Common Fixed Point Results Related to Generalized F-Contractions in Extended Cone b-Metric Spaces

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Abstract: - This paper presents a new class of F-functions defined on a cone and proves some theorems showing the uniqueness and existence of common fixed points for two functions satisfying a generalized F nonlinear contractions condition in extended cone b-metric spaces. Several examples illustrate the main theorems and demonstrate the applicable side of theoretical results.

Key-Words: - Generalized F-contraction, Fixed point, Cauchy sequences, Extended cone metric space, Convergent sequences.

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1 Introduction

In 2007, [1], restructured the concept of metric spaces by introducing cone metric spaces, wherein the traditional real numbers were replaced with an ordering Banach space. Through their pioneering work, they established several fixed-point theorems for contractive mappings within these spaces, effectively extending analogous results previously established in conventional metric spaces. This innovative approach not only broadened the scope of metric space theory but also provided a fresh perspective on the convergence properties of mappings in the realm of cone metric spaces.
Based on the metric cone spaces, many authors have generalized them and studied the results of fixed points in them, as in [2], [3], [4].

Among the generalized cone metric spaces are the extended b-metric cone spaces conceived by [5], and [6]. The authors relied on the concept of extended b-metric spaces given by [7] and studied in them the existence and uniqueness of fixed points for Kannan contractions, [8]. Recently, these spaces have been placed as the focus of study for some mathematicians, such as in [9] and [10].

In 2012, [11], introduced a new contraction to fixed point theory by introducing the concept of F-contraction. This new contraction was studied by many other authors in different metric spaces such as in [12], [13], [14], [15], [16], [17], [18], [19], [20].

In this paper, a generalization of F-contractions in extended cone metric spaces is given and the existence and uniqueness of common fixed points for two functions that complete this generalized F-contraction are studied. Also, a result on the existence and uniqueness of a fixed point for a contraction where an ultra-altering function is used is verified. The methodology used throughout this paper is proof. To prove the main results, we use Cauchy and convergent sequences, respectively. Concrete examples accompany the main results of the paper. In addition, our results generalize some theorems of given references.

2 Preliminaries

Definition 2.1. [1] Let P be a non-empty subset of E, where E is an ordered Banach space. The set P is called *cone* if and only if:
(i) P is closed, nonempty, and $P \neq \{0\}$,
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P$ implies $ax + by \in P$,
(iii) $x \in P$ and $-x \in P$ implies $x = 0$.

When a cone $P \subset E$ is given, a partial ordering $\leq$ concerning P is defined by the relation $x \leq y$ if and only if $y - x \in P$. To indicate $x \leq y$ but $x \neq y$, we denote $x \leq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. The cone P is called normal if, there is a positive real number K such that, for all x, y in P we have:

$$0 \leq x \leq y \text{ implies } \|x\| \leq K\|y\|$$

The last positive number satisfying the above is called the normal constant of P.

The cone P is called regular if every increasing sequence that is bounded from above is convergent. That is, if $\{x_n\}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y$ for some $y \in E$, $x \in E$ such that $\|x_n - x\| \to 0$ $(n \to \infty)$. Equivalently, the cone P is regular if and only if any decreased sequence that is bounded from below is convergent. A regular cone is a normal cone.

In the following, we always suppose E is a Banach space, P is a cone in E with $\operatorname{int} P \neq \emptyset$, and $\leq$ is partial ordering concerning P.

Giving generalizations of metric spaces has been an open challenge for mathematicians. One of the most interesting generalizations of metric spaces was introduced in [1].

Definition 2.2. [2], Let P be a cone and X a non-empty set. The function $d: X \times X \to P$ is called a *cone metric* if it satisfies the following conditions:
(c_1) $d(x, y) \in P$ that is $0 \leq d(x, y)$ for$x, y \in X$, and $d(x, y) = 0$ iff $x = y$,
(c_2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(c_3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.
The pair (X, d) is called a *cone metric space*.

The concept of cone metric space and fixed point theory on these spaces has been developed by many authors in their works.

In 1998, [21], introduced the following interesting concept.

Definition 2.3. [21], Let X be a non-empty set and $s \geq 1$ be a given real number. A function $d: X \times X \to \mathbb{R}^+$ is called a b-metric if, for all $x, y, z \in X$ it satisfies the conditions:
($b1$) $d(x, y) = 0$ iff $x = y$,
($b2$) $d(x, y) = d(y, x)$,
($b3$) $d(x, z) \leq s[d(x, y) + d(y, z)]$ where $s \geq 1$.

The pair (X, d) is called b-metric space with parameter s.

Authors in [6], introduced a new type of generalized metric space by taking a two-variables function $\theta(x, y)$ instead of the parameter s.

Definition 2.4. [6], Let X be a nonempty set and $\theta: X \times X \to [1, +\infty)$. A function $d_\theta: X \times X \to [0, +\infty)$ is an extended b-metric, if for all $x, y, z \in X$ it satisfies:
$(d_\theta 1)$ $d_\theta(x, y) = 0$ iff $x = y$,
$(d_\theta 2)$ $d_\theta(x, y) = d_\theta(y, x)$, for all $x, y \in X$,
$(d_\theta 3)$ $d_\theta(x, z) \leq \theta(x, z)(d_\theta(x, y) + d_\theta(y, z))$ for all $x, y, z \in X$.

The pair (X, d_θ) is called an extended b-metric space.

Authors in [5], in their generalization, extended the domain of the function θ from $X \times X$ to $X \times X \times X$ thus giving this definition as follows.

Definition 2.5. [5], Let X be a non-empty set, and $\theta: X \times X \times X \to [1 + \infty)$. Let $d_\theta: X \times X \to \square^+$ be a function that satisfies the following conditions:
$(d_\theta 1)$ $d_\theta(x, y) \geq 0$ for all x, y and $d_\theta(x, y) = 0$ iff $x = y$,
$(d_\theta 2)$ $d_\theta(x, y) = d_\theta(y, x)$ for all x, y in X,
$(d_\theta 3)$ $d_\theta(x, y) \leq \theta(x, y, z)[d_\theta(x, z) + d_\theta(z, y)]$ for all x, y, z in X.

The function d_θ is called extended cone metric on X and the pair (X, d_θ) is called extended cone metric space.

Convergence, Cauchy sequences, continuity, and completeness on extended cone metric spaces are defined as follows:

Definition 2.6. [5], Consider a sequence $\{x_n\}$ in an extended cone metric space (X, d) and let P be a normal cone in E with normal constant K.
Then
(i) $\{x_n\}$ converges to x if for every $c \in E$ with $c > 0$, there exists $N > 0$ such that for all $n \geq N, d(x_n, x) < c$. Denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \xrightarrow[n\to\infty]{} x$.
(ii) $\{x_n\}$ is said to be Cauchy in X if for every $c \in E$ with $c > 0$, there exists a positive integer N such that for all $n, m \geq N \Rightarrow d(x_n, x_m) < c$.
(iii) The mapping $T: X \to X$ is said to be continuous at a point $x \in X$if for every sequence $\{x_n\}$ converging to x it follows that $\lim_{n\to\infty} Tx_n = T(\lim_{n\to\infty} x_n = Tx.)$
(iv) (X, d) is said to be a complete cone metric space if every Cauchy sequence is convergent in X.

3 Main Results

In this section, we present a new class of functions denoted $\mathfrak{F}_\theta$ where every element $F \in \mathfrak{F}_\theta$ satisfies the following conditions:

(1) $F: P \to \mathbb{R}$ is strictly increasing

(2) For every sequence $\{t_n\} \subset P$, the following equivalence holds:
$$\lim_{n\to+\infty} F(t_n) = -\infty \text{ iff } \lim_{n\to+\infty} t_n = 0$$

(3) For every sequence $\{t_n\} \subset P$, where $\lim_{n\to+\infty} t_n = 0$, there exists a number $s \in (0,1)$ such that $\lim_{n\to+\infty} \|t_n\|^s F(t_n) = -\infty$.

(4) If $\{t_n\} \subset P$ is a sequence that $\tau + F(\theta_n t_n) \le F(t_{n-1})$ for $n \in \mathbb{N}$ and $\tau \ge 0$ then $\tau + F(\prod_{i=1}^{n} \theta_i\, t_n) \le F\left(\prod_{i=1}^{n-1} \theta_i\, t_{n-1}\right)$.

Theorem 3.1. Let $(X,\ d_\theta)$ be a complete extended cone metric space and $T, S: X \to X$ two functions that satisfy the following implication:

If $d(x,y) > 0$ then
$$\tau + F(\theta(x,y,z) \cdot d_\theta(Tx,Sy) < F(M(x,y) + L \cdot m(x,y)) \quad (1)$$

where $\tau > 0,\ L > 0$,
$$M(x,y) = max\begin{Bmatrix} d_\theta(x,y), d_\theta(Tx,x), d_\theta(y,Sy), \\ \frac{d_\theta(Tx,y)+d_\theta(x,Ty)}{2\theta(x,y,z)} \end{Bmatrix},$$
$$m(x,y) = min\begin{Bmatrix} d_\theta(x,y), d_\theta(Tx,x), d_\theta(Sy,y), \\ d_\theta(x,y), d_\theta(x,Sy) \end{Bmatrix}$$
and θ is a convergent function used in extended b-metric for each $x, y, z \in X$.

Then *S* and *T* have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in *X*. Define the sequence $\{x_n\}$ by taking $x_1 = Tx_0$, $x_2 = Sx_1$, $x_3 = Tx_2$, $x_4 = Sx_3$ and so on, or, more generally, $x_{2n-1} = Tx_{2n-2}$ and $x_{2n} = Sx_{2n-1}$ for all $n \in \mathbb{N}$.
Beginning with the inequality expressed in inequality (1), we proceed:
$$\tau + F(\theta(x_{2n-1}, x_{2n-2}, z) \cdot d_\theta(x_{2n}, x_{2n-1}) < F(M(x_{2n-1}, x_{2n-2}) + Lm(x_{2n-1}, x_{2n-2})). \quad (2)$$

We observe that $M(x_{2n-1}, x_{2n-2})$ and $m(x_{2n-1}, x_{2n-2})$ are respectively:
$$M(x_{2n-1}, x_{2n-2}) = max\begin{Bmatrix} d_\theta(x_{2n-1}, x_{2n-2}), d_\theta(x_{2n-1}, x_{2n}), \\ d_\theta(x_{2n-2}, x_{2n-1}), \\ \frac{d_\theta(x_{2n},x_{2n-1})+d_\theta(x_{2n-1},x_{2n-2})}{2\theta(x_{2n-1},x_{2n-2},z)} \end{Bmatrix} \quad (3)$$

$$= max\begin{Bmatrix} d_\theta(x_{2n-1}, x_{2n-2}), d_\theta(x_{2n-1}, x_{2n}), \\ \frac{d_\theta(x_{2n},x_{2n-1})}{2\theta(x_{2n-1},x_{2n-2},z)} \end{Bmatrix}$$

$$= max\{d_\theta(x_{2n-1}, x_{2n-2}), d_\theta(x_{2n-1}, x_{2n})\}, \quad (4)$$
for all $n \in \mathbb{N}$

meanwhile,
$$m(x_{2n-1}, x_{2n-2}) = min\begin{Bmatrix} d_\theta(x_{2n-1}, x_{2n-2}), d_\theta(x_{2n-1}, x_{2n}), \\ d_\theta(x_{2n-2}, x_{2n-1}), d_\theta(x_{2n}, x_{2n-1}), \\ d_\theta(x_{2n-1}, x_{2n-1}) \end{Bmatrix} = 0 \quad (5)$$
for all $n \in \mathbb{N}$.

Applying equalities (3) and (4) over (1) we derive the following inequality
$$\tau + F(\theta(x_{2n-1}, x_{2n-2}, z) \cdot d_\theta(x_{2n}, x_{2n-1}) < F(max\{(d_\theta x_{2n-1}, x_{2n-2}), d_\theta(x_{2n-1}, x_{2n})\}) \quad (6)$$
for all $n \in \mathbb{N}$.

We distinguish the following cases:

Case 1.
If $max\{d_\theta(x_{2n-1}, x_{2n-2}), d_\theta(x_{2n-1}, x_{2n})\} = d_\theta(x_{2n-1}, x_{2n-2})$,
inequality (5) takes the form:
$$\tau + F(\theta(x_{2n-1}, x_{2n-2}, z) \cdot d_\theta(x_{2n}, x_{2n-1}) < F(d_\theta(x_{2n-1}, x_{2n-2})) \text{ for all } n \in \mathbb{N}.$$

Thus, for $n = 1$, our last inequality is derived:
$$\tau + F(\theta(x_1, x_0, z) \cdot d_\theta(x_2, x_1) < F(d_\theta(x_1, x_0)).$$

For $n = 2$, the following assessments hold:
$$F(\theta(x_2, x_1, z) \cdot \theta(x_1, x_0, z) d_\theta(x_3, x_2)) < F(\theta(x_1, x_0, z) d_\theta(x_2, x_1) - \tau < F(d_\theta(x_1, x_0)) - 2\tau.$$

Continuing iteratively, it is observed that the function *F* satisfies the following condition:
$$F\left(\prod_{i=1}^{2n} \theta(x_i, x_{i-1}, z) d_\theta(x_{2n-1}, x_{2n})\right) < F(d_\theta(x_1, x_0)) - n\tau. \quad (7)$$

Taking the limit on both sides as n tends toward infinity, we ascertain that:
$$\lim_{n\to\infty} F\left(\prod_{i=1}^{2n} \theta(x_i, x_{i-1,} z) d_\theta(x_{2n-1}, x_{2n})\right) = -\infty.$$
\Exploiting the condition (2) from the determination of the function *F*, it follows that:
$$\lim_{n\to+\infty} \theta(x_i, x_{i-1}, z) d_\theta(x_{2n-1}, x_{2n}) = 0 \quad \text{and}$$
$$\lim_{n\to+\infty} \prod_{i=1}^{2n} \theta(x_i, x_{i-1,}, z) \|d_\theta(x_{2n-1}, x_{2n})\| = 0 \quad (8)$$

The fulfillment of this equation along with the condition (3) of the function *F* implies that:

$$\lim_{n\to+\infty}\left(\prod_{i=1}^{2n}\theta(x_i,x_{i-1},z)\,\|d_\theta(x_{2n-1},x_{2n})\|\right)^s \cdot F\left(\prod_{i=1}^{2n}\theta(x_i,x_{i-1,}z)\,d_\theta(x_{2n-1},x_{2n})\right)=0. \quad (9)$$

Now, we multiply both sides of the above inequality (6) by $\left[\left(\prod_{i=1}^{2n}\theta(x_i,x_{i-1,}z)\right)\|d_\theta(x_{2n-1,}x_{2n})\|\right]^s$, thus obtaining:

$$\left[\left(\prod_{i=1}^{2n}\theta(x_i,x_{i-1},z)\right)\|d_\theta(x_{2n-1,}x_{2n})\|\right]^s \cdot F(\theta(x_{2n-1},x_{2n-2})\cdot d_\theta(x_{2n},x_{2n-1}) < \left[\left(\prod_{i=1}^{2n}\theta(x_i,x_{i-1,}z)\right)\|d_\theta(x_{2n-1,}x_{2n})\|\right]^s \cdot F\big(d_\theta(x_{2n-1},x_{2n-2})\big) - \tau\left[\left(\prod_{i=1}^{2n}\theta(x_i,x_{i-1},z)\right)\|d_\theta(x_{2n-1,}x_{2n})\|\right]^s.$$

Taking limits on both sides of this inequality and using the equalities (7) and (8) it follows that:

$$\lim_{n\to+\infty} n\left[\prod_{i=1}^{2n}\theta(x_i,x_{i-1},z)\,\|d_\theta(x_{2n-1,}x_{2n})\|\right]^s=0.$$

By employing the definition of the convergent sequence, we note that for $\varepsilon=1$, there exists $n_0\in\mathbb{N}$, such that for all $n>n_0$ we have:

$n\left(\prod_{i=1}^{2n}\theta(x_i,x_{i-1},z)\,\|d_\theta(x_{2n-1,}x_{2n})\|\right)^s<1.$

Therefore, we obtain this inequality:

$$\prod_{i=1}^{2n}\theta(x_i,x_{i-1},z)\,\|d_\theta(x_{2n-1,}x_{2n})\|<\frac{1}{n^{\frac{1}{s}}} \qquad (10)$$

for each $n\in\mathbb{N}$.

Below, we demonstrate that $\{x_{2n}\}$ is a Cauchy sequence on the extended cone metric space (X,d_θ). To do so, for $n>m$ we can derive that:

$$\begin{aligned} d_\theta(x_{2n},x_{2m}) &\le \theta(x_{2n},x_{2n-1},x_{2m})(d_\theta(x_{2n},x_{2n-1}) + d_\theta(x_{2n-1},x_{2m}))\\ &\le \theta(x_{2n},x_{2n-1},x_{2m})d_\theta(x_{2n},x_{2n-1}) + \theta(x_{2n},x_{2n-1},x_{2m})\theta(x_{2n-1},x_{2n-2},x_{2m})\big(d_\theta(x_{2n-1},x_{2n-2})+d_\theta(x_{2n-2},x_{2m})\big)\\ &= \theta(x_{2n},x_{2n-1},x_{2m})d_\theta(x_{2n},x_{2n-1}) + \theta(x_{2n},x_{2n-1},x_{2m})\theta(x_{2n-1},x_{2n-2},x_{2m})d_\theta(x_{2n-1},x_{2n-2}) + \theta(x_{2n},x_{2n-1},x_{2m})\theta(x_{2n-1},x_{2n-2},x_{2m})d_\theta(x_{2n-2},x_{2m})\\ &\le \cdots \le \sum_{n=1}^{\infty}\left(\prod_{i=1}^{n}\theta(x_{2i},x_{2i-1}x_{2m})\right)d_\theta(x_{2n},x_{2n-1}). \end{aligned}$$

Taking the norm of both sides, we obtain

$$\|d_\theta(x_{2n},x_{2m})\| \le \sum_{n=1}^{\infty}\left(\prod_{i=1}^{n}\theta(x_{2i},x_{2i-1}x_{2m})\right)K\|d_\theta(x_{2n},x_{2n-1})\| \le K\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{s}}}.$$

As a consequence:

$$\lim_{\substack{n\to\infty\\ m\to\infty}}\|d_\theta(x_{2n},x_{2m})\|=0.$$

Then, $\{x_{2n}\}$ is a Cauchy sequence on (X,d_θ). Given the completeness of (X,d_θ), there exists a point $x^*\in X$, such that $d_\theta(x_{2n},x^*)\xrightarrow[n\to\infty]{}0$.

Subsequently, we must demonstrate that x^* is a common fixed point of the functions S and T. Since F is strictly increasing, we can write that $\tau+F(\theta(x_{2n-1},x^*,z)\cdot d_\theta(x_{2n},Sx^*))\le F(M(x_{2n-1},x^*)+Lm(x_{2n-1},x^*))$. From here it follows that:

$$\theta(x_{2n-1},x^*,z)\cdot d_\theta(x_{2n},Sx^*)\le M(x_{2n-1},x^*)+Lm(x_{2n-1},x^*) \qquad (11)$$

where

$$M(x_{2n-1}x^*)=\max\left\{\begin{matrix} d_\theta(x_{2n-1},x^*),d_\theta(x_{2n},x_{2n-1}),d_\theta(Sx^*,x^*),\\ \frac{d_\theta(x_{2n,}x^*)+d_\theta(x_{2n-1,}Sx^*)}{2\theta(x_{2n-1},x*)}\end{matrix}\right\}$$

and

$$m(x_{2n-1},x^*)=\min\left\{\begin{matrix} d_\theta(x_{2n-1},x^*),d_\theta(x_{2n},x_{2n-1}),d_\theta(Sx^*,x^*),\\ d_\theta(x_{2n-1},x^*),d_\theta(x_{2n-1},Sx^*)\end{matrix}\right\}.$$

Taking the limits on both sides of (10) we derive that:

$$\lim_{n\to+\infty}\theta(x_{2n-1},x^*,z)\cdot d_\theta(x_{2n},Sx^*\le \lim_{n\to+\infty}(M(x_{2n-1},x^*+Lm(x_{2n-1},x^*))$$

or,

$$\theta_0 d_\theta(x^*,Sx^*)\le d_\theta(x^*,Sx^*)+L\cdot 0.$$

Consequently, $(\theta_0-1)d_\theta(x^*,Sx^*)\le 0$. Hence, $d_\theta(x^*,Sx^*)=0$ or $Sx^*=x^*$.

Similarly, if we see the inequality $\tau+F\big(\theta(x^*,x_{2n-2},z)\cdot d_\theta(Tx^*,x_{2n-1})\big)\le F\big(M(x^*,x_{2n-2})+Lm(x^*,x_{2n-2})\big)$, we derive that $d_\theta(Tx^*,x^*)=0$ or $Tx^*=x^*$. As a consequence, $Sx^*=x^*=Tx^*$thus, x^* is a common fixed point of S and T. Finally, we must show that x^* is the unique common fixed point for S and T. Suppose that there exists another point $y^*\in X,$ where $x^*\neq y^*$ such that $Ty^*=y^*=Sy^*$. We start from the inequation

$\tau + F(\theta(x^*, y^*, z) \cdot d_\theta(Tx^*, Sy^*)) \leq F\big(M(x^*, y^*) + Lm(x^*, y^*)\big)$ where, $M(x^*, y^*) = max\begin{Bmatrix} d_\theta(x^*, y^*), d_\theta(Tx^*, x^*), d_\theta(Sy^*, y^*), \\ \frac{d_\theta(Tx^*, y^*) + d_\theta(x^*, Ty^*)}{2\theta(x^*, y*, z)} \end{Bmatrix} = d_\theta(x^*, y^*)$
and
$m(x^*, y^*) = min\begin{Bmatrix} d_\theta(x^*, y^*), d_\theta(Tx^*, x^*), d_\theta(Sy^*, y^*), \\ d_\theta(x_{2n-1}, y^*), d_\theta(x_{2n-1}, Sy^*) \end{Bmatrix} = 0.$

After replacing $M(x^*, y^*)$ and $m(x^*, y^*)$ in the last inequality, we get:

$$\tau + F\big(\theta(x^*, y^*, z) \cdot d_\theta(Tx^*, Sy^*)\big) \leq F(d_\theta(x^*, y^*) + L \cdot 0)$$

which implies:
$d_\theta(x^*, y^*) \leq \theta(x^*, y^*, z) \cdot d_\theta(Tx^*, Sy^*) = \theta(x^*, y^*, z) \cdot d_\theta(x^*, y^*)$.
Thus $d_\theta(x^*, y^*) \leq \theta(x^*, y^*, z) \cdot d_\theta(x^*, y^*)$. This inequality can be true only if $d_\theta(x^*, y^*) = 0$ or $x^* = y^*$. Then the fixed point x^* is unique.

Example 3.2. Given the sets $X = \left\{\frac{1}{n}, n \in \mathbb{N}\right\} \cup \{0\}$, $E = \mathbb{R}$ and $P = \{x \in E\colon\ x \geq 0\}$. Define $d_\theta\colon X \times X \to P$ by $d_\theta(x, y) = \frac{1}{p}(x - y)^2$ where $p \geq 5$, $\theta(x, y, z) = 1 + x + y + z$ and $F\colon P \to \mathbb{R}$, such that $F(t) = ln\, t$, $\tau = \ln p$. Also, let T and S be respectively given by $T(x) = \frac{x}{p}$, and $S(y) = \frac{y}{p^2}$.
Firstly, we derive that $d_\theta(T(x), S(y)) = d_\theta(\frac{x}{p}, \frac{y}{p^2}) = \frac{1}{p}(\frac{x}{p} - \frac{y}{p^2})^2 = \frac{1}{p^3}(x - \frac{y}{p})^2$. Evaluating the distances between respective points, we have $d_\theta(x, y) = \frac{1}{p}(x - y)^2$.
On the other hand, we notice that
$d_\theta(T(x), x) = \frac{1}{p}(\frac{x}{p} - x)^2 = \frac{(1-p)^2 x^2}{p^2}$,
$d_\theta(y, S(y)) = \frac{(1-p^2)^2 y^2}{p^3}$, $d_\theta(T(x), y) = \frac{1}{p}(\frac{x}{p} - y)^2$, and $d_\theta(x, S(y)) = \frac{1}{p}(x - \frac{y}{p^2})^2$,
Then, make the following comparison between

$F(\theta(x, y, z) \cdot d_\theta(T(x), S(y)) = \ln \theta(x, y, z) \frac{1}{p^3}(x - \frac{y}{p})^2$

$= \ln(1 + x + y + z) \frac{1}{p^3}\left(x - \frac{y}{p}\right)^2 \leq \ln 4 \frac{1}{p^3}\left(x - \frac{y}{p}\right)^2 \leq \ln \frac{1}{p^2}\left(x - \frac{y}{p}\right)^2$

and

$F(M(x, y) + L \cdot m(x, y)) = ln(M(x, y) + L \cdot m(x, y)) >$ $\ln(\frac{1}{p}(x - y)^2 + L \cdot m(x, y) > \ln \frac{1}{p}\left(x - \frac{y}{p}\right)^2 = \ln \frac{1}{p^2}\left(x - \frac{y}{p}\right)^2 +$ $\ln p > F(\theta(x, y, z) \cdot d_\theta\big(T(x), S(y)\big) + \tau$.

Hence, the inequality (1) holds, then the functions have a common fixed point $x = 0$.

Theorem 3.3. Let $(X,\ d_\theta)$ be a complete extended cone metric space and $T\colon X \to X$ a function that satisfies the following implication
If $d(x, y) > 0$ then $\tau + F(\theta(x, y, z) \cdot d_\theta(Tx, Ty) < F(M(x, y) + L \cdot m(x, y))$ (12)

where $\tau > 0$, $L > 0$, $M(x, y) = max\left\{d_\theta(x, y), d_\theta(Tx, x), d_\theta(y, Ty), \frac{d_\theta(Tx, y) + d_\theta(x, Ty)}{2\theta(x, y, z)}\right\}$,
$m(x, y) = min\{d_\theta(x, y), d_\theta(Tx, x), d_\theta(Ty, y), d_\theta(x, y), d_\theta(x, Ty)\}$
and θ is a convergent function used in extended b-metric for each $x, y, z \in X$.
Then T has a unique fixed point.

Proof. Taking the function $S = T$ in inequality (1) we obtain the condition (12). As a result, the function T has a unique fixed point in X.

Example 3.4 Let us take the sets $X = [0,1]$, $E = \mathbb{R}$ and $P = \{x \in E\colon\ x \geq 0\}$. Define $d_\theta\colon X \times X \to P$ by $d_\theta(x, y) = (x - y)^2$, $\theta(x, y, z) = 1 + x + y + z$ and $F\colon P \to \mathbb{R}$, such that $F(t) = t + ln\, t$, $\tau = ln4$. Let T be given by $T(x) = \frac{x+1}{5}$.
Initially we see that for $x, y \in [0,1)$ a $d_\theta(T(x), T(y)) = d_\theta(\frac{x+1}{5}, \frac{y+1}{5}) = (\frac{x-y}{5})^2 = \frac{(x-y)^2}{25}$.

Calculating the distances between respective points x, y, Tx, Ty we have $d_\theta(x, y) = (x - y)^2$, $d_\theta(T(x), x) = (\frac{x-1}{5} - x)^2 = \frac{1+4x}{25}$, $d_\theta(y, T(y)) = \frac{1+4y}{25}$, $d_\theta(T(x), y) = (\frac{x}{5} - y)^2$, and $d_\theta(x, T(y)) = (x - \frac{y}{5})^2$,

Then, making the following comparison, we obtain

$$\begin{aligned} &\tau + F(\theta(x, y, z) \cdot d_\theta(T(x), S(y)) \\ &= ln4 + \theta(x, y, z)\frac{(x-y)^2}{25} + \ln \theta(x, y, z)\frac{(x-y)^2}{25} \\ &= (1 + x + y + z)\frac{(x-y)^2}{25} \\ &\quad + \ln 4(1 + x + y + z)\frac{(x-y)^2}{25} \\ &\leq 5\frac{(x-y)^2}{25} + \ln 20\frac{(x-y)^2}{25} \\ &= \frac{(x-y)^2}{5} + \ln\frac{4(x-y)^2}{5} \\ &\leq (x-y)^2 + \ln(x-y)^2 \\ &\leq F\big(M(x, y) + L \cdot m(x, y)\big) \end{aligned}$$

Since the inequality (12) holds, then the function T has a unique fixed point $x = \frac{1}{4}$.

Theorem 3.5 Let $(X,\ d_\theta)$ be a complete extended cone metric space and $T, S: X \to X$ two functions that satisfy the following inequality
If $d(x,y) > 0$ then $\tau + F(\theta(x,y,z) \cdot d_\theta(Tx,Sy)) < F(\varphi(M(x,y)))$ (13)

for every $x,y,z \in X$, where $\varphi: P \to P$ is a function which satisfies $\varphi(t) < t$, $\tau > 0$, $M(x,y) = max\begin{Bmatrix} d_\theta(x,y), d_\theta(Tx,x), d_\theta(y,Sy), \\ \frac{d_\theta(Tx,y)+d_\theta(x,Ty)}{2\theta(x,y,z)} \end{Bmatrix}$, and θ is a convergent function used in extended b-metric for each $x,y,z \in X$.
Then S and T have a unique common fixed point.

Proof. We use $\varphi(M(x,y)) < M(x,y)$ and the fact that the function F is strictly increasing in inequality (13). As a consequence, we derive

$$\tau + F(\theta(x,y,z) \cdot d_\theta(Tx,Sy)) < F\left(\varphi(M(x,y))\right) < F(M(x,y))$$

for every $x,y \in X$. Using the same scheme of proof as Theorem 3.1, the result is clear.

Theorem 3.6 Let (X, d_θ) be an extended cone b-metric space and T,S two functions that satisfy the following inequality
$\tau + F(\varphi(\theta(x,y,z) \cdot d_\theta(Tx,Sy)) \leq F(\varphi(M(x,y)) - \psi(M(x,y) + L \cdot m(x,y))$ (14)

for all $x,y,z \in X$, $\tau > 0$, $L > 0$ and φ is a sublinear altering, $\varphi: P \to P$, $\psi: P \to P$ with $\psi(0) = 0$.
Then S and T have a common fixed point.

Proof. Let's construct the sequence $\{x_n\}$ as in the proofing procedure of Theorem 3.1 by choosing x_0 an arbitrary point in X. Define the sequence $\{x_n\}$ by taking $x_1 = Tx_0$, $x_2 = Sx_1$, $x_3 = Tx_2$, $x_4 = Sx_3$ and so on, or, more generally, $x_{2n-1} = Tx_{2n-2}$ and $x_{2n} = Sx_{2n-1}$ for all $n \in \mathbb{N}$.
We can see that for each $n \in \mathbb{N}$, the following inequality holds

$$\tau + F(\varphi(\theta(x_{2n-1}, x_{2n-2}, z) \cdot d_\theta(x_{2n-1}, x_{2n-2})) \leq F\left(\varphi(M(x_{2n-1}, x_{2n-2})\right) - \psi(M(x_{2n-1}, x_{2n-2})) + L \cdot m(x_{2n-1}, x_{2n-2})$$

where $M(x_{2n-1}, x_{2n-2}) = max\begin{Bmatrix} d_\theta(x_{2n-1}, x_{2n-2}), d_\theta(x_{2n-1}, x_{2n}), \\ d_\theta(x_{2n-2}, x_{2n-1}), \\ \frac{d_\theta(x_{2n}, x_{2n-1}) + d_\theta(x_{2n-1}, x_{2n-2})}{2\theta(x_{2n-1}, x_{2n-2}, z)} \end{Bmatrix} = max\{d_\theta(x_{2n-1}, x_{2n-2}), d_\theta(x_{2n-1}, x_{2n})\}$
for all $n \in \mathbb{N}, m(x_{2n-1}, x_{2n-2}) = min\begin{Bmatrix} d_\theta(x_{2n-1}, x_{2n-2}), d_\theta(x_{2n-1}, x_{2n}), \\ d_\theta(x_{2n-2}, x_{2n-1}), d_\theta(x_{2n}, x_{2n-1}), \\ d_\theta(x_{2n-1}, x_{2n-1}) \end{Bmatrix} = 0$
Thus, we obtain this inequality

$$\tau + F(\varphi(\theta(x_{2n-1}, x_{2n-2}, z) \cdot d_\theta(x_{2n-1}, x_{2n})) \leq F\left(\varphi\left(d_\theta(x_{2n-1}, x_{2n-2})\right) - \psi\left(d_\theta(x_{2n-1}, x_{2n-2})\right)\right) \leq F(\varphi(d_\theta(x_{2n-1}, x_{2n-2})))$$

Replacing $n = 1$ in the final inequality, we derive

$$\tau + F(\varphi(\theta(x_1, x_0, z) \cdot d_\theta(x_2, x_1)) \leq F(\varphi(d_\theta(x_1, x_0)))$$

which implies

$$F(\varphi(\theta(x_1, x_0, z) \cdot d_\theta(x_2, x_1)) \leq F(\varphi(d_\theta(x_1, x_0))) - \tau \quad (15)$$

For $n = 2$, the following assessments, hold

$$\tau + F(\varphi(\theta(x_2, x_1, z) \cdot \theta(x_1, x_0, z) \cdot d_\theta(x_3, x_2))) \leq F(\varphi(\theta(x_1, x_0, z) \cdot d_\theta(x_1, x_2))) \leq F(\varphi(d_\theta(x_1, x_0))) - \tau. \quad (16)$$

To summarize, we have:

$$F(\varphi(\theta(x_2, x_1, z) \cdot \theta(x_1, x_0, z) \cdot d_\theta(x_3, x_2))) \leq F(\varphi(d_\theta(x_1, x_0))) - 2\tau.$$

By using this procedure iteratively, we obtain:

$$F\left(\varphi\left(\prod_{i=1}^{2n} \theta(x_i, x_{i-1}, z) d(x_{2n}, x_{2n+1})\right)\right) \leq F\left(\varphi\left(d_\theta(x_1, x_0)\right)\right) - 2n\tau \quad (17)$$

Taking the limit when $n \to +\infty$ in inequality (17)

$$\lim_{n\to\infty} F\left(\varphi\left(\prod_{i=1}^{2n} \theta(x_i, x_{i-1}, z) d(x_{2n}, x_{2n+1})\right)\right) = -\infty \quad (18)$$

we have:

$$\lim_{n\to\infty} \varphi\left(\prod_{i=1}^{2n} \theta(x_i, x_{i-1}, z) d(x_{2n}, x_{2n+1})\right) = 0 \quad (19)$$

Now, by leveraging the third property of function F, there exists $s \in (0,1)$ such that:

$$\lim_{n\to\infty}\varphi\left(\prod_{i=1}^{2n}\theta(x_i,x_{i-1},z)\cdot d_\theta(x_{2n},x_{2n+1})\right)^s F\left(\varphi\left(\prod_{i=1}^{2n}\theta(x_i,x_{i-1},z)d_\theta(x_{2n},x_{2n+1})\right)\right) = -\infty. \quad (19)$$

Multiplying (19) in both sides with: $\varphi\left(\prod_{i=1}^{2n}\theta(x_i,x_{i-1},z)\cdot\|d_\theta(x_{2n},x_{2n+1})\|\right)^s$ we get

$$\left[\varphi\left(\prod_{i=1}^{2n}\theta(x_i,x_{i-1},z)\cdot\|d_\theta(x_{2n},x_{2n+1})\|\right)^s\right] F\left(\varphi\left(\prod_{i=1}^{2n}\theta(x_i,x_{i-1},z)d(x_{2n},x_{2n+1})\right)\right) - F\left(\varphi\left(d_\theta(x_1,x_0)\right)\right) \le -2n\tau\left[\varphi\left(\prod_{i=1}^{2n}\theta(x_i,x_{i-1},z)\cdot\|d_\theta(x_{2n},x_{2n+1})\|\right)^s\right] \le 0. \quad (20)$$

Since the map φ does not take negative values, as a result, we conclude that $\lim_{n\to\infty} n\left[\varphi\left(\prod_{i=1}^{2n}\theta(x_i,x_{i-1},z)\cdot\|d_\theta(x_{2n},x_{2n+1})\|\right)^s\right]=0$ from which we can say that for $\varepsilon=1$, there exists $n_1\in\mathbb{N}$, such that, for all $n\in\mathbb{N}$ with $n>n_1$, we get

$$n\left[\varphi\left(\prod_{i=1}^{2n}\theta(x_i,x_{i-1},z)\cdot\|d_\theta(x_{2n},x_{2n+1})\|\right)^s\right]<1$$

or,

$$\varphi\left(\prod_{i=1}^{2n}\theta(x_i,x_{i-1},z)\cdot\|d_\theta(x_{2n},x_{2n+1})\|\right) < \frac{1}{n^{1/s}} \quad (21)$$

Next, let's show that the sequence $\{x_{2n}\}$ is a Cauchy sequence. Supposing $n>m$ we have

$$\begin{aligned}
&\varphi\left(d(x_{2n},x_{2m})\right)\\
&\le \varphi\left(\theta(x_{2n},x_{2n-1},x_{2m})\left(d_\theta(x_{2n},x_{2n-1}) + d_\theta(x_{2n-1},x_{2m})\right)\right)\\
&\le \varphi\big(\theta(x_{2n},x_{2n-1},x_{2m})\cdot d_\theta(x_{2n},x_{2n-1}) + \theta(x_{2n},x_{2n-1})\cdot\theta(x_{2n-1},x_{2n-2},x_{2m}) \cdot d_\theta(x_{2n-1},x_{2n-2})\cdot d_\theta(x_{2n-2},x_{2m})\big)\\
&\le \cdots\\
&\le \varphi\left(\sum_{k=m}^{n}\left(\prod_{i=m}^{k}\theta(x_{2i},x_{2i-1},x_{2m})\right)\right)\cdot d_\theta(x_{2k},x_{2k+1})\\
&\le \varphi\left(\sum_{n=1}^{\infty}\left(\prod_{i=1}^{n}\theta(x_{2i},x_{2i-1},x_{2m})\right)\right)\cdot d_\theta(x_{2n},x_{2n+1}) \le \sum_{n=1}^{\infty}\frac{1}{n^{1/s}}.
\end{aligned} \quad (22)$$

Applying the property of φ we obtain:

$$\sum_{n=1}^{\infty}\varphi\left(\prod_{i=1}^{n}\theta(x_{2i},x_{2i-1},x_{2m})\right)\cdot d_\theta(x_{2n},x_{2n+1}) \le \sum_{n=1}^{\infty}\frac{1}{n^{1/s}} \quad (23)$$

which implies:

$$\lim_{n\to\infty}\varphi\left(\sum_{n=1}^{\infty}\left(\prod_{i=1}^{n}\theta(x_{2i},x_{2i-1},x_{2m})\right)\right)\cdot d_\theta(x_{2n},x_{2n+1}) = 0 \text{ and } \lim_{n\to\infty}\varphi\left(d_\theta(x_{2n},x_{2m})\right) = 0. \quad (24)$$

From the above result, it follows that $\lim_{n\to\infty} d_\theta(x_{2n},x_{2m})=0$, which means that $\{x_{2n}\}$ is Cauchy. By using the completeness of $(X,\, d_\theta)$, there exists $x^*\in X$, such that $\lim_{n\to\infty} x_{2n}=x^*$.

Now, let us show that x^* is a common fixed point of S and T.

Applying the inequality from the condition of the theorem for the triple (x_{2n-1},x^*,z) we have

$$\tau + F(\varphi(\theta(x_{2n-1},x^*,z)\cdot d_\theta(x_{2n},Sx^*)) \le F\left(\varphi\left(M(x_{2n-1},x^*)\right)\right) - \psi\left(M(x_{2n-1},x^*) + L\cdot m(x_{2n-1},x^*)\right)$$

which implies:

$\varphi\left(\theta(x_{2n-1},x^*,z)\cdot d_\theta(x_{2n},Sx^*)\right) \le \varphi\left(M(x_{2n-1},x^*)\right) - \psi\left(M(x_{2n-1},x^*) + L\cdot m(x_{2n-1},x^*)\right)$. (25)

Taking the limits on both sides in the inequality (23) for $n\to\infty$, we obtain:

$\varphi\left(\theta_0\cdot d_\theta(x^*,Sx^*)\right) \le \varphi\left(d_\theta(x^*,Sx^*)\right) - \psi(d_\theta(x^*,Sx^*) + L\cdot 0) \le \varphi\left(d_\theta(x^*,Sx^*)\right)$.Since φ is increasing, it follows that $\theta_0\cdot d_\theta(x^*,Sx^*) \le d_\theta(x^*,Sx^*)$ or $(\theta_0-1)\cdot d_\theta(x^*,Sx^*)\le 0$.

Next, using the fact that $\theta_0 > 1$ we obtain $d_\theta(x^*, Sx^*) = 0$, and $Sx^* = x^*$.
Similarly, applying the inequality for the triple (x^*, x_{2n-2}, z) we have

$$\tau + F(\varphi(\theta(x^*, x_{2n-2}, z) \cdot d_\theta(Tx^*, x_{2n-2})) \leq F\left(\varphi\big(M(x^*, x_{2n-2})\big)\right) - \psi\big(M(x^*, x_{2n-2}) + L \cdot m(x^*, x_{2n-2})\big).$$

Following the same procedure as in the case of the function S, we obtain $Tx^* = x^* = Sx^*$ showing that x^* is a common fixed point of S and T.

To complete the proof, we must show the uniqueness of x^*.

If $y^* \in X$ is another common fixed point of S and T, then $Sy^* = y^* = Ty^*$. Using the condition of theorem we have

$$\tau + F(\varphi(\theta(x^*, y^*, z) \cdot d_\theta(Tx^*, Sy^*)) \leq F(\varphi\big(M(x^*, y^*)\big)) - \psi\big(M(x^*, y^*) + L \cdot m(x^*, y^*)\big).$$

which implies:

$$F\left(\varphi\big(\theta(x^*, y^*, z) \cdot d_\theta(x^*, y^*)\big)\right) \leq F\left(\big(\varphi\big(d_\theta(x^*, y^*)\big)\right) - \psi(d_\theta(x^*, y^*) + L \cdot 0) - \tau \leq F\left(\varphi\big(d_\theta(x^*, y^*)\big)\right) - \psi\big(d_\theta(x^*, y^*)\big).$$

and

$$\varphi\big(\theta(x^*, y^*, z) \cdot d_\theta(x^*, y^*)\big) \leq \varphi\big(d_\theta(x^*, y^*)\big)$$

which follows $\theta(x^*, y^*, z) \cdot d_\theta(x^*, y^*) \leq d_\theta(x^*, y^*)$. Since $\theta(x^*, y^*, z) \geq 1$, the last inequality holds only when $d_\theta(x^*, y^*) = 0$ which implies $x^* = y^*$, and the proof is done.

4 Conclusions

In this paper, we present some fixed point results for functions that satisfy several inequalities related to generalized F-contraction in extended cone b-metric space are studied. Theorem 3.1 and Theorem 3.6 are the highlights of this study. Theorem 3.1 is a generalization of Theorem 2.4 in [16], since we have proved the existence and uniqueness of a common fixed point for two functions in extended cone b-metric space, which is more general than metric space. Theorem 3.3 generalizes Theorem 2.8 in [22], because of the use of maximum and minimum respective distances in the function F in extended cone b-metric space. Theorem 3.6 is an extension of Theorem 2.1 in [23], using the generalized function. In future work, authors recommend the applications of their results to Integral Equations to prove the existence and uniqueness of various equations.

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