On the Integral Representations of the k-Fibonacci and k-Lucas Numbers

WEERAYUTH NILSRAKOO[1], ACHARIYA NILSRAKOO[2]
[1] Department of Mathematics, Faculty of Science,
Ubon Ratchathani University, Ubon Ratchathani, 34190
THAILAND
[2]Department of Mathematics, Faculty of Science,
Ubon Ratchathani Rajabhat University, Ubon Ratchathani, 34000
THAILAND

Abstract: In this paper, we propose the integral representations of the k-Fibonacci and k-Lucas numbers. We use the Binet's formulas to establish some identities and use simple integral calculus to prove them. Our results are also deduced with the Fibonacci, Lucas, Pell, and Pell-Lucas numbers.

Key-Words: k-Fibonacci number, k-Lucas number, Fibonacci number, Lucas number, integral representation

Received: May 14, 2024. Revised: September 28, 2024. Accepted: October 21, 2024. Published: November 28, 2024.

1 Introduction

Sequences of special numbers have been studied over several years, with the greatest numbers on studies of well-known Fibonacci and Lucas sequences that are related to the golden ratio; for instance, see, [1], [2], [3], [4], [5], [6], [7], [8], [9], [10].

Recall that the Fibonacci numbers F_n are defined via the recurrence relation

$$F_n = F_{n-1} + F_{n-2}$$

for $n \geq 2$ with $F_0 = 0$ and $F_1 = 1$. The Lucas numbers L_n are defined via the recurrence relation

$$L_n = L_{n-1} + L_{n-2}$$

for $n \geq 2$ with $L_0 = 2$ and $L_1 = 1$.

Like Fibonacci and Lucas numbers, the Pell family is widely used. Recall that Pell number P_n is defined by the recurrence relation

$$P_n = 2P_{n-1} + P_{n-2}$$

for $n \geq 2$ with $P_0 = 0$ and $P_1 = 1$. The Pell-Lucas number Q_n is defined by the recurrence relation

$$Q_n = 2Q_{n-1} + Q_{n-2}$$

for $n \geq 2$ with $Q_0 = 2$ and $Q_1 = 2$. The Binet's formulas for the Pell and Pell-Lucas numbers are related to the silver ratio $\varphi = 1 + \sqrt{2}$.

The generalization of Fibonacci and Pell numbers were introduced by [11], in 2007 as follows: the k-Fibonacci numbers $F_{k,n}$ is defined by the recurrence relation

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2} \tag{1}$$

for $n \geq 2$ with $F_{k,0} = 0$ and $F_{k,1} = 1$, where k and n are non-negative integers with $k \neq 0$. In 2011, [12], introduced and studied a generalization of Lucas and Pell-Lucas numbers as follows: the k-Lucas numbers $L_{k,n}$ is defined by the recurrence relation

$$L_{k,n} = kL_{k,n-1} + L_{k,n-2}, \tag{2}$$

for $n \geq 2$ with $L_{k,0} = 2$ and $L_{k,1} = k$. The initial terms of the k-Fibonacci numbers $F_{k,n}$ and k-Lucas numbers $L_{k,n}$ for selected values of k presented as in Table 1 and Table 2.

Table 1. The initial terms of the k -Fibonacci numbers

n	0	1	2	3	4	5	6	7
$F_{1,n}$	0	1	1	2	3	5	8	13
$F_{2,n}$	0	1	2	5	12	29	70	169
$F_{3,n}$	0	1	3	10	33	109	360	1189
$F_{4,n}$	0	1	4	17	72	305	1292	5473
$F_{5,n}$	0	1	5	26	135	701	3640	18901
$F_{6,n}$	0	1	6	37	228	1405	8658	53353
$F_{7,n}$	0	1	7	50	357	2549	18200	129949

Table 2. The initial terms of the k -Lucas numbers

n	0	1	2	3	4	5	6	7
$L_{1,n}$	2	1	3	4	7	11	18	29
$L_{2,n}$	2	2	6	14	34	82	198	478
$L_{3,n}$	2	3	11	36	119	393	1298	4287
$L_{4,n}$	2	4	18	76	322	1364	5778	24476
$L_{5,n}$	2	5	27	140	727	3775	19602	101785
$L_{6,n}$	2	6	38	234	1442	8886	54758	337434
$L_{7,n}$	2	7	51	364	2599	18557	132498	946043

We can see that the classical Fibonacci and classical Lucas numbers are obtained for $k = 1$.

And then the classical Pell and classical Pell–Lucas numbers are appeared if $k = 2$. Moreover, sequences $\{F_{3,n}\}$, $\{F_{4,n}\}$ and $\{F_{6,n}\}$ are listed in The Online Encyclopaedia of Integer Sequences, [13], under the symbols A006190, A001076 and A005668, respectively, while sequences $\{L_{3,n}\}$, $\{L_{4,n}\}$, $\{L_{5,n}\}$, $\{L_{6,n}\}$ and $\{L_{7,n}\}$ under the symbols A006497, A014448, A087130, A085447 and A086902, respectively.

The recurrence relations (1) and (2) generate characteristic equation of the form

$$r^2 - kr - 1 = 0$$

Since $k \geq 1$, this equation has two roots $r_1 = \frac{1}{2}\left(k + \sqrt{k^2+4}\right)$ and $r_2 = \frac{1}{2}\left(k - \sqrt{k^2+4}\right)$. Therefore, the Binet's formulas for the k-Fibonacci numbers $\{F_{k,n}\}$ and the k-Lucas numbers $\{L_{k,n}\}$ are

$$F_{k,n} = \frac{1}{\Delta_k}\left(\varphi_k^n - \frac{(-1)^n}{\varphi_k^n}\right) \tag{3}$$

and

$$L_{k,n} = \varphi_k^n + \frac{(-1)^n}{\varphi_k^n} \tag{4}$$

where $\Delta_k = \sqrt{k^2+4}$ and $\varphi_k = \frac{1}{2}(k + \Delta_k)$, see also, [12, Theorem 2.2], [14, Proposition 2].

Some identities have been proposed to represent and extend of spacial numbers in recent years, [15], [16], [17], [18], [19], [20], [21], [22], [23], [24]. Integral representations are important tools available in their analysis (see, for example, [1], [3], [4], [8], [9], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34]).

The first example for the integral representations of the Fibonacci numbers upon even orders by using the hypergeometric function showed in 2000 by [3], as follows:

$$F_{2n} = \frac{n}{2}\left(\frac{3}{2}\right)^{n-1}\int_0^\pi \left(1 + \frac{\sqrt{5}}{3}\cos x\right)^{n-1}\sin x dx.$$

In 2015, [4], worked out an explicit integral representation for F_n involving trigonometric functions. Indeed, the main result in their paper is the representation of the form

$$F_n = \frac{\alpha^n}{\sqrt{5}} - \frac{2}{\pi}\int_0^\infty \left(\frac{\sin(x/2)}{x}\right) \times \left(\frac{\cos(2nx) - 2\sin(nx)\sin x}{5\sin^2 x + \cos^2 x}\right) dx$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Another representation is given by [1].

In a recent year, [8], derived some appealing integral representations for Fibonacci numbers F_n and Lucas numbers L_n. For instance, he proved the representations

$$F_{\ell n} = \frac{nF_\ell}{2^n}\int_{-1}^1 (L_\ell + \sqrt{5}F_\ell x)^{n-1} dx$$

and

$$L_{\ell n} = \frac{1}{2^n}\int_{-1}^1 (L_\ell + \sqrt{5}F_\ell x)^{n-1} \times (L_\ell + \sqrt{5}(n+1)F_\ell x) dx,$$

where ℓ and n are non-negative integers. The special case of this identity for $\ell = 1$ is also discussed in [9], from 2023.

In this paper, we give new integral representations of the k-Fibonacci and the k-Lucas numbers. To prove them, we propose some identities relied on the Binet's formulas and simple integral calculus.

2 Preliminaries

We employ the technique of [8], to obtain new integral representations for the k-Fibonacci numbers and the k-Lucas numbers. We start with the following some identities relied on the Binet's formulas that we will require.

Lemma 1. *Let k and n be non-negative integers with $k \neq 0$, $\Delta_k = \sqrt{k^2+4}$ and $\varphi_k = \frac{1}{2}(k + \Delta_k)$. Then the following hold:*

(i) $L_{k,n} + \Delta_k F_{k,n} = 2\varphi_k^n$;

(ii) $L_{k,n} - \Delta_k F_{k,n} = 2\frac{(-1)^n}{\varphi_k^n}$;

(iii) $L_{k,n}^2 - \Delta_k^2 F_{k,n}^2 = 4(-1)^n$.

Proof. (i) Combining Binet's formulas (3) and (4) gives

$$\begin{aligned} &L_{k,n} + \Delta_k F_{k,n} \\ &= \left(\varphi_k^n + \frac{(-1)^n}{\varphi_k^n}\right) + \left(\varphi_k^n - \frac{(-1)^n}{\varphi_k^n}\right) \\ &= 2\varphi_k^n. \end{aligned}$$

(ii) Subtracting Binet's formulas (3) and (4) gives

$$\begin{aligned} &L_{k,n} - \Delta_k F_{k,n} \\ &= \left(\varphi_k^n + \frac{(-1)^n}{\varphi_k^n}\right) - \left(\varphi_k^n - \frac{(-1)^n}{\varphi_k^n}\right) \\ &= 2\frac{(-1)^n}{\varphi_k^n}. \end{aligned}$$

(iii) From (i) and (ii), we have

$$
\begin{aligned}
&L_{k,n}^2 - \Delta_k^2 F_{k,n}^2 \\
&= L_{k,n}^2 - (\Delta_k F_{k,n})^2 \\
&= (L_{k,n} + \Delta_k F_{k,n})(L_{k,n} - \Delta_k F_{k,n}) \\
&= (2\varphi_k^n)\left(2\frac{(-1)^n}{\varphi_k^n}\right) \\
&= 4(-1)^n.
\end{aligned}
$$

This completes the proof. □

Remark 2. Lemma 1 (iii) is presented in [12, Theorem 2.3].

Lemma 3. *Let k, m and n be non-negative integers $k \neq 0$ and $\Delta_k = \sqrt{k^2+4}$. Then the following hold:*

(i) $2F_{k,m+n} = F_{k,m}L_{k,n} + F_{k,n}L_{k,m}$;

(ii) $2L_{k,m+n} = L_{k,m}L_{k,n} + \Delta_k^2 F_{k,m}F_{k,n}$.

Proof. (i) Using Binet's formulas (3) and (4), we obtain

$$
\begin{aligned}
&F_{k,m}L_{k,n} \\
&= \left(\frac{1}{\Delta_k}\left(\varphi_k^m - \frac{(-1)^m}{\varphi_k^m}\right)\right)\left(\varphi_k^n + \frac{(-1)^n}{\varphi_k^n}\right) \\
&= \frac{1}{\Delta_k}\left(\varphi_k^{m+n} - \frac{(-1)^m\varphi_k^n}{\varphi_k^m}\right) \\
&\quad + \frac{1}{\Delta_k}\left(\frac{(-1)^n\varphi_k^m}{\varphi_k^n} - \frac{(-1)^{m+n}}{\varphi_k^{m+n}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&F_{k,n}L_{k,m} \\
&= \left(\frac{1}{\Delta_k}\left(\varphi_k^n - \frac{(-1)^n}{\varphi_k^n}\right)\right)\left(\varphi_k^m + \frac{(-1)^m}{\varphi_k^m}\right) \\
&= \frac{1}{\Delta_k}\left(\varphi_k^{m+n} - \frac{(-1)^n\varphi_k^m}{\varphi_k^n}\right) \\
&\quad + \frac{1}{\Delta_k}\left(\frac{(-1)^m\varphi_k^n}{\varphi_k^m} - \frac{(-1)^{m+n}}{\varphi_k^{m+n}}\right).
\end{aligned}
$$

So, we get

$$
\begin{aligned}
&F_{k,m}L_{k,n} + F_{k,n}L_{k,m} \\
&\qquad = \frac{2}{\Delta_k}\left(\varphi_k^{m+n} - \frac{(-1)^{m+n}}{\varphi_k^{m+n}}\right) \\
&\qquad = 2F_{k,m+n}.
\end{aligned}
$$

(ii) Using Binet's formulas (4), we obtain

$$
\begin{aligned}
&L_{k,m}L_{k,n} \\
&= \left(\varphi_k^m + \frac{(-1)^m}{\varphi_k^m}\right)\left(\varphi_k^n + \frac{(-1)^n}{\varphi_k^n}\right) \\
&= \varphi_k^{m+n} + \frac{(-1)^m\varphi_k^n}{\varphi_k^m} + \frac{(-1)^n\varphi_k^m}{\varphi_k^n} + \frac{(-1)^{m+n}}{\varphi_k^{m+n}}.
\end{aligned}
$$

Using Binet's formulas (3), we obtain

$$
\begin{aligned}
&\Delta_k^2 F_{k,m}F_{k,n} \\
&= \Delta_k^2\left(\frac{1}{\Delta_k}\left(\varphi_k^m - \frac{(-1)^m}{\varphi_k^m}\right)\right) \\
&\qquad \times \left(\frac{1}{\Delta_k}\left(\varphi_k^n - \frac{(-1)^n}{\varphi_k^n}\right)\right) \\
&= \left(\varphi_k^m - \frac{(-1)^m}{\varphi_k^m}\right)\left(\varphi_k^n - \frac{(-1)^n}{\varphi_k^n}\right) \\
&= \varphi_k^{m+n} - \frac{(-1)^m\varphi_k^n}{\varphi_k^m} - \frac{(-1)^n\varphi_k^m}{\varphi_k^n} + \frac{(-1)^{m+n}}{\varphi_k^{m+n}}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&L_{k,m}L_{k,n} + \Delta_k^2 F_{k,m}F_{k,n} \\
&\quad = 2\left(\varphi_k^{m+n} + \frac{(-1)^{m+n}}{\varphi_k^{m+n}}\right) \\
&\quad = 2L_{k,m+n}.
\end{aligned}
$$

Hence, (i) and (ii) complete the proof. □

Setting $k = 1$ and $k = 2$ in Lemma 3, we have the following.

Remark 4. Let m and n be non-negative integers. Then the following hold:

(i) $2F_{m+n} = F_mL_n + F_nL_m$;

(ii) $2L_{m+n} = L_mL_n + 5F_mF_n$;

(iii) $2P_{m+n} = P_mQ_n + P_nQ_m$;

(iv) $2Q_{m+n} = Q_mQ_n + 8P_mP_n$.

3 Main Results

In this section, we now present that the integral representation for the k-Fibonacci numbers $F_{k,\ell n}$ can be found by employing other known relations between the two numbers $F_{k,\ell}$ and $L_{k,\ell}$.

Theorem 5. *For k, ℓ and n are non-negative integers with $k \neq 0$, the k-Fibonacci numbers $F_{k,\ell n}$ can be represented by the integral*

$$
F_{k,\ell n} = \frac{nF_{k,\ell}}{2^n}\int_{-1}^{1}(L_{k,\ell} + \Delta_k F_{k,\ell}x)^{n-1}dx, \quad (5)
$$

where $\Delta_k = \sqrt{k^2+4}$.

Proof. For $n = 0$ or $\ell = 0$, it is obvious. Let us assume that $\ell, n > 0$. Let $u(x) = L_{k,\ell} + \Delta_k F_{k,\ell}x$. Then $du = \Delta_k F_{k,\ell}dx$. Using integration by substitution leads to

$$
\begin{aligned}
&\int_{-1}^{1}(L_{k,\ell} + \Delta_k F_{k,\ell}x)^{n-1}dx \\
&\qquad = \frac{1}{\Delta_k F_{k,\ell}}\int_{u(-1)}^{u(1)} u^{n-1}du
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{n\Delta_k F_{k,\ell}} (u^n)\Big|_{u(-1)}^{u(1)} \\
&= \frac{1}{n\Delta_k F_{k,\ell}} \Big[(L_{k,\ell} + \Delta_k F_{k,\ell} x)^n\Big]_{-1}^{1} \\
&= \frac{1}{n\Delta_k F_{k,\ell}} (L_{k,\ell} + \Delta_k F_{k,\ell})^n \\
&\quad - \frac{1}{n\Delta_k F_{k,\ell}} (L_{k,\ell} - \Delta_k F_{k,\ell})^n. \qquad (6)
\end{aligned}
$$

Applying (i) and (ii) of Lemma 1 in (6) with n replaced with ℓ, we get

$$
\begin{aligned}
&\int_{-1}^{1} (L_{k,\ell} + \Delta_k F_{k,\ell} x)^{n-1} dx \\
&= \frac{1}{n\Delta_k F_{k,\ell}} \left[\left(2\varphi_k^{\ell}\right)^n - \left(2\frac{(-1)^{\ell}}{\varphi_k^{\ell}}\right)^n\right] \\
&= \frac{2^n}{nF_{k,\ell}} \left[\frac{1}{\Delta_k}\left(\varphi_k^{\ell n} - \frac{(-1)^{\ell n}}{\varphi_k^{\ell n}}\right)\right].
\end{aligned}
$$

It follows from (3) with replace n by ℓn that

$$
\int_{-1}^{1} (L_{k,\ell} + \Delta_k F_{k,\ell} x)^{n-1} dx = \frac{2^n}{nF_{k,\ell}} F_{k,\ell n}.
$$

Then (5) which completes the proof. □

The integral representations of the k-Fibonacci numbers for even and odd orders are shown as follows:

Theorem 6. *Let k and n be non-negative integers with $k \neq 0$ and $\Delta_k = \sqrt{k^2+4}$.*

(i) The k-Fibonacci numbers $F_{k,2n}$ can be represented by the integral

$$
F_{k,2n} = \frac{kn}{2^n} \int_{-1}^{1} (k^2 + 2 + k\Delta_k x)^{n-1} dx. \quad (7)
$$

(ii) The k-Fibonacci numbers $F_{k,2n+1}$ can be represented by the integral

$$
\begin{aligned}
F_{k,2n+1} &= \frac{1}{2^{n+1}} \int_{-1}^{1} \left(k^2 + 2 + k\Delta_k x\right)^{n-1} \\
&\times \left(k^2 n + k^2 + 2 + k(n+1)\Delta_k x\right) dx.
\end{aligned}
$$

Proof. (i) Notice that $F_{k,2} = k$ and $L_{k,2} = k^2 + 2$. Setting $\ell = 2$ in (5), we have

$$
F_{k,2n} = \frac{kn}{2^n} \int_{-1}^{1} (k^2 + 2 + k\Delta_k\, x)^{n-1} dx.
$$

(ii) Re-indexing of n by $n+1$ in (7), we get

$$
F_{k,2n+2} = \frac{k(n+1)}{2^{n+1}} \int_{-1}^{1} (k^2 + 2 + k\Delta_k x)^n dx.
$$

This together with (7) and

$$
F_{k,2n+2} = kF_{k,2n+1} + F_{k,2n}
$$

gives

$$
\begin{aligned}
F_{k,2n+1} &= \frac{F_{k,2n+2}}{k} - \frac{F_{k,2n}}{k} \\
&= \frac{(n+1)}{2^{n+1}} \int_{-1}^{1} (k^2 + 2 + k\Delta_k x)^n dx \\
&\quad - \frac{n}{2^n} \int_{-1}^{1} (k^2 + 2 + k\Delta_k\, x)^{n-1} dx \\
&= \frac{1}{2^{n+1}} \int_{-1}^{1} \left(k^2 + 2 + k\Delta_k x\right)^{n-1} \times \\
&\qquad \left(k^2 n + k^2 + 2 + k(n+1)\Delta_k x\right) dx.
\end{aligned}
$$

This completes the proof. □

Setting $k = 1$ in Theorems 5 and 6, we have the following corollaries.

Corollary 7 ([8], Theorem 2.1)**.** *For ℓ and n are non-negative integers, the Fibonacci numbers $F_{\ell n}$ can be represented by the integral*

$$
F_{\ell n} = \frac{nF_\ell}{2^n} \int_{-1}^{1} (L_\ell + \sqrt{5} F_\ell x)^{n-1} dx.
$$

Proof. Notice that $F_{1,\ell n} = F_{\ell n}$, $F_{1,\ell} = F_\ell$, $L_{1,\ell} = L_\ell$ and $\Delta_1 = \sqrt{5}$. Then, by Theorem 5, the conclusion follows. □

Corollary 8 ([8], Remark 2.1)**.** *Let n be a non-negative integer.*

(i) The Fibonacci numbers F_{2n} can be represented by the integral

$$
F_{2n} = \frac{n}{2^n} \int_{-1}^{1} (3 + \sqrt{5}\, x)^{n-1} dx.
$$

(ii) The Fibonacci numbers F_{2n+1} can be represented by the integral

$$
\begin{aligned}
F_{2n+1} &= \frac{1}{2^{n+1}} \int_{-1}^{1} \left(3 + \sqrt{5} x\right)^{n-1} \\
&\quad \times \left(n + 3 + \sqrt{5}(n+1)x\right) dx.
\end{aligned}
$$

Setting $k = 2$ in Theorems 5 and 6, we have the following corollaries.

Corollary 9 ([30], Theorem 3.1)**.** *For ℓ and n are non-negative integers with $k \neq 0$, the Pell numbers $P_{\ell n}$ can be represented by the integral*

$$P_{\ell n} = \frac{nP_\ell}{2^n} \int_{-1}^{1} (Q_\ell + \sqrt{8}P_\ell x)^{n-1} dx.$$

Proof. Notice that $F_{2,\ell n} = P_{\ell n}$, $F_{2,\ell} = P_\ell$, $L_{2,\ell} = Q_\ell$ and $\Delta_2 = \sqrt{8}$. Then, by Theorem 5, the conclusion follows. □

Corollary 10 ([30], Corollary 3.2)**.** *Let n be a non-negative integer.*

(i) The Pell numbers P_{2n} can be represented by the integral

$$P_{2n} = n \int_{-1}^{1} (3 + \sqrt{8}\, x)^{n-1} dx.$$

(ii) The Pell numbers P_{2n+1} can be represented by the integral

$$P_{2n+1} = \frac{1}{2} \int_{-1}^{1} \left(3 + \sqrt{8}x\right)^{n-1} \times \left(2n + 3 + \sqrt{8}(n+1)x)\right) dx.$$

Proof. By Theorem 6, we get

$$\begin{aligned} P_{2n} = F_{2,2n} &= \frac{2n}{2^n} \int_{-1}^{1} (6 + 2\sqrt{8}\, x)^{n-1} dx \\ &= n \int_{-1}^{1} (3 + \sqrt{8}\, x)^{n-1} dx \end{aligned}$$

and

$$\begin{aligned} P_{2n+1} &= F_{2,2n+1} \\ &= \frac{1}{2^{n+1}} \int_{-1}^{1} \left(6 + 2\sqrt{8}x\right)^{n-1} \\ &\qquad \times \left(4n + 6 + 2\sqrt{8}(n+1)x)\right) dx \\ &= \frac{1}{2} \int_{-1}^{1} \left(3 + \sqrt{8}x\right)^{n-1} \\ &\qquad \times \left(2n + 3 + \sqrt{8}(n+1)x)\right) dx. \end{aligned}$$

This completes the proof. □

Setting $k = 3$ in Theorem 5, we have the following numerical example.

Example 11. The 3-Fibonacci numbers $F_{3,\ell n}$ can be represented by the integral

$$F_{3,\ell n} = \frac{nF_{3,\ell}}{2^n} \int_{-1}^{1} (L_{3,\ell} + \sqrt{13}F_{3,\ell}x)^{n-1} dx.$$

It is known that $F_{3,2} = 3$ and $L_{3,2} = 11$. Then we can find $F_{3,4}$ and $F_{3,6}$ as follows:

$$\begin{aligned} F_{3,4} &= F_{3,2(2)} \\ &= \frac{2F_{3,2}}{2^2} \int_{-1}^{1} (L_{3,2} + \sqrt{13}F_{3,2}x)^{2-1} dx \\ &= \frac{3}{2} \int_{-1}^{1} (11 + 3\sqrt{13}x) dx \\ &= \frac{3}{2} \left(11x + \frac{3\sqrt{13}x^2}{2}\right) \Big|_{-1}^{1} \\ &= 33 \end{aligned}$$

and

$$\begin{aligned} F_{3,6} &= F_{3,2(3)} \\ &= \frac{3F_{3,2}}{2^3} \int_{-1}^{1} (L_{3,2} + \sqrt{13}F_{3,2}x)^{3-1} dx \\ &= \frac{9}{8} \int_{-1}^{1} (11 + 3\sqrt{13}x)^2 dx \\ &= \frac{9}{8} \int_{-1}^{1} (121 + 66\sqrt{13}x + 117x^2) dx \\ &= \frac{9}{8} \left(121x + 33\sqrt{13}x^2 + \frac{117x^3}{3}\right) \Big|_{-1}^{1} \\ &= 360. \end{aligned}$$

In another way, we can find $F_{3,6}$ when we known that $F_{3,3} = 10$ and $L_{3,3} = 36$ as follows:

$$\begin{aligned} F_{3,6} &= F_{3,3(2)} \\ &= \frac{2F_{3,3}}{2^2} \int_{-1}^{1} (L_{3,3} + \sqrt{13}F_{3,3}x)^{2-1} dx \\ &= 5 \int_{-1}^{1} (36 + 10\sqrt{13}x) dx \\ &= 5 \left(36x + 5\sqrt{13}x^2\right) \Big|_{-1}^{1} \\ &= 360. \end{aligned}$$

Moreover, we obtain

$$\begin{aligned} F_{3,9} &= F_{3,3(3)} \\ &= \frac{3F_{3,3}}{2^3} \int_{-1}^{1} (L_{3,3} + \sqrt{13}F_{3,3}x)^{3-1} dx \\ &= \frac{15}{4} \int_{-1}^{1} (36 + 10\sqrt{13}x)^2 dx \\ &= 15 \int_{-1}^{1} (324 + 45\sqrt{13}x + 325x^2) dx \\ &= 15 \left(324x + \frac{45\sqrt{13}x^2}{2} + \frac{325x^3}{3}\right) \Big|_{-1}^{1} \\ &= 12970. \end{aligned}$$

Next, we provide the integral representations for the k-Lucas numbers $L_{k,\ell n}$ based on the two numbers $F_{k,\ell}$ and $L_{k,\ell}$.

Theorem 12. *For k, ℓ and n are non-negative integers with $k \neq 0$, the k-Lucas numbers $L_{k,\ell n}$ can be represented by the integral*

$$L_{k,\ell n} = \frac{1}{2^n}\int_{-1}^{1}(L_{k,\ell}+\Delta_k F_{k,\ell}x)^{n-1} \times (L_{k,\ell}+(n+1)\Delta_k F_{k,\ell}x)dx, \quad (8)$$

where $\Delta_k = \sqrt{k^2+4}$.

Proof. We will solve this integral (8) using the integration by parts. Let u and v be such that

$$u(x) = L_{k,\ell}+(n+1)\Delta_k F_{k,\ell}x$$

and

$$dv = (L_{k,\ell}+\Delta_k F_{k,\ell}x)^{n-1}dx.$$

Then $du = (n+1)\Delta_k F_{k,\ell}dx$ and from (6) gives

$$\begin{aligned} v &= \int (L_{k,\ell}+\Delta_k F_{k,\ell}x)^{n-1}dx \\ &= \frac{1}{n\Delta_k F_{k,\ell}}(L_{k,\ell}+\Delta_k F_{k,\ell})^n. \end{aligned}$$

It follows that

$$\begin{aligned} I &= \frac{1}{2^n}\int_{-1}^{1}(L_{k,\ell}+\Delta_k F_{k,\ell}x)^{n-1} \\ &\quad \times (L_{k,\ell}+(n+1)\Delta_k F_{k,\ell}x)dx \\ &= \frac{1}{n2^n\Delta_k F_{k,\ell}} \\ &\quad \times \Big[(L_{k,\ell}+\Delta_k F_{k,\ell}x)^n (L_{k,\ell}+(n+1)\Delta_k F_{k,\ell}x) \Big]_{-1}^{1} \\ &\quad - \frac{n+1}{n2^n}\int_{-1}^{1}(L_{k,\ell}+\Delta_k P_{k,\ell}x)^n dx. \end{aligned} \quad (9)$$

Replacing n by $n+1$ in (5) becomes

$$F_{k,\ell n+\ell} = \frac{(n+1)F_{k,\ell}}{2^{n+1}}\int_{-1}^{1}(L_{k,\ell}+\Delta_k\, F_{k,\ell}x)^n dx.$$

and so

$$\frac{2F_{k,\ell n+\ell}}{nF_{k,\ell}} = \frac{(n+1)}{n2^n}\int_{-1}^{1}(L_{k,\ell}+\Delta_k\, F_{k,\ell}x)^n dx.$$

This together with (9) gives

$$\begin{aligned} I &= \frac{1}{n2^n\Delta_k F_{k,\ell}} \\ &\quad \times \Big[(L_{k,\ell}+\Delta_k F_{k,\ell})^n (L_{k,\ell}+(n+1)\Delta_k F_{k,\ell}) \\ &\quad - (L_{k,\ell}-\Delta_k F_{k,\ell})^n (L_{k,\ell}-(n+1)\Delta_k F_{k,\ell}) \Big] \\ &\quad - \frac{2F_{k,\ell n+\ell}}{nF_{k,\ell}}. \end{aligned}$$

Applying (i) and (ii) of Lemma 1 with n replaced by ℓ to the righthand side of the above equation gives

$$\begin{aligned} I &= \frac{1}{n2^n\Delta_k F_{k,\ell}}\Big[2^n\varphi_k^{\ell n}(L_{k,\ell}+(n+1)\Delta_k F_{k,\ell}) \\ &\quad - 2^n\frac{(-1)^{\ell n}}{\varphi_k^{\ell n}}(L_{k,\ell}-(n+1)\Delta_k F_{k,\ell})\Big] \\ &\quad - \frac{2F_{k,\ell n+\ell}}{nF_{k,\ell}} \\ &= \frac{1}{nF_{k,\ell}}\Big[\frac{1}{\Delta_k}\left(\varphi_k^{\ell n}-\frac{(-1)^{\ell n}}{\varphi_k^{\ell n}}\right)L_{k,\ell} \\ &\quad + (n+1)F_{k,\ell}\left(\varphi_k^{\ell n}+\frac{(-1)^{\ell n}}{\varphi_k^{\ell n}}\right)\Big] \\ &\quad - \frac{2F_{k,\ell n+\ell}}{nF_{k,\ell}}. \end{aligned}$$

Applying both Binet's formulas (3) and (4) with n replaced by ℓn and Lemma 3 (i) leads to

$$\begin{aligned} I &= \frac{1}{nF_{k,\ell}}[F_{k,\ell n}L_{k,\ell}+(n+1)F_{k,\ell}L_{k,\ell n}] \\ &\quad - \frac{2F_{k,\ell n+\ell}}{nF_{k,\ell}} \\ &= \frac{1}{nF_{k,\ell}}(F_{k,\ell n}L_{k,\ell}+F_{k,\ell}L_{k,\ell n}) + L_{k,\ell n} \\ &\quad - \frac{2F_{k,\ell n+\ell}}{nF_{k,\ell}} \\ &= L_{k,\ell n}, \end{aligned}$$

which completes the proof. □

Setting $k = 1$ in Theorem 12, we have the following corollary.

Corollary 13 ([8], Theorem 2.2)**.** *For ℓ and n are non-negative integers, the Lucas numbers $L_{\ell n}$ can be represented by the integral*

$$L_{\ell n} = \frac{1}{2^n}\int_{-1}^{1}(L_\ell+\sqrt{5}F_\ell x)^{n-1} \times (L_\ell+\sqrt{5}(n+1)F_\ell x)dx.$$

Setting $k = 2$ in Theorem 12, we have the following corollary.

Corollary 14 ([30], Theorem 3.4)**.** *For ℓ and n are non-negative integers, the Pell-Lucas numbers $Q_{\ell n}$ can be represented by the integral*

$$Q_{\ell n} = \frac{1}{2^n}\int_{-1}^{1}\times(Q_\ell+\sqrt{5}P_\ell x)^{n-1} (Q_\ell+\sqrt{8}(n+1)P_\ell x)dx.$$

Setting $k = 3$ in Theorem 12, we have the following numerical example.

Example 15. The 3-Lucas numbers $L_{3,\ell n}$ can be represented by the integral

$$L_{3,\ell n} = \frac{1}{2^n}\int_{-1}^{1}(L_{3,\ell}+\sqrt{13}F_{3,\ell}x)^{n-1} \times (L_{3,\ell}+\sqrt{13}(n+1)F_{3,\ell}x)dx.$$

Since $F_{3,2} = 3$ and $L_{3,2} = 11$, we can find $L_{3,4}$ and $L_{3,6}$ as follows:

$$\begin{aligned} L_{3,4} &= L_{3,2(2)} \\ &= \frac{1}{2^2}\int_{-1}^{1}(L_{3,2}+\sqrt{13}F_{3,2}x)^{2-1} \\ &\quad \times (L_{3,2}+3\sqrt{13}F_{3,2}x)dx \\ &= \frac{1}{4}\int_{-1}^{1}(11+3\sqrt{13}x)(11+9\sqrt{13}x)dx \\ &= \frac{1}{4}\int_{-1}^{1}(121+132\sqrt{13}x+351x^2)dx \\ &= \frac{1}{4}\left(121x+66\sqrt{13}x^2+117x^3\right)\Big|_{-1}^{1} \\ &= 119 \end{aligned}$$

and

$$\begin{aligned} L_{3,6} &= L_{3,2(3)} \\ &= \frac{1}{2^3}\int_{-1}^{1}(L_{3,2}+\sqrt{13}F_{3,2}x)^{3-1} \\ &\quad \times (L_{3,2}+4\sqrt{13}F_{3,2}x)dx \\ &= \frac{1}{8}\int_{-1}^{1}(11+3\sqrt{13}x)^2(11+12\sqrt{13}x)dx \\ &= \frac{1}{8}\int_{-1}^{1}(1331+2178\sqrt{13}x \\ &\quad +11583x^2+1404\sqrt{13}x^3)dx \\ &= \frac{1}{8}(1331x+1089\sqrt{13}x^2+3861x^3+351\sqrt{13}x^4)\Big|_{-1}^{1} \\ &= 1298. \end{aligned}$$

Moreover, we can find $L_{3,6}$ when we known that $F_{3,3} = 10$ and $L_{3,3} = 36$ as follows:

$$\begin{aligned} L_{3,6} &= F_{3,3(2)} \\ &= \frac{1}{2^2}\int_{-1}^{1}(L_{3,3}+\sqrt{13}F_{3,3}x)^{2-1} \\ &\quad \times (L_{3,3}+3\sqrt{13}F_{3,3}x)dx \\ &= \frac{1}{4}\int_{-1}^{1}(36+10\sqrt{13}x)(36+30\sqrt{13}x)dx \\ &= \int_{-1}^{1}(324+360\sqrt{13}x+975x^2)dx \\ &= \left(324x+180\sqrt{13}x^2+325x^3\right)\Big|_{-1}^{1} \\ &= 1298. \end{aligned}$$

Finally, both $F_{k,\ell n}$ and $L_{k,\ell n}$ are then used to establish the generalized forms of integral representations for the k-Fibonacci numbers $F_{k,\ell n+r}$ and k-Lucas numbers $L_{k,\ell n+r}$ as the following theorems.

Theorem 16. *For k, ℓ, n and r are non-negative integers with $k \neq 0$, the k-Fibonacci number $F_{k,\ell n+r}$ can be represented by the integral*

$$F_{k,\ell n+r} = \frac{1}{2^{n+1}}\int_{-1}^{1}(L_{k,\ell}+\Delta_k F_{k,\ell}x)^{n-1} \times (nF_{k,\ell}L_{k,r}+F_{k,r}L_{k,\ell}+(n+1)\Delta_k F_{k,\ell}F_{k,r}x)dx,$$

where $\Delta_k = \sqrt{k^2+4}$.

Proof. Using Lemma 3 (i) with m and n replaced by ℓn and r respectively, we get

$$F_{k,\ell n+r} = \frac{1}{2}F_{k,\ell n}L_{k,r} + \frac{1}{2}F_{k,r}L_{k,\ell n}.$$

Applying Theorems 5 and 12 leads to

$$\begin{aligned} &F_{k,\ell n+r} \\ &= \frac{1}{2}\left(\frac{nF_{k,\ell}}{2^n}\int_{-1}^{1}(L_{k,\ell}+\Delta_k F_{k,\ell}x)^{n-1}dx\right)L_{k,r} \\ &\quad + \frac{1}{2}F_{k,r}\Big(\frac{1}{2^n}\int_{-1}^{1}(L_{k,\ell}+(n+1)\Delta_k F_{k,\ell}x) \\ &\qquad \times (L_{k,\ell}+\Delta_k F_{k,\ell}x)^{n-1}dx\Big) \\ &= \frac{1}{2^{n+1}}\int_{-1}^{1}(L_{k,\ell}+\Delta_k F_{k,\ell}x)^{n-1} \\ &\quad \times (nF_{k,\ell}L_{k,r}+F_{k,r}L_{k,\ell}+(n+1)\Delta_k F_{k,\ell}F_{k,r}x)dx. \end{aligned}$$

This completes the proof. □

Remark 17. Notice that the results for the integral representations of the even and odd k-Fibonacci numbers given in Theorem 6 are recovered from Theorem 16 on setting $(\ell, r) = (2, 0)$ and $(\ell, r) = (2, 1)$, respectively.

Setting $k = 1$ in Theorem 16, we have the following corollary.

Corollary 18 ([8], Theorem 2.3)**.** *For ℓ, n and r are non-negative integers, the Fibonacci numbers $F_{\ell n+r}$*

can be represented by the integral

$$F_{\ell n+r} = \frac{1}{2^{n+1}} \int_{-1}^{1} (L_\ell + \sqrt{5}\, F_\ell x)^{n-1} \times \left(nF_\ell L_r + F_r L_\ell + \sqrt{5}(n+1)\, F_\ell F_r x\right) dx.$$

Setting $k = 2$ in Theorem 16, we have the following corollary.

Corollary 19 ([30], Theorem 3.5)**.** *For ℓ, n and r are non-negative integers, the Pell numbers $P_{\ell n+r}$ can be represented by the integral*

$$P_{\ell n+r} = \frac{1}{2^{n+1}} \int_{-1}^{1} (Q_\ell + \sqrt{8}\, P_\ell x)^{n-1} \times \left(nP_\ell Q_r + P_r Q_\ell + \sqrt{8}(n+1)\, P_\ell P_r x\right) dx.$$

Theorem 20. *For k, ℓ, n and r are non-negative integers with $k \neq 0$, the k-Lucas numbers $L_{k,\ell n+r}$ can be represented by the integral*

$$L_{k,\ell n+r} = \frac{1}{2^{n+1}} \int_{-1}^{1} (L_{k,\ell} + \Delta_k F_{k,\ell} x)^{n-1} \times \begin{pmatrix} n\Delta_k^2 F_{k,\ell} F_{k,r} + L_{k,\ell} L_{k,r} \\ +(n+1)\Delta_k\, F_{k,\ell} L_{k,r} x \end{pmatrix} dx,$$

where $\Delta_k = \sqrt{k^2+4}$.

Proof. Using Lemma 3 (ii) with m and n replaced by ℓn and r respectively, we get

$$L_{k,\ell n+r} = \frac{1}{2} L_{k,\ell n} L_{k,r} + \frac{\Delta_k^2}{2} F_{k,\ell n} F_{k,r}.$$

This together with Theorems 5 and 12 gives that the proof is finish. □

Using the same idea as in Theorem 6, or setting $(\ell, r) = (2, 0)$ and $(\ell, r) = (2, 1)$ in Theorem 20, we also have the following integral representations of the k-Lucas numbers for even and odd orders.

Theorem 21. *Let k and n be non-negative integers with $k \neq 0$ and $\Delta_k = \sqrt{k^2+4}$.*

(i) The k-Lucas numbers $L_{k,2n}$ can be represented by the integral

$$L_{k,2n} = \frac{1}{2^n} \int_{-1}^{1} (k^2 + 2 + k\Delta_k x)^{n-1} \times (k^2 + 2 + k(n+1)\Delta_k x) dx.$$

(ii) The k-Lucas numbers $L_{k,2n+1}$ can be represented by the integral

$$L_{k,2n+1} = \frac{k}{2^{n+1}} \int_{-1}^{1} \left(k^2 + 2 + k\Delta_k x\right)^{n-1} \times \left(n\Delta_k^2 + k^2 + 2 + k(n+1)\Delta_k x)\right) dx.$$

Setting $k = 1$ in Theorems 20 and 21, we have the following corollaries.

Corollary 22 ([8], Theorem 2.4)**.** *For ℓ, n and r are non-negative integers, the Lucas numbers $L_{\ell n+r}$ can be represented by the integral*

$$L_{\ell n+r} = \frac{1}{2^{n+1}} \int_{-1}^{1} (L_\ell + \sqrt{5}\, F_\ell x)^{n-1} \times \left(5nF_\ell F_r + L_\ell L_r + \sqrt{5}(n+1)\, F_r L_r x\right) dx.$$

Corollary 23 ([8], Remark 2.4)**.** *Let n be a non-negative integer.*

(i) The Lucas numbers L_{2n} can be represented by the integral

$$L_{2n} = \frac{1}{2^n} \int_{-1}^{1} (3+\sqrt{5}\, x)^{n-1} (3+\sqrt{5}(n+1)\, x) dx.$$

(ii) The Lucas numbers L_{2n+1} can be represented by the integral

$$L_{2n+1} = \frac{1}{2^{n+1}} \int_{-1}^{1} (3 + \sqrt{5}x)^{n-1} \times (5n + 3 + \sqrt{5}(n+1)x) dx.$$

Setting $k = 2$ in Theorems 20 and 21, we have the following corollaries.

Corollary 24 ([30], Theorem 3.6)**.** *For ℓ, n and r are non-negative integers, the Pell-Lucas numbers $Q_{\ell n+r}$ can be represented by the integral*

$$Q_{\ell n+r} = \frac{1}{2^{n+1}} \int_{-1}^{1} (Q_\ell + \sqrt{8}\, P_\ell x)^{n-1} \times \left(8nP_\ell P_r + Q_\ell Q_r + \sqrt{8}(n+1) P_\ell Q_r x\right) dx.$$

Corollary 25. *Let n be a non-negative integer.*

(i) The Pell-Lucas numbers Q_{2n} can be represented by the integral

$$Q_{2n} = \int_{-1}^{1} (3 + \sqrt{8}x)^{n-1} (3 + \sqrt{8}(n+1)x) dx.$$

(ii) The Pell-Lucas numbers Q_{2n+1} can be represented by the integral

$$Q_{2n+1} = \int_{-1}^{1} (3 + \sqrt{8}x)^{n-1} \times (4n + 3 + \sqrt{8}(n+1)x) dx.$$

Setting $k = 3$ and $r = 1$ in Theorems 16 and 20, we have the following numerical example.

Example 26. The 3-Fibonacci numbers $F_{3,\ell n+1}$ and 3-Lucas numbers $L_{3,\ell n+1}$ can be represented by the integral

$$F_{3,\ell n+1} = \frac{1}{2^{n+1}}\int_{-1}^{1}(L_{3,\ell} + \sqrt{13}F_{3,\ell}x)^{n-1} \times (3nF_{3,\ell} + L_{3,\ell} + \sqrt{13}(n+1)F_{3,\ell}x)dx$$

and

$$L_{3,\ell n+r} = \frac{1}{2^{n+1}}\int_{-1}^{1}(L_{3,\ell} + \sqrt{13}F_{3,\ell}x)^{n-1} \times \left(13nF_{3,\ell} + 3L_{3,\ell} + 3\sqrt{13}(n+1)\,F_{3,\ell}x\right)dx.$$

It is known that $F_{3,2} = 3$ and $L_{3,2} = 11$. Then we can find $F_{3,3}$ and $L_{3,3}$ as follows:

$$\begin{aligned} F_{3,3} &= F_{3,2(1)+1} \\ &= \frac{1}{2^2}\int_{-1}^{1}(3F_{3,2} + L_{3,2} + 2\sqrt{13}F_{3,2}x)dx \\ &= \frac{1}{2}\int_{-1}^{1}(10 + 3\sqrt{13}x)dx \\ &= \frac{1}{2}\left(10x + \frac{3\sqrt{13}x^2}{2}\right)\Big|_{-1}^{1} \\ &= 10 \end{aligned}$$

and

$$\begin{aligned} L_{3,3} &= L_{3,2(1)+1} \\ &= \frac{1}{2^2}\int_{-1}^{1}\left(13F_{3,2} + 3L_{3,2} + 6\sqrt{13}\,F_{3,2}x\right)dx \\ &= \frac{1}{2}\int_{-1}^{1}(36 + 9\sqrt{13}x)dx \\ &= \frac{1}{2}\left(36x + \frac{9\sqrt{13}x^2}{2}\right)\Big|_{-1}^{1} \\ &= 36. \end{aligned}$$

Moreover, we obtain $F_{3,5}$ and $L_{3,5}$ as follows:

$$\begin{aligned} F_{3,5} &= F_{3,2(2)+1} \\ &= \frac{1}{2^3}\int_{-1}^{1}(L_{3,2} + \sqrt{13}F_{3,2}x) \\ &\qquad \times (6F_{3,2} + L_{3,2} + 3\sqrt{13}F_{3,2}x)dx \\ &= \frac{1}{8}\int_{-1}^{1}(11 + 3\sqrt{13}x)(29 + 9\sqrt{13}x)dx \\ &= \frac{1}{8}\int_{-1}^{1}(319 + 186\sqrt{13}x + 351x^2)dx \\ &= \frac{1}{8}\left(319x + 93\sqrt{13}x^2 + 117x^3\right)\Big|_{-1}^{1} \\ &= 109 \end{aligned}$$

and

$$\begin{aligned} L_{3,5} &= L_{3,2(2)+1} \\ &= \frac{1}{2^3}\int_{-1}^{1}(L_{3,2} + \sqrt{13}F_{3,2}x) \\ &\qquad \times (26F_{3,2} + 3L_{3,2} + 9\sqrt{13}F_{3,2}x)dx \\ &= \frac{1}{8}\int_{-1}^{1}(11 + 3\sqrt{13}x)(111 + 27\sqrt{13}x)dx \\ &= \frac{1}{8}\int_{-1}^{1}(1221 + 630\sqrt{13}x + 1053x^2)dx \\ &= \frac{1}{8}\left(1221x + 315\sqrt{13}x^2 + 351x^3\right)\Big|_{-1}^{1} \\ &= 393. \end{aligned}$$

4 Conclusion

In this paper, new integral representations of the k-Fibonacci numbers and the k-Lucas numbers have been introduced and studied. Many of the properties of these numbers are proved by using Binet's formulas. We also establish some identities and simple integral calculus to prove them. The approach primarily builds on mathematical skills for deriving integral representations and provides formulas for both even and odd terms in these sequences. Indeed, we present that the integral representation for the k-Fibonacci numbers $F_{k,\ell n}$ and k-Lucas numbers $L_{k,\ell n}$ can be found by employing other known relations between the two numbers $F_{k,\ell}$ and $L_{k,\ell}$. And then both $F_{k,\ell n}$ and $L_{k,\ell n}$ are used to establish the generalized forms of integral representations for the k-Fibonacci numbers $F_{k,\ell n+r}$ and k-Lucas numbers $L_{k,\ell n+r}$. Moreover, we deduce results applicable to related number sequences like Fibonacci, Lucas, Pell, and Pell-Lucas numbers. Finally, we give some numerical examples of 3-Fibonacci and 3-Lucas numbers.

References:

[1] D. Andrica and O. Bagdasar, *Recurrent Sequences: Key Results, Applications, and Problems*, Springer, Cham, 2020.

[2] M. Bicknell, A primer on the Pell sequence and related sequences, *Fibonacci Quart.*, Vol.13, No.4, 1915, pp. 345–349.

[3] K. Dilcher, Hypergeometric functions and Fibonacci numbers, *Fibonacci Quart.*, Vol.38, No.4, 2000, pp. 342–363.

[4] M. L. Glasser and Y. Zhou, An integral representation for the Fibonacci numbers and their generalization, *Fibonacci Quart.*, Vol.53, No.4, 2015, pp. 313–318.

[5] A. F. Horadam, Pell identities, *Fibonacci Quart.*, Vol.9, No.3, 1971, pp. 245–263.

[6] A. F. Horadam and J. M. Mahon, Pell and Pell-Lucas polynomials, *Fibonacci Quart.*, Vol.23, No.1, 1985, pp. 7–20.

[7] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, 2nd, NJ: John Wiley & Sons, 2018.

[8] S. M. Stewart, Simple integral representations for the Fibonacci and Lucas numbers, *Aust. J. Math. Anal. Appl.*, Vol.19, No.2, 2022, pp. 1–5.

[9] S. M. Stewart, A simple integral representation of the Fibonacci numbers, *Mathematical Gazette*, Vol.107, No.568, 2023, pp. 120–123.

[10] S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications*, Dover Press, 2008.

[11] S. Falcón and A. Plaza, On the Fibonacci k-numbers, *Chaos Solitons Fractals*, Vol.32, No.5, 2007, pp. 1615–1624.

[12] S. Falcón, On the k-Lucas numbers, *Int. J. Contemp. Math. Sci.*, Vol. 6, No.21, 2011, pp. 1039–1050.

[13] OEIS Foundation Inc, The on-line encyclopedia of integer sequences, *http://oeis.org*, 2024.

[14] S. Falcón and A. Plaza, The k-Fibonacci sequence and the Pascal 2-triangle, *Chaos Solitons Fractals*, Vol.33, No. 1, 2007, pp. 38–49.

[15] P. Chumket, P. Singavananda, R. Chinram and I. Thongsomnuk, On generalization k-Fibonacci and k-Lucas numbers, *ICIC Express Letters*, Vol. 18, No.5, 2024, pp. 461–468.

[16] M. Kumari, K. Prasad, B. Kuloğlu and E. Özkan, The k-Fibonacci group and periods of the k-step Fibonacci sequences, *WSEAS Transactions on Mathematics*, vol. 21, 2022, pp. 838–843.

[17] R. Potůček, On one series of the reciprocals of the product of two Fibonacci numbers whose indices differ by an even number, *Equations*, Vol.4, 2024, pp. 24–31.

[18] P. Puangjumpa, Some k-Fibonacci and k-Lucas identities by a matrix approach with applications, *Thai J. Math.*, Vol. 20 No.1, 2022, pp. 417–423.

[19] S. E. Rihane, k-Fibonacci and k-Lucas numbers as product of two repdigits, *Results Math.*, Vol.76, 2021, Article ID. 208, 20 pp.

[20] S. E. Rihane, On k-Fibonacci and k-Lucas numbers written as a product of two Pell numbers, *Bol. Soc. Mat. Mex.*, Vol.30, 2024, Article ID. 20, 25 pp.

[21] P. Singavananda, H. Kusa-A, S. Chakapi and A. Denphetnong, On generalized Fibonacci and k-generalized Fibonacci numbers, *ICIC Express Letters*, Vol. 18, No.8, 2024, pp. 801–809.

[22] S. Somprom, W. Nimnual and W. Hongthong, Some identities for an alternating sum of Fibonacci and Lucus numbers of order k, *WSEAS Transactions on Mathematics*, Vol. 21, 2022, pp. 580–584.

[23] S. Somprom, P. Puangjumpa and A. Sichiangha, On the properties of Generalized Jacobsthal and Generalized Jacobsthal-Lucas sequences, *WSEAS Transactions on Mathematics*, Vol. 22, 2023, pp. 634–640.

[24] N. Yilmaz, A. Aydoğdu and E. Özkan, Some properties of k-generalized Fibonacci numbers, *Mathematica Montisnigri*, Vol. 50, 2021, pp. 73–79.

[25] T. Dana-Picard, Sequences of definite integrals, *Internat. J. Math. Ed. Sci. Tech.* Vol.38 No.3, 2007, pp. 393–401.

[26] T. Dana-Picard, Integral presentations of Catalan numbers, *Internat. J. Math. Ed. Sci. Tech.*, Vol.41, No.1, 2010, pp. 63–69.

[27] T. Dana-Picard, Integral presentations of Catalan numbers and Wallis formula, *Internat. J. Math. Ed. Sci. Tech.*, Vol.42, No. 1, 2011, pp. 122–129.

[28] A. Ipek, Integral representations of the Jacobsthal and Jacobsthal-Lucas numbers, *Asia Mathematika*, Vol.8, No.2, 2024, pp. 1–15.

[29] W.-H. Li, O. Kouba, I. Kaddoura and F. Qi, A further generalization of the Catalan numbers and its explicit formula and integral representation, *Filomat*, Vol.37, No.19, 2023, pp. 6505–6524.

[30] A. Nilsrakoo, Integral representations of the Pell and Pell-Lucas numbers, *J. Science and Science Education*, Vol. 7, No.2, 2024, pp. 272–281.

[31] K. A. Penson and J. -M. Sixdeniers, Integral representations of Catalan and related numbers, *J. Integer Seq.*, Vol.4, No.2, 2001, Article 01.2.5.

[32] F. Qi, Parametric integrals, the Catalan numbers, and the beta function, *Elem. Math.*, Vol.72, No.3, 2017, pp. 103–110.

[33] F. Qi, X. -T. Shi and F. -F. Liu, An integral representation, complete monotonicity, and inequalities of the Catalan numbers, *Filomat*, Vol.32, No.2, 2018, pp. 575–587.

[34] X. -T. Shi, F. -F. Liu and F. Qi, An integral representation of the Catalan numbers, *Glob. J. Math. Anal.*, Vol.3, No.3, 2015, pp. 130–133.

Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

Weerayuth Nilsrakoo is responsible for the conceptualization of the research problem, formal analysis, and the supervision of the work. Achariya Nilsrakoo is responsible for the formal analysis, validation, and corresponding author.

Sources of funding for research presented in a scientific article or scientific article itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.