Intrinsically Hölder sections in metric spaces

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Abstract: - We introduce the notion of intrinsically Hölder graphs in metric spaces that generalized the one of intrinsically Lipschitz sections. This concept is relevant because it has many properties similar to Hölder maps but is profoundly different from them. We prove some relevant results as the Ascoli-Arzelà compactness Theorem, Ahlfors-David regularity and the Extension Theorem for this class of sections. In the first part of this note, thanks to Cheeger theory, we define suitable sets in order to obtain a vector space over $\mathbb{R}$ or $\mathbb{C}$, a convex set and an equivalence relation for intrinsically Hölder graphs. These last three properties are new also in the Lipschitz case.

Key-Words: - Hölder graphs, Ahlfors-David regularity, Extension theorem, Ascoli-Arzelà compactness theorem, vector space, equivalence relation, Metric spaces

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1 Introduction

Starting to the seminal papers [1, 2, 3] (see also [4, 5]), in [6] we generalize the notion of intrinsically Lipschitz maps introduced in subRiemannian Carnot groups [7, 8, 9]. This concept was introduced in order to give a good definition of rectifiability in subRiemannian geometry after the negative result shown in [10] (see also [11]) regarding the classical definition of rectifiability using Lipschitz maps given by [12]. The notion of rectifiable sets is a key one in Calculus of Variations and in Geometric Measure Theory. The reader can see [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23].

In this paper, we give a new natural definition of intrinsically Hölder sections (see Definition 2.1) which includes intrinsically Lipschitz ones. More precisely, this paper differs from the related ones because the settings are metric spaces which are more general than Carnot groups and because intrinsically Hölder maps has many properties similar to Hölder maps but are profoundly different from them (see [4, Example 4.58]). There are two main reasons for the importance of these mappings. First, their definition is extremely simple and widely applicable, and so they can be found in abundance on any metric space without any assumptions of smoothness. Second, despite the the simplicity of their definition, they often possess many rigidity properties and therefore their study can yield surprising analytic and geometric conclusions.

We prove the following results using basic mathematical tools.

1. Theorem 2.1, i.e., Compactness Theorem a lá Ascoli-Arzelà for the intrinsically Hölder sections.
2. Theorem 2.2, i.e., Ahlfors-David regularity for the intrinsically Hölder sections.
3. Proposition 3.3 states that the class of the intrinsically Hölder sections is a convex set.
4. Theorem 3.1 states that a suitable class of the intrinsically Hölder sections is a vector space over $\mathbb{R}$ or $\mathbb{C}$.
5. Theorem 4.1 gives an equivalence relation for a suitable class of the intrinsically Hölder sections.
6. Theorem 5.1, i.e., Extension Theorem for the intrinsically Hölder sections.

The points $(3)-(4)-(5)$ are new results also in the context of Lipschitz sections.

2 Intrinsically Hölder sections: definition and basic properties

Definition 2.1 (Intrinsic Hölder section) *Let (X,d) be a metric space and let Y be a topological space. We say that a map $\varphi : Y \to X$ is a* section *of a quotient map $\pi : X \to Y$ if $\pi \circ \varphi = id_Y$. Moreover, we say that φ is an* intrinsically (L,α)-Hölder section *with constant $L>0$ and $\alpha \in (0,1)$ if in addition*

$$d(\varphi(y_1),\varphi(y_2)) \le Ld(\varphi(y_1),\pi^{-1}(y_2))^{\alpha} + d(\varphi(y_1),\pi^{-1}(y_2)), \tag{1}$$

for all $y_1, y_2 \in Y$.

Equivalently, we are requesting that $d(x_1,x_2) \leq Ld(x_1,\pi^{-1}(\pi(x_2)))^\alpha + d(x_1,\pi^{-1}(\pi(x_2)))$, *for all* $x_1,x_2 \in \varphi(Y)$.

The standard example for us is when X is a metric Lie group G (meaning that the Lie group G is equipped with a left-invariant distance that induces the manifold topology), for example a subRiemannian Carnot group, and Y is the space of left cosets G/H, where $H < G$ is a closed subgroup and $\pi : G \to G/H$ is the projection modulo H, $g \mapsto gH$.

When $\alpha = 1$, a section φ is intrinsically Lipschitz in the sense of [6]. Moreover, we underline that, in the case $\alpha = 1$ and π is a Lipschitz quotient or submetry [24, 25], the results trivialize, since in this case being intrinsically Lipschitz is equivalent to biLipschitz embedding, see Proposition 2.4 in [6].

We further rephrase the definition as saying that $\varphi(Y)$, which we call the *graph* of φ, avoids some particular sets (which depend on α, L and φ itself):

Proposition 2.1 *Let* $\pi : X \to Y$ *be a quotient map between a metric space and a topological space,* $\varphi : Y \to X$ *be a section of* π*,* $\alpha \in (0,1)$ *and* $L > 0$*. Then* φ *is intrinsically* (L,α)*-Hölder if and only if* $\varphi(Y) \cap R_{x,L} = \emptyset$*, for all* $x \in \varphi(Y)$*, where* $R_{x,L} := \{x' \in X \mid Ld(x',\pi^{-1}(\pi(x)))^\alpha + d(x',\pi^{-1}(\pi(x))) < d(x',x)\}$.

Proposition 2.1 is a triviality, still its purpose is to stress the analogy with the intrinsically Lipschitz sections theory introduced in [6] when $\alpha = 1$. In particular, the sets $R_{x,L}$ are the intrinsic cones in the sense of Franchi, Serapioni and Serra Cassano when X is a subRiemannian Carnot group and $\alpha = 1$.

Definition 2.1 it is very natural if we think that what we are studying graphs of appropriate maps. However, in the following proposition, we introduce an equivalent condition of (1) when Y is a compact set.

Proposition 2.2 *Let* $\pi : X \to Y$ *be a quotient map between a metric space* X *and a topological and compact space* Y *and let* $\alpha \in (0,1)$*. The following are equivalent:*

1. *there is* $L > 0$ *such that* $d(\varphi(y_1),\varphi(y_2)) \leq Ld(\varphi(y_1),\pi^{-1}(y_2))^\alpha + d(\varphi(y_1),\pi^{-1}(y_2))$*, for all* $y_1,y_2 \in Y$.
2. *there is* $K \geq 1$ *such that*
$$d(\varphi(y_1),\varphi(y_2)) \leq Kd(\varphi(y_1),\pi^{-1}(y_2))^\alpha, \quad (2)$$
for all $y_1,y_2 \in Y$.

(1) $\Rightarrow$ (2). This is trivial when $d(\varphi(y_1),\pi^{-1}(y_2)) \leq 1$. On the other hand, if we consider $y_1,y_2 \in Y$ and $\bar{x} \in X$ such that $d(\varphi(y_1),\pi^{-1}(y_2)) = d(\varphi(y_1),\bar{x}) > 1$, then it is possible to consider ℓ equidistant points $x_1,\dots,x_\ell \in X$ such that

$$d(\varphi(y_1),\bar{x}) = d(\varphi(y_1),x_1) + \sum_{i=1}^{\ell-1} d(x_i,x_{i+1}) + d(x_\ell,\bar{x}),$$

with $d(\varphi(y_1),x_1) = d(x_i,x_{i+1}) = d(x_\ell,\bar{x}) \in (\frac{1}{2},1)$. Here, $\ell \leq \lfloor d(\varphi(y_1),\bar{x}) \rfloor + 1$ depends on y_1,y_2 and $\lfloor k \rfloor$ denotes the integer part of k. However, it is possible to choose $k \in \mathbb{R}^+$ defined as

$$k := \sup_{y_1,y_2 \in Y} d(\varphi(y_1),\pi^{-1}(y_2)), \quad (3)$$

such that k not depends on the points and $\ell \leq k$. We notice that this constant is finite because, on the contrary, we get the contradiction $\infty = d(\varphi(y_1),\pi^{-1}(y_2)) \leq d(\varphi(y_1),\varphi(y_2))$. Hence,
$$\begin{aligned}
&\mathrm{d}(\varphi(y_1),\varphi(y_2)) \leq Ld(\varphi(y_1),\bar{x})^\alpha + d(\varphi(y_1),\bar{x})\\
&= Ld(\varphi(y_1),\bar{x})^\alpha\\
&+ d(\varphi(y_1),x_1) + \textstyle\sum_{i=1}^{\ell-1} d(x_i,x_{i+1}) + d(x_\ell,\bar{x})\\
&\leq Ld(\varphi(y_1),\bar{x})^\alpha\\
&+ d(\varphi(y_1),x_1)^\alpha + \textstyle\sum_{i=1}^{\ell-1} d(x_i,x_{i+1})^\alpha + d(x_\ell,\bar{x})^\alpha\\
&\leq (L + 3(\lfloor k \rfloor + 1))d(\varphi(y_1),\bar{x})^\alpha\\
&=: Kd(\varphi(y_1),\bar{x})^\alpha.
\end{aligned}$$
(2) $\Rightarrow$ (1). This is a trivial implication.

Definition 2.2 (Intrinsic Hölder with respect to a section) *Given sections* $\varphi,\psi : Y \to X$ *of* π*. We say that* φ *is* intrinsically (L,α)-Hölder with respect to ψ at point $\hat{x}$*, with* $L > 0, \alpha \in (0,1)$ *and* $\hat{x} \in X$*, if*

1. $\hat{x} \in \psi(Y) \cap \varphi(Y)$;
2. $\varphi(Y) \cap C^\psi_{\hat{x},L} = \emptyset$,

where $C^\psi_{\hat{x},L} := \{x \in X : d(x,\psi(\pi(x))) > Ld(\hat{x},\psi(\pi(x)))^\alpha + d(\hat{x},\psi(\pi(x)))\}$.

Remark 1 *Definition 2.2 can be rephrased as follows. A section* φ *is intrinsically* (L,α)*-Hölder with respect to* ψ *at point* $\hat{x}$ *if and only if there is* $\hat{y} \in Y$ *such that* $\hat{x} = \varphi(\hat{y}) = \psi(\hat{y})$ *and*

$$d(x,\psi(\pi(x))) \leq Ld(\hat{x},\psi(\pi(x)))^\alpha + d(\hat{x},\psi(\pi(x))), \quad (4)$$

for all $x \in \varphi(Y)$*, which equivalently means*

$$d(\varphi(y),\psi(y)) \leq Ld(\psi(\hat{y}),\psi(y))^\alpha + d(\psi(\hat{y}),\psi(y)), \quad (5)$$

for all $y \in Y$.

Notice that Definition 2.2 does not induce an equivalence relation because of lack of symmetry in the right-hand side of (5). In Section 4 we give a stronger definition in order to obtain an equivalence relation.

Finally, following Cheeger theory [26] (see also [27, 28]), we give another equivalent property of Hölder section. Here it is fundamental that Y is a compact set.

Proposition 2.3 *Let X be a metric space, Y a topological and compact space, $\pi : X \to Y$ a quotient map, $L > 0$ and $\alpha, \beta, \gamma \in (0,1)$. Assume that every point $x \in X$ is contained in the image of an intrinsic (L, α)-Hölder section ψ_x for π. Then for every section $\varphi : Y \to X$ of π the following are equivalent:*

1. *for all $x \in \varphi(Y)$ the section φ is intrinsically (L_1, β)-Hölder with respect to ψ_x at x;*
2. *the section φ is intrinsically (L_2, γ)-Hölder.*

The proof of the last statement is an immediately consequence of the following result.

Proposition 2.4 *Let X be a metric space, Y a topological and compact space, and $\pi : X \to Y$ a quotient map. Let $L > 0$ and $y_0 \in Y$. Assume $\varphi_0 : Y \to X$ is an intrinsically (L, α)-Hölder section of π. Let $\varphi : Y \to X$ be a section of π such that $x_0 := \varphi(y_0) = \varphi_0(y_0)$. Then the following are equivalent:*

1. *For some $L_1 > 0$ and $\beta \in (0,1)$, φ is intrinsically (L_1, β)-Hölder with respect to φ_0 at x_0;*
2. *For some $L_2 \geq 1$ and $\gamma \in (0,1)$, φ satisfies*

$$d(x_0, \varphi(y)) \leq L_2 d(x_0, \pi^{-1}(y))^{\gamma}, \quad \forall y \in Y. \quad (6)$$

Moreover, the constants L_1 and L_2 are quantitatively related in terms of L.

We begin recall that, by Proposition 2.2, (1) is equivalent to (2).

(1) $\Rightarrow$ (2). For every $y \in Y$, it follows that
$d(\varphi(y), x_0) \leq d(\varphi(y), \varphi_0(y)) + d(\varphi_0(y), x_0)$
$\leq L_1 d(\varphi_0(y), x_0)^{\beta} + d(\varphi_0(y), x_0)$
$\leq L_1 L d(x_0, \pi^{-1}(y))^{\beta\alpha} + L d(x_0, \pi^{-1}(y))^{\alpha}$
$\leq L(L_1+1) \max\{d(x_0, \pi^{-1}(y))^{\beta\alpha}, d(x_0, \pi^{-1}(y))^{\alpha}\}$
where in the first inequality we used the triangle inequality, and in the second one the intrinsic Hölder property (1) of φ. Then, in the third inequality we used the intrinsic Hölder property of φ_0. Here, noticing that $\beta\alpha < \alpha$, we have that if $d(x_0, \pi^{-1}(y)) \leq 1$, then $\max\{d(x_0, \pi^{-1}(y))^{\beta\alpha}, d(x_0, \pi^{-1}(y))^{\alpha}\} = d(x_0, \pi^{-1}(y))^{\beta\alpha}$. On the other hand, if $d(x_0, \pi^{-1}(y)) > 1$, then using a similar technique using in Proposition 2.2 we obtain the same maximum with additional constant $K := L + 3(\lfloor k \rfloor + 1)$ where $k \in \mathbb{R}$ is given by costuniversale. Definitely, $d(\varphi(y), x_0) \leq LK(L_1+1) d(x_0, \pi^{-1}(y))^{\beta\alpha}$.

(2) $\Rightarrow$ (1). For every $y \in Y$, we have that $d(\varphi(y), \varphi_0(y)) \leq d(\varphi(y), x_0) + d(x_0, \varphi_0(y))$
$\leq L_2 d(\varphi_0(y), x_0)^{\gamma} + d(\varphi_0(y), x_0)$, where in the first equality we used the triangle inequality, and in the second one we used (6).

It is easy to see that if $\alpha = 1$, then we get $\beta = \gamma$ and so we have the following corollary.

Corollary 2.1 *Let X be a metric space, Y a topological and compact space, $\pi : X \to Y$ a quotient map, $L \geq 1$ and $\beta \in (0,1)$. Assume that every point $x \in X$ is contained in the image of an intrinsically L-Lipschitz section ψ_x for π. Then for every section $\varphi : Y \to X$ of π the following are equivalent:*

1. *for all $x \in \varphi(Y)$ the section φ is intrinsically (L_1, β)-Hölder with respect to ψ_x at x;*
2. *the section φ is intrinsically (L_2, β)-Hölder.*

2.1 Continuity

An intrinsically (L, α)-Hölder section $\varphi : Y \to X$ of π is a continuous section. Indeed, fix a point $y \in Y$. Since π is open, for every $\varepsilon \in (0,1)$ and every $x \in X$ such that $x = \varphi(y)$ we know that there is an open neighborhood U_ε of $\pi(x) = y$ such that $U_\varepsilon \subset \pi(B(x, [\varepsilon/(L+1)]^{1/\alpha}))$. Hence, if $y' \in U_\varepsilon$ then there is $x' \in B(x, [\varepsilon/(L+1)]^{1/\alpha})$ such that $\pi(x') = y'$. That means $x' \in \pi^{-1}(y')$ and, consequently, $d(\varphi(y), \varphi(y')) \leq L d(\varphi(y), \pi^{-1}(y'))^{\alpha} + d(\varphi(y), \pi^{-1}(y'))$
$\leq (L+1) d(x, x')^{\alpha}$
$\leq \varepsilon$, i.e., $\varphi(U_\varepsilon) \subset B(x, \varepsilon)$.

2.2 An Ascoli-Arzelà compactness theorem

Similar to the Lipschitz case, we have the following theorem a lá Ascoli-Arzelá.

Theorem 2.1 (Equicontinuity and Compactness Theorem) *Let $\pi : X \to Y$ be a quotient map between a metric space X for which closed balls are compact and a topological space Y. Then,*
(i) *For all $K' \subset Y$ compact, $L \geq 1$, $\alpha \in (0,1), K \subset X$ compact, and $y_0 \in Y$ the set $A_0 := \{\varphi_{|_{K'}} : K' \to X \mid \varphi : Y \to X$ is intrinsically (L, α)-Hölder section of $\pi, \varphi(y_0) \in K\}$ is equibounded, equicontinuous, and closed in the uniform convergence topology.*
(ii) *For all $L \geq 1$, $\alpha \in (0,1), K \subset X$ compact, and $y_0 \in Y$ the set $\{\varphi : Y \to X : \varphi$ is intrinsically (L, α)-Hölder section of π, $\varphi(y_0) \in K\}$ is compact with respect to the topology of uniform convergence on compact sets.*

The proof is similar to the one of [6, Theorem 1.2].

2.3 Ahlfors-David regularity

Following again [6], we can prove an Ahlfors-David regularity for the intrinsically Hölder sections. Recall that in Euclidean case $\mathbb{R}^s$, there are (L,α)-Hölder maps such that the $(s+1-\alpha)$-Hausdorff measure of their graphs is not zero and the $(s+1)$-Hausdorff measure of their graphs is zero, we give the following result.

Theorem 2.2 (Ahlfors-David regularity) *Let $\pi : X \to Y$ be a quotient map between a metric space X and a topological space Y such that there is a measure μ on Y such that for every $r_0 > 0$ and every $x, x' \in X$ with $\pi(x) = \pi(x')$ there is $C > 0$ such that*

$$\mu(\pi(B(x,r))) \leq C\mu(\pi(B(x',r))), \quad \forall r \in (0,r_0). \tag{7}$$

Let $\ell \in (0,\infty)$. We also assume that there is an intrinsically (L,α)-Hölder section $\varphi : Y \to X$ of π such that $\varphi(Y)$ is locally $(\ell+1-\alpha)$-Ahlfors-David regular with respect to the measure $\varphi_*\mu$.

Then, for every intrinsically (L,α)-Hölder section $\psi : Y \to X$ of π, the set $\psi(Y)$ is locally Q-Ahlfors-David regular with respect to the measure $\psi_*\mu$, where $Q = \alpha(\ell+1-\alpha)$ when the radius of the balls is small than 1 and $Q = \ell+1-\alpha$ when the radius of the balls is larger than 1.

Namely, locally Q-Ahlfors-David regularity means that the measure $\varphi_*\mu$ is such that for each point $x \in \varphi(Y)$ there exist $r_0 > 0$ and $C > 0$ so that $C^{-1}r^Q \leq \varphi_*\mu\big(B(x,r) \cap \varphi(Y)\big) \leq Cr^Q, forallr \in (0,r_0)$. The same inequality will hold for $\psi_*\mu$ with a possibly different value of C and Q.

The proof of this statement is similar to the one of [6, Theorem 1.3]. We notice that in the Lipschitz case we use [6, Proposition 2.12 (iii)]; here, the corresponding result is the following one: Let X be a metric space, Y a topological space, and $\pi : X \to Y$ a quotient map. If $\varphi : Y \to X$ is an intrinsically (L,α)-Hölder section of π with $\alpha \in (0,1)$ and $L > 0$, then $\pi(B(p,r)) \subset \pi(B(p, Lr^\alpha + r) \cap \varphi(Y)) \subset \pi(B(p, Lr^\alpha + r))$, for all $p \in \varphi(Y)$ and $r > 0$.

3 Properties of linear and quotient map

In order to give some relevant properties as convexity and being vector space over $\mathbb{R}$ we need to ask that π is also a linear map. Notice that this fact is not too restrictive because in our idea π is the 'usual' projection map. More precisely, throughout the section we will consider π a linear and quotient map between a normed space X and a topological space Y.

3.1 Basic properties

In this section we give two simple results in the particular case when π is a linear map.

Proposition 3.1 *Let $\pi : X \to Y$ be a linear and quotient map between normed spaces X and Y. The set of all section of π is a convex set.*

Fix $t \in [0,1]$ and let $\varphi, \psi : Y \to X$ sections of π. By the simply fact $\pi(t\varphi(y) + (1-t)\psi(y)) = t\pi(\varphi(y)) + (1-t)\pi(\psi(y)) = y,$ we get the thesis.

Proposition 3.2 *Let $\pi : X \to Y$ be a linear and quotient map between normed spaces X and Y. If $\varphi : Y \to X$ is an intrinsically Hölder section of π, then for any $\lambda \in \mathbb{R} - \{0\}$ the section $\lambda\varphi$ is also intrinsic Hölder for $1/\lambda\pi$ with the same Lipschitz constant up to the constant $|\lambda|^{1-\alpha}$.*

Fix $\lambda \in \mathbb{R} - \{0\}$. The fact that $\lambda\varphi$ is a section is trivial using the similar argument of Proposition 3.1. On the other hand, for any $y_1, y_2 \in Y$ $\|\lambda\varphi(y_1) - \lambda\varphi(y_2)\| \leq |\lambda| L d(\varphi(y_1), \pi^{-1}(y_2))^\alpha$
$= |\lambda|^{1-\alpha} L d(\lambda\varphi(y_1), (1/\lambda\pi)^{-1}(y_2))^\alpha,$ i.e., the thesis holds. This fact follows by these observations:

1. if $d(\varphi(y_1), \pi^{-1}(y_2)) = d(\varphi(y_1), a)$ then $|\lambda|^\alpha d(\varphi(y_1), \pi^{-1}(y_2))^\alpha = \|\lambda\varphi(y_1) - \lambda a\|^\alpha$.
2. $\lambda a \in \pi^{-1}(\lambda y)$.
3. $\pi^{-1}(\lambda y) = (1/\lambda\pi)^{-1}(y)$.

The second point is true because using the linearity of π we have that $\pi(\lambda a) = \lambda\pi(a) = \lambda y$. Finally, the third point holds because $\pi^{-1}(\lambda y) = \{x \in X : \pi(x) = \lambda y\}$
$= \{x \in X : 1/\lambda\pi(x) = y\}$
$= (1/\lambda\pi)^{-1}(y),$ as desired.

3.2 Convex set

In this section we show that the set of all intrinsically Hölder sections is a convex set. We underline that the hypothesis on boundness of Y is not necessary.

Definition 3.1 (Intrinsic Hölder set with respect to ψ) *Let $\alpha \in (0,1]$ and $\psi : Y \to X$ a section of π. We define the set of all intrinsically Hölder sections of π with respect to ψ at point $\hat{x}$ as $H_{\psi,\hat{x},\alpha} := \{\varphi : Y \to X$ a section of $\pi : \varphi$ is intrinsically $(\tilde{L},\alpha)$-Hölder w.r.t. ψ at point $\hat{x}$ for some $\tilde{L} > 0\}$.*

Proposition 3.3 *Let $\pi : X \to Y$ be a linear and quotient map between normed spaces X and Y. Assume also that $\alpha \in (0,1]$, $\psi : Y \to X$ a section of π and $\hat{x} \in \psi(Y)$. Then, the set $H_{\psi,\hat{x},\alpha}$ is a convex set.*

Let $\varphi, \eta \in H_{\psi,\hat{x},\alpha}$ and let $t \in [0,1]$. We want to show that

$$w := t\varphi + (1-t)\eta \in H_{\psi,\hat{x},\alpha}.$$

Notice that, by Proposition 3.1, w is a section of π and $w(\bar{y}) = \varphi(\bar{y}) = \eta(\bar{y}) = \hat{x}$ for some $\bar{y} \in Y$. On the other hand, for every $y \in Y$ we have $\|w(y) - \psi(y)\| = \|t(\varphi(y) - \psi(y)) + (1-t)(\eta(y) - \psi(y))\|$, and so $\|w(y) - \psi(y)\| \leq t\|\varphi(y) - \psi(y)\| + (1-t)\|\eta(y) - \psi(y)\|$. Hence, $d(w(y),\psi(y)) \leq tL_\varphi d(\psi(\bar{y}), \psi(y))^\alpha$
$+ (1-t)L_\psi d(\psi(\bar{y}), \psi(y))^\alpha + d(\psi(\bar{y}), \psi(y))$
$= [t(L_\varphi - L_\psi) + L_\psi] d(\psi(\bar{y}), \psi(y))^\alpha$
$+ d(\psi(\bar{y}), \psi(y))$,
for every $y \in Y$, as desired.

3.3 Vector space

In this section we show that a 'large' class of intrinsically Hölder sections is a vector space over $\mathbb{R}$ or $\mathbb{C}$. Notice that it is no possible to obtain the statement for $H_{\psi,\hat{x},\alpha}$ since the simply observation that if $\psi(\hat{y}) = \hat{x}$ then $\psi(\hat{y}) + \psi(\hat{y}) \neq \hat{x}$.

Theorem 3.1 *Let $\pi : X \to Y$ is a linear and quotient map between normed spaces X and Y. Assume also that $\psi : Y \to X$ is a section of π.*

Then, for any $\alpha \in (0,1]$, the set $\bigcup_{\lambda\in\mathbb{R}^+} H_{\lambda\psi,\lambda\hat{x},\alpha} \cup \{0\}$ is a vector space over $\mathbb{R}$ or $\mathbb{C}$.

Let $\varphi, \eta \in \bigcup_{\lambda\in\mathbb{R}^+} H_{\lambda\psi,\lambda\hat{x},\alpha}$ and $\beta \in \mathbb{R} - \{0\}$. We want to show that

1. $w = \varphi + \eta \in \bigcup_{\lambda\in\mathbb{R}^+} H_{\lambda\psi,\lambda\hat{x},\alpha}$.
2. $\beta\varphi \in \bigcup_{\lambda\in\mathbb{R}^+} H_{\lambda\psi,\lambda\hat{x},\alpha}$.

(1). If $\varphi \in H_{\delta_1\psi,\delta_1\hat{x},\alpha}$ and $\eta \in H_{\delta_2\psi,\delta_2\hat{x},\alpha}$ for some $\delta_1, \delta_2 \in \mathbb{R}^+$ it holds

$$w \in H_{(\delta_1+\delta_2)\psi,(\delta_1+\delta_2)\hat{x},\alpha}.$$

For simplicity, we choose $\varphi, \eta \in H_{\psi,\hat{x},\alpha}$ and so it remains to prove

$$w \in H_{2\psi,2\hat{x},\alpha}.$$

By linear property of π, w is a section of $1/2\pi$. On the other hand, if $\psi(\bar{y}) = \hat{x}$, then $w(\bar{y}) = \varphi(\bar{y}) + \eta(\bar{y}) = 2\psi(\bar{y}) \in X$. Moreover, using (5), we deduce
$\|w(y) - 2\psi(y)\| = \|\varphi(y) + \eta(y) - 2\psi(y)\|$
$\leq \|\varphi(y) - \psi(y)\| + \|\eta(y) - \psi(y)\|$
$\leq 2\max\{L_\varphi, L_\eta\}\|\psi(\bar{y}) - \psi(y)\|^\alpha$
$+ 2\|\psi(\bar{y}) - \psi(y)\|$
$= 2^{1-\alpha}\max\{L_\varphi, L_\eta\}\|2\psi(\bar{y}) - 2\psi(y)\|^\alpha$
$+ \|2\psi(\bar{y}) - 2\psi(y)\|$, for any $y \in Y$, as desired.

(2). Let $\beta \in \mathbb{R} - \{0\}$ and $\varphi \in H_{\psi,\hat{x},\alpha}$. By linear property of $\pi, \beta\varphi$ is a section of $1/\beta\pi$. On the other hand, $\beta\varphi(\bar{y}) = \beta\psi(\bar{y}) = \beta\hat{x} \in X$. Moreover, using (5), we deduce $\|\beta\varphi(y) - \beta\psi(y)\| = |\beta|\|\varphi(y) - \psi(y)\|$
$\leq |\beta| L_\varphi \|\psi(\bar{y}) - \psi(y)\|^\alpha$
$+ |\beta|\|\psi(\bar{y}) - \psi(y)\|$
$= L_\varphi \|\beta\psi(\bar{y}) - \beta\psi(y)\|^\alpha$
$+ \|\beta\psi(\bar{y}) - \beta\psi(y)\|$,
for any $y \in Y$. Hence $\beta\varphi \in H_{\beta\psi,\beta\hat{x},\alpha}$ and the proof is complete.

Remark 2 *Theorem 3.1 holds also if we consider $\lambda \in \mathbb{R}^-$ instead of $\mathbb{R}^+$.*

3.4 Examples

In this section, π is a linear map. Here, we present some examples of linear sections and intrinsically Lipschitz sections.

1. Let the general linear group $X = GL(n,\mathbb{R})$ or $X = GL(n,\mathbb{C})$ of degree n which is the set of $n \times n$ invertible matrices, together with the operation of ordinary matrix multiplication. We consider $Y = \mathbb{R}^* = GL(n,\mathbb{R})/SL(n,\mathbb{R})$ or $Y = \mathbb{C}^* = GL(n,\mathbb{C})/SL(n,\mathbb{C})$ where the special linear group $SL(n,\mathbb{R})$ (or $SL(n,\mathbb{C})$) is the subgroup of $GL(n,\mathbb{R})$ (or $GL(n,\mathbb{C})$) consisting of matrices with determinant of 1. Here the linear map $\pi = det : GL(n,\mathbb{R}) \to \mathbb{R}^*$ is a surjective homomorphism where $Ker(\pi) = SL(n,\mathbb{R})$.

2. Let $X = GL(n,\mathbb{R})$ as above and $Y = GL(n,\mathbb{R})/O(n,\mathbb{R})$ where $O(n,\mathbb{R})$ is the orthogonal group in dimension n. Recall that Y is diffeomorphic to the space of upper-triangular matrices with positive entries on the diagonal, the natural map $\pi : X \to Y$ is linear.

3. Let $X = \mathbb{R}^2, Y = \mathbb{R}$ and $\pi : \mathbb{R}^2 \to \mathbb{R}$ defined as $\pi((x_1,x_2)) := x_1 + x_2$ for any $(x_1,x_2) \in \mathbb{R}^2$. An easy example of sections of π is the following one: let $\varphi : \mathbb{R} \to \mathbb{R}^2$ given by $\varphi(y) = (by + af(y), (1-b)y - af(y))$, $\forall y \in \mathbb{R}$, where $a, b \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous map.

4. Let $X = \mathbb{R}^{2\kappa}, Y = \mathbb{R}$ and $\pi : \mathbb{R}^{2\kappa} \to \mathbb{R}$ defined as $\pi((x_1,\dots,x_{2\kappa})) := x_1 + \dots + x_{2\kappa}$ for any $(x_1,\dots,x_{2\kappa}) \in \mathbb{R}^{2\kappa}$. An easy example of sections of π is the following one: let $\varphi : \mathbb{R} \to \mathbb{R}^{2\kappa}$ given by $\varphi(y) =$
$(y + a_1 f_1(y), -a_1 f_1(y), a_2 f_2(y), -a_2 f_2(y),$
$\dots, a_\kappa f_\kappa, -a_\kappa f_\kappa)$, for all $y \in \mathbb{R}$, where $a_i \in \mathbb{R}$ and $f_i : \mathbb{R} \to \mathbb{R}$ are continuous maps for any $i = 1, \dots, \kappa$.

5. Regarding examples of intrinsically Lipschitz sections the reader can see [4, Example 4.58].

4 An equivalence relation

In this section X is a metric space, Y a topological space and $\pi : X \to Y$ a quotient map (we do *not* ask that π is a linear map). We stress that Definition 2.2 does not induce an equivalence relation, because of lack of symmetry in the right-hand side of (5). As a consequence we must ask a stronger condition in order to obtain an equivalence relation.

Definition 4.1 *[Intrinsic Hölder with respect to a section in strong sense] Given sections $\varphi, \psi : Y \to X$ of π. We say that φ is* intrinsically (L,α)-Hölder with respect to ψ at point $\hat{x}$ in strong sense*, with $L > 0, \alpha \in (0,1]$ and $\hat{x} \in X$, if*

1. *$\hat{x} \in \psi(Y) \cap \varphi(Y)$;*
2. *it holds $d(\varphi(y), \psi(y)) \leq \min\{Ld(\psi(\hat{y}), \psi(y))^\alpha + d(\psi(\hat{y}), \psi(y)), Ld(\psi(\hat{y}), \varphi(y))^\alpha + d(\psi(\hat{y}), \varphi(y))\}$, for every $y \in Y$.*

Now we are able to give the main theorem.

Theorem 4.1 *Let $\alpha \in (0,1]$ and $\pi : X \to Y$ be a quotient map from a metric space X to a topological space Y. Assume also that $\psi : Y \to X$ is a section of π and $\hat{x} \in X$. Then, being intrinsically Hölder with respect to ψ at point $\hat{x}$ in strong sense induces an equivalence relation. We will write the class of equivalence of ψ at point $\hat{x}$ as $[H_{\psi,\hat{x},\alpha}] := \{\varphi : Y \to X$ a section of π : φ is intrinsically $(\tilde{L},\alpha)$-Hölder with respect to ψ at point $\hat{x}$ in strong sense, for some $\tilde{L} > 0\}$.*

An interesting observation is that, considering $H_{\psi,\hat{x},\alpha}$, the intrinsic constants L can be change but it is fundamental that the point $\hat{x}$ is a common one for the every section.

We need to show:

1. reflexive property;
2. symmetric property;
3. transitive property.

(1). It is trivial that $\varphi \backsim \varphi$.

(2). If $\varphi \backsim \psi$, then $\psi \backsim \varphi$. This follows from Definition 4.1.

(3). We know that $\varphi \backsim \psi$ and $\psi \backsim \eta$. Hence, $\hat{x} = \varphi(\hat{y}) = \psi(\hat{y}) = \eta(\hat{y})$. Moreover, by Definition 4.1, it holds $d(\varphi(y), \psi(y)) \leq \min\{L_1 d(\psi(\hat{y}), \psi(y))^\alpha + d(\psi(\hat{y}), \psi(y)), L_1 d(\psi(\hat{y}), \varphi(y))^\alpha + d(\psi(\hat{y}), \varphi(y))\}$,
$d(\psi(y), \eta(y)) \leq \min\{L_2 d(\eta(\hat{y}), \eta(y))^\alpha + d(\eta(\hat{y}), \eta(y)), L_2 d(\eta(\hat{y}), \psi(y))^\alpha + d(\eta(\hat{y}), \psi(y))\}$,
for any $y \in Y$ and, consequently, if $\tilde{L} = 2\max\{L_1, L_2\}$, then $d(\varphi(y), \psi(y)) \leq d(\varphi(y), \psi(y)) + d(\psi(y), \eta(y))$
$\leq \min\{\tilde{L}d(\eta(\hat{y}), \eta(y))^\alpha + d(\eta(\hat{y}), \eta(y)), \tilde{L}d(\psi(\hat{y}), \varphi(y))^\alpha + d(\psi(\hat{y}), \varphi(y))\}$,
$= \min\{\tilde{L}d(\eta(\hat{y}), \eta(y))^\alpha + d(\eta(\hat{y}), \eta(y)), \tilde{L}d(\eta(\hat{y}), \varphi(y))^\alpha + d(\eta(\hat{y}), \varphi(y))\}$,
forany y$\in Y$. This means that $\varphi \backsim \eta$, as desired.

5 Level sets and extensions

A crucial property of Hölder sections is that under suitable assumptions they can be extended. This property is much studied in the context of metric spaces if we consider the Hölder maps; the reader can see [29, 30, 31] and their references. We need to mention several earlier partial results on extensions of Lipschitz graphs in the context of Carnot groups, as for example in [32, 33], [34, Proposition 4.8], [35, Theorem 1.5]), [36, Proposition 3.4], [5, Theorem 4.1].

Our proof follows using the link between Hölder sections and level sets of suitable maps. This idea is widespread in the context of subRiemannian Carnot groups (see, for instance, [37, 38, 39, 35]). In next result, we say that a map f on X is *L-biLipschitz on fibers* (of π) if on each fiber of π it restricts to an L-biLipschitz map.

Theorem 5.1 (Extensions as level sets) *Let $\pi : X \to Y$ be a quotient map between a metric space X and a topological space Y.*
(5.1.i) *If Z is a metric space, $z_0 \in Z$ and $f : X \to Z$ is (λ,β)-Hölder and λ-biLipschitz on fibers, with $\lambda > 0$ and $\beta \in (0,1)$, then there exists an intrinsically (λ^2,β)-Hölder section $\varphi : Y \to X$ of π such that $\varphi(Y) = f^{-1}(z_0)$.*
(5.1.ii) *Vice versa, assume that X is geodesic and that there exist $k \geq 1, \alpha \in (0,1)$, $\rho : X \times X \to \mathbb{R}$ k-biLipschitz equivalent to the distance of X, and $\tau : X \to \mathbb{R}$ is (k,α)-Hölder and k-biLipschitz on fibers such that*

1. *for all $\tau_0 \in \mathbb{R}$ the set $\tau^{-1}(\tau_0)$ is an intrinsically (k,α)-Hölder graph of a section $\varphi_{\tau_0} : Y \to X$;*
2. *for all $x_0 \in \tau^{-1}(\tau_0)$ the map $X \to \mathbb{R}, x \mapsto \delta_{\tau_0}(x) := \rho(x_0, \varphi_{\tau_0}(\pi(x)))$ is k-Lipschitz on the set $\{|\tau| \leq \delta_{\tau_0}\}$.*

Let $Y' \subset Y$ a set and $L \geq 1$. Then for every intrinsically (L,α)-Hölder section $\varphi : Y' \to \pi^{-1}(Y')$ of $\pi|_{\pi^{-1}(Y')} : \pi^{-1}(Y') \to Y'$, there exists a map $f : X \to \mathbb{R}$ that is (K,α)-Hölder and K-biLipschitz on fibers, with $K \geq 1$, such that $\varphi(Y') \subseteq f^{-1}(0)$. In particular, each 'partially defined' intrinsically Hölder graph $\varphi(Y')$ is a subset of a 'globally defined' intrinsically Hölder graph $f^{-1}(0)$.

We underline that an important point is that the constant β in (5.1.i) does not change.

[Proof of Theorem 5.1] The proof is the same to [6, Theorem 1.4]. The only difference that we want to notice is the "good" map in the second point is defined as follows: Fix $x_0 \in \tau^{-1}(\tau_0)$. We consider the map $f_{x_0} : X \to \mathbb{R}$ defined as

$$\begin{cases} 2(\Gamma - \gamma(\delta_{\tau_0}(x)^\alpha + \delta_{\tau_0}(x)) & \text{if } |\Gamma| \leq 2\gamma[\delta_{\tau_0}(x)^\alpha + \delta_{\tau_0}(x)] \\ \Gamma & \text{if } \Gamma > 2\gamma[\delta_{\tau_0}(x)^\alpha + \delta_{\tau_0}(x)] \\ 3\Gamma & \text{if } \Gamma < -2\gamma[\delta_{\tau_0}(x)^\alpha + \delta_{\tau_0}(x)] \end{cases}$$

where $\Gamma := \tau(x) - \tau(x_0)$ and $\gamma := 2kL + 1$.

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