ &\quad + \left(\frac{1}{2(\alpha p + 2\alpha + 1)} - \lambda\right) a_{p+1}^2 \\ &= \frac{\beta p[C_1^\gamma(x)b_2 + C_2^\mu(t)b_1^2]}{2(\alpha p + 2\alpha + 1)} \\ &\quad + \frac{\beta p(1 - 2\lambda(\alpha p + 2\alpha + 1))[C_1^\mu(t)]^2 b_1^2}{2(\alpha p + 2\alpha + 1)(\alpha p + \alpha + 1)^2} \\ &= \frac{\gamma\beta p x}{\alpha p + 2\alpha + 1}\left\{b_2 + A b_1^2\right\}. \end{aligned}$$

Hence, using Lemma 2.2, we get

$$|a_{p+2} - \lambda a_{p+1}^2| \leq \frac{\gamma p|\beta||x|}{\alpha p + 2\alpha + 1} \max\{1, |A|\}.$$

For $x > 0$, if $|A| \leq 1$, then

$$\left|\frac{C_2^\gamma(x)}{C_1^\gamma(t)} + \frac{\beta p(1 - 2\lambda(\alpha p + 2\alpha + 1)[C_1^\gamma(t)]}{(\alpha p + \alpha + 1)^2}\right| \leq 1.$$

Therefore, solving for λ we get

$$\begin{aligned} &\frac{-[C_2^\gamma(x) + C_1^\gamma(x)](\alpha p + \alpha + 1)^2}{\beta p[C_1^\gamma(x)]^2} \\ &\leq 1 - 2\lambda(\alpha p + 2\alpha + 1) \\ &\leq \frac{[C_1^\gamma(x) - C_2^\gamma(x)](\alpha p + \alpha + 1)^2}{\beta p[C_1^\gamma(x)]^2}. \end{aligned}$$

Hence, simple calculations give us the following inequality

$$\begin{aligned} &\frac{(C_2^\gamma(x) - C_1^\gamma(x))(\alpha p + \alpha + 1)^2 + \beta p[C_1^\gamma(x)]^2}{2\beta p(\alpha p + 2\alpha + 1)[C_1^\gamma(x)]^2} \leq \lambda \\ &\leq \frac{(C_2^\gamma(x) + C_1^\gamma(x))(\alpha p + \alpha + 1)^2 + \beta p[C_1^\gamma(x)]^2}{2\beta p(\alpha p + 2\alpha + 1)[C_1^\gamma(x)]^2}. \\ &\iff \lambda_1 \leq \lambda \leq \lambda_2. \end{aligned}$$

Therefore, in view of the initial values (4), if $|A| \leq 1$, then $\lambda \in [\lambda_1, \lambda_2]$ and hence we get

$$|a_{p+2} - \lambda a_{p+1}^2| \leq \frac{\gamma p|\beta||x|}{\alpha p + 2\alpha + 1}.$$

Moreover, if $|A| > 1$, then $\lambda \notin [\lambda_1, \lambda_2]$ and hence we get

$$|a_{p+2} - \lambda a_{p+1}^2| \leq \frac{\gamma p|\beta||x||A|}{\alpha p + 2\alpha + 1}.$$

This completes the Theorem's proof. □

The following corollary is just consequences of Theorem 4.1. Taking $\gamma = 1$, we get the Fekete-Szegö inequality that is related to Chebyshev polynomials of the second kind.

Corollary 4.2. *If a function $f \in \mathcal{A}_p$ is given by (1) belong to the class $\mathcal{A}_p(\alpha, \beta, 1)$, then for some $\lambda \in \mathbb{R}$ and for $x \in (0, 1]$,*

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \frac{px|\beta|}{\alpha p + 2\alpha + 1}, & \text{if } \lambda \in [\zeta_1, \zeta_2] \\ \frac{px|\beta||B|}{\alpha p + 2\alpha + 1}, & \text{if } \lambda \notin [\zeta_1, \zeta_2], \end{cases}$$

where

$$\zeta_1 = \frac{(4x^2 - 2x - 1)(\alpha p + \alpha + 1)^2 + 4\beta p x^2}{8\beta p x^2(\alpha p + 2\alpha + 1)},$$

$$\zeta_2 = \frac{(4x^2 + 2x - 1)(\alpha p + \alpha + 1)^2 + 4\beta p x^2}{8\beta p x^2(\alpha p + 2\alpha + 1)},$$

and

$$B = \frac{4x^2 - 1}{2x} + \frac{2x\beta(1 - 2\lambda(\alpha p + 2\alpha + 1))}{(\alpha p + \alpha + 1)^2}.$$

Corollary 4.3. *If a function $f \in \mathcal{A}_p$ is given by (1) satisfies the subordination (12), then for some $\lambda \in \mathbb{R}$ and for $x \in (0,1]$,*

$$|a_3 - \lambda a_2^2| \leq \begin{cases} p\gamma x|\beta|, & \text{if } \lambda \in [\zeta_3, \zeta_4] \\ p\gamma x|\beta||K|, & \text{if } \lambda \notin [\zeta_3, \zeta_4], \end{cases}$$

where

$$\zeta_3 = \frac{(4\gamma\beta p + 2\gamma + 2)x^2 - 2x - 1}{8\gamma\beta p x^2},$$

$$\zeta_4 = \frac{(4\gamma\beta p + 2\gamma + 2)x^2 + 2x - 1}{8\gamma\beta p x^2},$$

and

$$K = \frac{(4\gamma\beta(1-2\lambda) + 2\gamma + 2)x^2 - 1}{2x}.$$

Corollary 4.4. *If a function $f \in \mathcal{A}_p$ is given by (1) satisfies the subordination (13), then for some $\lambda \in \mathbb{R}$ and for $x \in (0,1]$,*

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \frac{p\gamma x}{\alpha p + 2\alpha + 1}, & \text{if } \lambda \in [\zeta_5, \zeta_6] \\ \frac{p\gamma x|\Delta|}{\alpha p + 2\alpha + 1}, & \text{if } \lambda \notin [\zeta_5, \zeta_6], \end{cases}$$

where

$$\zeta_5 = \frac{(2(\gamma+1)x^2 - 2x - 1)(\alpha p + \alpha + 1)^2 + 4\gamma p x^2}{8\gamma p x^2(\alpha p + 2\alpha + 1)},$$

$$\zeta_6 = \frac{(2(\gamma+1)x^2 + 2x - 1)(\alpha p + \alpha + 1)^2 + 4\gamma p x^2}{8\gamma p x^2(\alpha p + 2\alpha + 1)},$$

and

$$\Delta = \frac{2(\gamma+1)x^2 - 1}{2x} + \frac{2x\gamma(1 - 2\lambda(\alpha p + 2\alpha + 1))}{(\alpha p + \alpha + 1)^2}.$$

5 Conclusion

This research paper has investigated a new class of p-valent functions related to Gegenbauer polynomials. For functions belong to this function class, the author has derived estimates for the initial coefficients and Fekete-Szegö functional problem. The work presented in this paper will lead to many different results for subclasses defined by the means of Horadam polynomials and their special cases, such as: Fibonacci polynomials, Lucas Polynomials, Pell Polynomials, and Pell-Lucas Polynomials. Moreover, the presented work in this paper will inspire researchers to extend its concepts to harmonic functions and symmetric q-calculus such as q-Ruscheweyh and q-Salagean differential operators.

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