Complex Analytic Functions with Natural Boundary

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Abstract: The analytic functions with *natural boundaries* have been only occasionally mentioned in literature. They were defined mainly by lacunary power series of Hadamard type, except for the modular function which is the result of a laborious construction. The case of infinite Blaschke products which cannot be analytically continued over the unit circle is also known, yet the authors have no knowledge about any study devoted to these functions. The purpose of this article is to take a closer look upon these functions, to find new techniques of generating them and to bring this topic into the mainstream study of analytic functions. A special attention is devoted to the theory of Blaschke products, which is completed with new results related to their boundary behavior, making possible the study of the Blaschke products with natural boundary. We apply to them the same method of study as for ordinary infinite Blaschke products obtaining mirror functions with respect to the unit circle. The working tool is that of the fundamental domains, which are easily revealed by the technique of continuation over a curve, or lifting of a curve, having its origins in the differential geometry. Graphic illustrations contribute to a better understanding of the theoretical endeavors.

Key-Words: Blaschke product, the permanence of functional equations, lacunary power series, natural boundary, denseness theorems, Frostman condition.

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1 Introduction

The modular function $\lambda(\tau)$, [1], has been obtained starting with a domain Ω_1 bounded by the half-lines

$$\Re\tau = \pm 1, \quad \Im\tau \geq 0,$$

and the half-circles

$$\left|\tau \pm \frac{1}{2}\right| = \frac{1}{2}, \quad \Im\tau \geq 0.$$

By the Riemann mapping theorem, there is a unique conformal mapping $\lambda(\tau)$ of the domain Ω_1 onto the complex plane with a slit alongside the real axis from $-\infty$ to -1 and from 1 to $+\infty$ such that $\tau = 0, 1, \infty$ is mapped onto $\lambda = 1, \infty, 0$.

By the Schwartz symmetry principle this mapping can be extended analytically to the whole upper half plane, first into symmetric domains Ω_2 and Ω_3 of Ω_1 with respect to the two half-lines by using the functional equations

$$\lambda(\tau + 2) = \lambda(\tau)$$

and then into symmetric domains Ω_4 and Ω_5 of Ω_1 with respect to the two half-circles by using the functional equation

$$\lambda\left(\frac{\tau}{1-2\tau}\right) = \lambda(\tau).$$

The new half-lines and half-circles obtained as boundaries can be used to do analytic continuations of $\lambda(\tau)$ into the symmetric domains with respect to them and the process can be repeated indefinitely. Finally, $\lambda(\tau)$ will be defined in the whole upper half-plane. The real axis becomes a natural boundary of $\lambda(\tau)$, as it can be seen in Fig. 1a. It can be easily seen that

$$\widetilde{\lambda(\tau)} = \overline{\lambda(\overline{\tau})}, \quad \Im\tau < 0,$$

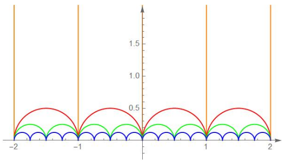
is an analytic function defined in the lower half-plane which has also as natural boundary the real axis. The two functions are mirror images one of each other in the sense that symmetric figures with respect to the real axis are mapped by the corresponding functions into symmetric figures with respect to the real axis, as it can be seen in Fig. 2a and Fig. 2b.

Let us notice that the Möbius transformation

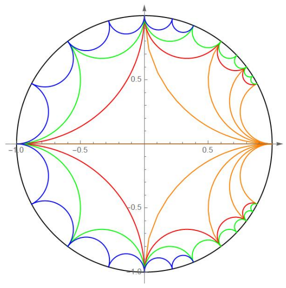
$$w = i\frac{(1+z)}{1-z}$$

maps conformally the unit disk onto the upper half-plane, carrying the unit circle onto the real axis, hence $\lambda(w(z)), |z| < 1$, will have the natural boundary the unit circle. This affirmation is illustrated in Fig. 1b. The same function maps the exterior of the unit disk onto the lower half-plane, hence $\overline{\lambda(w(\overline{z}))}$, $|z| > 1$, will have the same natural boundary, namely, $|z| = 1$.

Reciprocally, if the analytic function $f(\zeta)$ has the unit circle as natural boundary, then the function

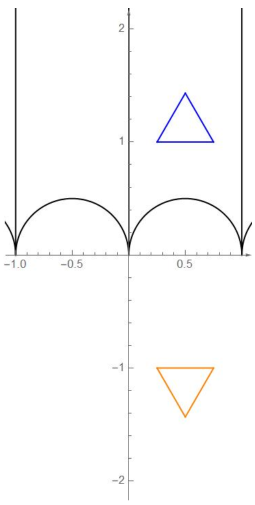


(a) The fundamental domains of the modular function

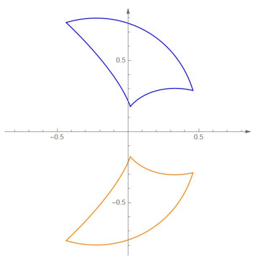


(b) The natural boundary of the modular function can be carried onto the unit circle

Fig. 1: T he modular function $\lambda(\tau)$ and the function $\lambda(i\frac{1+z}{1-z})$ have the natural boundaries the real axis and respectively the unit circle



(a) Symmetric triangles with respect to the real axis



(b) Images of triangles are symmetric with respect to the real axis

Fig. 2: The images by $\lambda(\tau)$ and by $\overline{\lambda\left(i\frac{1+\overline{z}}{1-\overline{z}}\right)}$ of symmetric figures with respect to the real axis are symmetric with respect to the real axis

$f(\frac{z-i}{z+i})$ has as natural boundary the real axis. A whole class of infinite Blaschke products have the unit circle as natural boundary. We will deal with them later, after presenting some new results related to Blaschke products.

2 A New Look on Blaschke Products

A Blaschke product is an expression of the form

$$B(z) = \prod_{n=1}^{m\leq\infty} e^{-i\theta_n}\frac{z-a_n}{1-\overline{a}_n z}, \tag{1}$$

where

$$a_n = r_n e^{i\theta_n}, \quad 0 \leq r_n < 1, \quad \theta_n \in \mathbb{R}.$$

When m is finite we have a finite Blaschke product of degree m. By relation (2), the finite Blaschke products of degree m are meromorphic functions in $\overline{\mathbb{C}}$ with the zeros a_n and the poles $\frac{1}{\overline{a_n}}$, $n = 1, 2, ..., m$. Infinite products (1) may be convergent or not. Blaschke proved that an infinite product (1) is convergent in

$$D = \{z \mid |z| < 1\}$$

if and only if $\Sigma_{n=1}^{\infty}(1-|a_n|)$ converges. This is known as the *Blaschke condition*. It shows that for a convergent Blaschke product the sequence (a_n) cannot have any accumulation point in the disk $|z| < 1$ and it should accumulate to points of the unit circle fast enough, i.e., the sequence $(1-|a_n|)$ should approach 0 fast enough to ensure the convergence of that series. When the Blaschke condition is fulfilled, the product (1) converges absolutely in $|z| < 1$ and uniformly on compact subsets of the unit disk, [2], hence $B(z)$ is an analytic function in D. By definition, for $|z| < 1$,

$$B(z) = \lim_{n\to\infty} B_n(z),$$

where

$$B_n(z) = \prod_{k=1}^{n} e^{-i\theta_k}\frac{z-a_k}{1-\overline{a}_k z}.$$

One can easily check that,

$$B_n(z) = \frac{1}{\overline{B_n\left(\frac{1}{\overline{z}}\right)}}. \tag{2}$$

It is also known, [2], that

$$B'(z) = B(z)\sum_{n=1}^{\infty}\frac{1-|a_n|^2}{(a_n-z)(1-\overline{a_n}z)} \tag{3}$$

Let us notice that

$$1-|a_n|^2=(1-|a_n|)(1+|a_n|)\le 2(1-|a_n|)$$

and for every $\delta>0$ there is a finite number K such that

$$\frac{1+|a_n|}{|a_n-z||1-\overline{a}_n z|}<K \text{ if } |z-a_n|>\delta,$$

hence the series (3) converges in the unit disk, except at the points a_n. Yet,

$$\begin{aligned} B'(a_n) &= \lim_{z\to a_n}\frac{B(z)}{z-a_n} \\ &= \frac{1}{1-\overline{a}_k a_n}\prod_{k\neq n} e^{i\theta_k}\frac{a_n-a_k}{1-\overline{a}_k a_n}, \end{aligned}$$

which is a convergent product since $\sum_{k\neq n}(1-|a_k|)$ converges. Thus, if $B(z)$ satisfies the Blaschke condition, then its derivative is given by (3) at every point of the unit disk.

We notice that if $B'(\zeta)$ exits for $\zeta=e^{i\theta}$, $\theta\in\mathbb{R}$, then since

$$\frac{1-|a_n|^2}{(a_n-e^{i\theta})(1-\overline{a}_n e^{i\theta})}=-e^{-i\theta}\frac{1-|a_n|^2}{|a_n-e^{i\theta}|^2}$$

we have

$$e^{i\theta}\sum_{n=1}^{\infty}\frac{1-|a_n|^2}{(a_n-\zeta)(1-\overline{a}_n\zeta)}<0,$$

hence $B'(\zeta)\neq 0$. Now we can formulate the following proposition.

Proposition 1. For any Blaschke product $B(z)$ (finite or infinite), the equation $B(z)=1$ has distinct roots, all located on the unit circle, and $B'(z)$ has no zero on the unit circle.

Indeed, every Blaschke factor

$$e^{-i\theta_n}\frac{z-a_n}{1-\overline{a}_n z}$$

maps the unit disk onto itself, the unit circle onto itself and the exterior of the unit disk onto itself, so the Blaschke product (1) does the same. Thus the solutions of the equation $B(z)=1$ must be all located on the unit circle. On the other hand since every multiple zero of $B(z)-1$ is also a zero of $B'(z)$, [3], and no zero of $B'(z)$ is located on the unit circle, no zero of $B(z)-1$ can be multiple zero.

By a theorem of Frostman, [4], [5], if

$$\sum_{n=1}^{\infty}\frac{1-|a_n|}{|e^{i\theta}-a_n|}<\infty$$

(we will call this inequality the *Frostman condition at* $e^{i\theta}$), then $B(z)$ has a radial limit at $e^{i\theta}$.

Theorem 1. Suppose that $\zeta_0\in\partial D\setminus E$, where E is the set of cluster points of zeros of the infinite Blaschke product $B(z)$ and let

$$B_n(z)=\prod_{k=1}^{n} e^{-i\theta_k}\frac{z-a_k}{1-\overline{a}_k z}.$$

Suppose that $\alpha<Arg(\zeta_0)<\beta$ and no point of E belongs to the arc of the unit circle between $e^{i\alpha}$ and $e^{i\beta}$. If $B(z)$ satisfies the Frostman condition at $e^{i\alpha}$ and $e^{i\beta}$, then $\lim_{n\to\infty}B_n(\zeta_0)$ exists and we can set

$$B(\zeta_0)=\lim_{n\to\infty}B_n(\zeta_0).$$

Proof. We will use the Cauchy integral formula for ζ_0 and the domain D bounded by the rays

$$\{z \mid Arg(z)=\alpha\} \text{ and } \{z \mid Arg(z)=\beta\}$$

and the arcs of the circles centered at the origin and of radius $r<1$ and $\frac{1}{r}$ between the two rays, such that no zero of $B(z)$ belongs to D. If γ is the boundary of D oriented counterclockwise, then for every n we have:

$$\begin{aligned} B_n(\zeta_0) &= \frac{1}{2\pi i}\int_{\gamma}\frac{B_n(\zeta)}{\zeta-\zeta_0}d\zeta \\ &= \frac{1}{2\pi i}\left[\int_r^{\frac{1}{r}}\frac{B_n(\rho e^{i\alpha})e^{i\alpha}d\rho}{\rho e^{i\alpha}-\zeta_0}\right. \\ &\quad+\int_{\alpha}^{\beta}\frac{B_n(\frac{1}{r}e^{i\theta})\frac{i}{r}e^{i\theta}id\theta}{\frac{1}{r}e^{i\theta}-\zeta_0} \\ &\quad+\int_{\frac{1}{r}}^{r}\frac{B_n(\rho e^{i\beta})e^{i\beta}d\rho}{\rho e^{i\beta}-\zeta_0} \\ &\quad\left.+\int_{\beta}^{\alpha}\frac{B_n(re^{i\theta})re^{i\theta}id\theta}{re^{i\theta}-\zeta_0}\right]. \end{aligned} \tag{4}$$

By relation (2) we have

$$B_n\left(\frac{1}{r}e^{i\theta}\right)=\frac{1}{\overline{B_n(re^{i\theta})}}$$

for $\alpha\le\theta\le\beta$, thus for $\theta=\alpha$ and $\theta=\beta$, after the change of variable $\rho = \frac{1}{t}$ we get

$$\int_1^{\frac{1}{r}} \frac{B_n(\rho e^{i\theta})e^{i\theta}d\rho}{\rho e^{i\theta}-\zeta_0} = -\int_1^{r} \frac{B_n(\frac{1}{t}e^{i\theta})\frac{e^{i\theta}}{t^2}dt}{\frac{1}{t}e^{i\theta}-\zeta_0} = \int_r^{1} \frac{\frac{1}{\overline{B_n(\rho e^{i\theta})}}\frac{e^{i\theta}}{\rho^2}d\rho}{\frac{1}{\rho}e^{i\theta}-\zeta_0},$$

hence

$$\int_r^{\frac{1}{r}} \frac{B_n(\rho e^{i\alpha})e^{i\alpha}d\rho}{\rho e^{i\alpha}-\zeta_0} = \int_r^1 \left[\frac{B_n(\rho e^{i\alpha})e^{i\alpha}}{\rho e^{i\alpha}-\zeta_0} + \frac{\frac{1}{\overline{B_n(\rho e^{i\alpha})}}\frac{e^{i\alpha}}{\rho^2}}{\frac{1}{\rho}e^{i\alpha}-\zeta_0}\right]d\rho.$$

Also, for $\rho = \frac{1}{r}$ we have

$$\int_\alpha^\beta \frac{B_n(\frac{1}{r}e^{i\theta})\frac{i}{r}e^{i\theta}id\theta}{\frac{1}{r}e^{i\theta}-\zeta_0} = \int_\alpha^\beta \frac{\frac{1}{\overline{B_n(re^{i\theta})}}\frac{i}{r}e^{i\theta}d\theta}{\frac{1}{r}e^{i\theta}-\zeta_0}$$

and

$$\begin{aligned} B_n(\zeta_0) &= \\ &= \frac{1}{2\pi i}\int_r^1 \left[\frac{B_n(\rho e^{i\alpha})e^{i\alpha}}{\rho e^{i\alpha}-\zeta_0} + \frac{\frac{1}{\overline{B_n(\rho e^{i\alpha})}}\frac{e^{i\alpha}}{\rho^2}}{\frac{1}{\rho}e^{i\alpha}-\zeta_0}\right]d\rho \\ &- \frac{1}{2\pi i}\int_r^1 \left[\frac{B_n(\rho e^{i\beta})e^{i\alpha}}{\rho e^{i\beta}-\zeta_0} + \frac{\frac{1}{\overline{B_n(\rho e^{i\beta})}}\frac{e^{i\beta}}{\rho^2}}{\frac{1}{\rho}e^{i\beta}-\zeta_0}\right]d\rho \\ &+ \frac{1}{2\pi i}\int_\alpha^\beta \frac{\frac{1}{\overline{B_n(re^{i\theta})}}\frac{i}{r}e^{i\theta}d\theta}{\frac{1}{r}e^{i\theta}-\zeta_0} \\ &- \frac{1}{2\pi i}\int_\alpha^\beta \frac{B_n(re^{i\theta})re^{i\theta}id\theta}{re^{i\theta}-\zeta_0}. \end{aligned} \tag{5}$$

Since $B_n(z)$ converges absolutely in $|z| < 1$ to $B(z)$, when taking the limit as $n \to \infty$ of $B_n(\zeta_0)$, we can switch the limit with the integral sign in (4). Also, by the Frostman condition at $e^{i\alpha}$ and $e^{i\beta}$, the integrals from $\rho = r$ to $\rho = 1$ of $B(z)$ exist for $z = \rho e^{i\alpha}$ and $z = \rho e^{i\beta}$. Then taking the limit as $n \to \infty$ in (4) we obtain an expression for $B(\zeta_0)$. Obviously, this is true for any $\zeta = e^{i\theta}$, $\alpha \le \theta \le \beta$. $\square$

Corollary 1. Let E the cluster points of zeros of the infinite Blaschke product $B(z)$, be a set of Lebesgue measure 0. Then $|B(e^{i\theta})| = 1$ at almost every point of $\partial D \setminus E$.

Proof. Indeed, $\left|\frac{B(z)}{B_n(z)}\right|$ being a subharmonic function in D, [3] (page 248), we have

$$\left|\frac{B(0)}{B_n(0)}\right| \le \frac{1}{2\pi}\int_{\partial D\setminus E} \frac{|B(e^{i\theta})|}{|B_n(e^{i\theta})|}d\theta = \frac{1}{2\pi}\int_{\partial D\setminus E} |B(e^{i\theta})|d\theta.$$

Letting $n \to \infty$, we get

$$\frac{1}{2\pi}\int_{\partial D\setminus E} |B(e^{i\theta})|d\theta = 1,$$

hence $|B(e^{i\theta})| = 1$ a.e. in $\partial D \setminus E$. In other words, if E is a set of Lebesgue measure 0, then $B(z)$ maps $\partial D \setminus E$ onto ∂D. $\square$

Proposition 2. For a Blaschke product $B(z)$ of degree m, the equation $B(z) = 1$ has exactly m distinct roots ζ_k, $k = 1, 2, ..., m$ all located on the unit circle. They partition the unit circle into m Jordan arcs γ_k such that, if we remove one end of each arc, it is mapped by $B(z)$ one to one onto the unit circle. In other words, when z travels once alongside the unit circle , its image $B(z)$ describes the unit circle m times.

For any Blaschke product $B(z)$ of degree m, the equation $B'(z) = 0$ has $m-1$ roots, counted with multiplicities, in the unit disk and other $m-1$ symmetric roots of these ones with respect to the unit circle, [6], [7], [8]. There is no root of this equation on the unit circle. When we perform simultaneous continuations, [2], by $B(z)$ over the real axis starting from ζ_k, $k = 1, 2, ..., m$ we obtain m Jordan arcs. These arcs can only meet at the zeros of $B'(z)$, since the points of intersection of these arcs are branch points of $B(z)$, [9], i.e., the zeros of $B'(z)$.

When they meet at some zeros of $B'(z)$, then the values of $B(z)$ at those zeros must be real. Suppose these values are all real. Then these Jordan arcs together with the symmetric ones with respect to the unit circle divide the complex plane into m fundamental domains of $w = B(z)$. Otherwise, we need another method to find fundamental domains for $B(z)$. Such a method has been described in the literature. Namely, if b_k are the zeros of $B'(z)$, we connect the points $B(b_k)$ and the point $w = 1$ by a non self-intersecting polygonal line L and we perform simultaneous continuations above L from the points

ζ_k. This time, all the Jordan arcs we obtain in this way meet at some points b_k and every b_k is an intersection point of some of these arcs. Together with their symmetric arcs with respect to the unit circle, they bound m fundamental domains of $B(z)$.

It can be easily checked that if $B(z)$ is a finite Blaschke product, for every $z \in \overline{\mathbb{C}}$ we have, [2],

$$B(z) = \frac{1}{\overline{B\left(\frac{1}{\overline{z}}\right)}} \tag{6}$$

For the infinite case, when the Blaschke condition is fulfilled, we set

$$\widetilde{B}(z) = \frac{1}{\overline{B\left(\frac{1}{\overline{z}}\right)}} \tag{7}$$

for $|z| > 1$.

Theorem 2. Let E be the set of cluster points of the zeros of the infinite Blaschke product $B(z)$. If $\partial D \setminus E$ includes an arc of the unit circle, then the function $\widetilde{B}(z)$ is a meromorphic continuation of $B(z)$ to $\overline{\mathbb{C}} \setminus E$.

Proof. Indeed, for $e^{i\theta} \in \partial D \setminus E$ we have

$$e^{i\theta} = \frac{1}{e^{-i\theta}} = \frac{1}{\overline{e^{i\theta}}}$$

and by Corollary 2, there is $\varphi \in \mathbb{R}$ such that $B(e^{i\theta}) = e^{i\varphi}$, for almost every $e^{i\theta} \in \partial D \setminus E$.

Then, for such $e^{i\theta} \in \partial D \setminus E$, we have

$$\widetilde{B}(e^{i\theta}) = \frac{1}{\overline{B\left(\frac{1}{\overline{e^{i\theta}}}\right)}} = \frac{1}{\overline{B(e^{i\theta})}} = \frac{1}{\overline{e^{i\varphi}}} = e^{i\varphi} = B(e^{i\theta}),$$

hence $B(z)$ and $\widetilde{B}(z)$ coincide a.e. on $\partial D \setminus E$, and by the permanence of functional equations, they coincide everywhere on $\partial D \setminus E$, which means that indeed $\tilde{B}(z)$ is a meromorphic continuation of $B(z)$. We keep the notation $B(z)$ for this function. It maps the unit disk onto itself, the exterior of the unit disk onto itself and $\partial D \setminus E$ onto ∂D. Since the zeros and the poles of $B(z)$ are symmetric points with respect to the unit circle, E is also the set of accumulation points of the poles of $B(z)$, hence every point of E is an essential non isolated singular point for $B(z)$. □

We have studied previously the fact that in any neighborhood of an isolated essential singular point of an analytic function $f(z)$ there are infinitely many fundamental domains of $f(z)$. This property should be true also for the points of E. A proof of this affirmation for the case where E is a single point, $E = \{z_0\}$, can be found in [10]. It is based on the fact that for every partial product $B_n(z)$ of $B(z)$ the equation $B_n(z) = 1$ has exactly n distinct solutions $\zeta_{n,k}$, $k = 1, 2, ..., n$ located all on the unit circle and counted counterclockwise starting from $z = 1$.

Theorem 3. Let $\zeta_k = e^{i\theta_k}$ be the roots of the equation $B_n(z) = 1$, for $k = 1, 2, ..., n$. If

$$a_{k+1} = r_{k+1} e^{i\alpha_{k+1}}$$

is a zero of $B_{k+1}(z)$, and $\alpha_{k+1} \neq \theta_j$, for $j = 1, 2, ..., n$, then the equations $B_n(z) = 1$ and $B_{n+1}(z) = 1$ do not have any common root.

Proof: Indeed, for such a root ζ we would have

$$B_{n+1}(\zeta) = B_n(\zeta),$$

hence

$$e^{-i\alpha_{n+1}} \frac{\zeta - a_{n+1}}{1 - \overline{a}_{n+1}\zeta} = 1,$$

or

$$e^{-i\alpha_{n+1}}\zeta - r_{n+1} = 1 - r_{n+1} e^{-i\alpha_{n+1}}\zeta,$$

thus

$$-e^{-i\alpha_{n+1}}\zeta(r_{n+1} + 1) = r_{n+1} + 1,$$

hence $\zeta = e^{i\alpha_{n+1}}$, which has been excluded.

We conclude that for different values of n, we have essentially different roots of the equation $B_n(z) = 1$ and the roots of all these equations form a discrete set. As there are infinitely many such roots on the unit circle for $n = 1, 2, ...$, they have at least one cluster point. □

Theorem 4. Let $B(z)$ be an infinite Blaschke product satisfying the Blaschke condition and let $B_n(z)$ be partial products of $B(z)$, $n = 1, 2, ...$ If $B(z)$ admits a meromorphic continuation to $\overline{\mathbb{C}} \setminus E$, then every solution of the equation $B(z) = 1$ is the limit of some solutions of the equations $B_n(z) = 1$.

Proof. Let P be the set of the poles of $B(z)$. Then, by using relations (6) and (7), it can be easily seen that $B_n(z)$ converges uniformly on compact subsets of $\overline{\mathbb{C}} \setminus (E \cup P)$ to $B(z)$. Let $\zeta_{n,k}$, $k = 1, 2, ..., n$, be the solutions of the equation $B_n(z) = 1$ and let ζ_0 be a solution of the equation $B(z) = 1$, hence

$$B_n(\zeta_{n,k}) = 1 = B(\zeta_0) = \lim_{n\to\infty} B_n(\zeta_0).$$

This shows that there is no $\delta_0 > 0$ such that $|B_n(\zeta_0) - B_n(\zeta_{n,k})| > \delta_0$ for n big enough, thus there should be a sequence (ζ_{n,k_n}) such that $\lim_{n\to\infty} \zeta_{n,k_n} = \zeta_0$. □

Simultaneous continuations over some polygonal lines through $B_n(z)$ from $\zeta_{n,k}$ are Jordan arcs which meet two by two into the zeros of $B_n'(z)$ and provide a partition of the complex plane into n fundamental domains $\Omega_{n,k}$ of $B_n(z)$ which are mapped by $B_n(z)$ onto the whole complex plane with some slits alongside these lines. When a_n are simple zeros of $B(z)$, hence of every $B_n(z)$, the domains $\Omega_{n,k}$ contain each one a unique zero a_k. If a_k is a multiple zero of order j of $B_n(z)$, then the boundaries of j fundamental domains will contain a_k.

The infinite Blaschke product $B(z)$ has a similar property, if it admits a meromorphic continuation to $\overline{\mathbb{C}} \setminus E$. We have the following result.

Theorem 5. The roots of the equation $B(z) = 1$ located in $\partial D \setminus E$ form a discrete set, having the cluster points in E.

Proof. Indeed, $\overline{\mathbb{C}} \setminus E$ is open and by the permanence of functional equations, if those roots had a cluster point in $\partial D \setminus E$, hence in $\overline{\mathbb{C}} \setminus E$, then $B(z)$ would be identically equal to 1, which is not possible. Therefore, the roots of $B(z) = 1$ can accumulate only on E. □

Let $z_{n,k}$ be the zeros $B_n'(z)$, $k = 1, 2, ..., n$ and let z_0 be a zero of $B'(z)$. Since

$$\lim_{n\to\infty} B_n(z) = B(z)$$

uniformly on compact subsets of the unit disk, we have $\lim_{n\to\infty} B_n'(z) = B'(z)$, for every z, $|z| < 1$, hence

$$\lim_{n\to\infty} B_n'(z_0) = B'(z_0) = 0$$
$$= B_n'(z_{n,k}) = \lim_{n\to\infty} B_n'(z_{n,k}),$$

which implies, due to the continuity of $B_n'(z)$, that there is a sequence (z_{n,k_n}) such that

$$\lim_{n\to\infty} z_{n,k_n} = z_0.$$

As, for every n, $B_n'(z)$ has $n-1$ zeros counted with multiplicities in the unit disk, $B'(z)$ must have infinitely many zeros in the unit disk. They form a discrete set, since otherwise $B'(z)$ would be identically zero. We can connect the points $B(z_n)$, where z_n are these zeros, with a non self-intersecting polygonal line L starting from $w = 1$, like in the finite case and then by simultaneous continuation from ζ_k over L, where ζ_k are the zeros of $B(z)-1$, obtain the boundaries of the fundamental domains of $B(z)$. This is true for any infinite Blaschke product satisfying the Blaschke condition and which admits a meromorphic continuation to $\overline{\mathbb{C}} \setminus E$.

Theorem 6. The fundamental domains of any infinite Blaschke product which admits meromorphic continuation to $\overline{\mathbb{C}} \setminus E$ accumulate to some points of E, in the sense that every neighborhood of such a point contains infinitely many such domains.

Proof. Indeed, these domains are disjoint connected open sets. Every one of them has a part in the unit disk and a symmetric part with respect to the unit circle, outside the unit disk. Considering those inside the unit disk, being infinitely many of them, they should accumulate to some point in the closed unit disk. This point cannot be in $\overline{\mathbb{C}} \setminus E$, since in every one of them, $B(z)$ assumes a given value w_0 and then, the points z_n with $B(z_n) = w_0$ would have a cluster point in $\overline{\mathbb{C}} \setminus E$, implying $B(z) \equiv w_0$, which is absurd. □

Theorem 7. If E is a discrete set, every point of E is a cluster point of zeros of $B(z)-1$, hence it is an accumulation point of fundamental domains of $B(z)$.

Proof. Let $\zeta_0 \in E$. Hence ζ_0 is an accumulation point of zeros of $B(z)$ and there is an open arc γ of the unit circle which contains ζ_0 and not other point of E. Let (a_{n_k}) be a sequence of zeros of $B(z)$ convergent to ζ_0 and let γ_{n_k} be the arcs obtained by continuation by $B(z)$ over the real axis starting from a_{n_k}. These arcs connect a_{n_k} with $\frac{1}{\overline{a_{n_k}}}$ and are orthogonal to the unit circle, since all have the same image, the real axis, which is orthogonal to the image by $B(z)$ of the unit circle, i.e. to the unit circle.

Let us notice that for every a_{n_k} there is a unique circle C_{n_k} passing through a_{n_k} and $\frac{1}{\overline{a_{n_k}}}$ and orthogonal to the unit circle. The disk bounded by this circle contains the arc γ_{n_k} since its image by $B(z)$ contains the image of that arc. Since

$$\lim_{n_k\to\infty} a_{n_k} = \zeta_0 = \lim_{n_k\to\infty} \frac{1}{\overline{a_{n_k}}},$$

the circles C_{n_k} will contract to ζ_0, and therefore will intersect γ, thus the arcs γ_{n_k} will do the same. These arcs intersect the unit circle at points ζ_{n_k}, where $B(\zeta_{n_k}) = 1$.These points must also accumulate to ζ_0, which proves the theorem. □

As shown in [10], there are occurrences, not quite unusual as it may seem, where E coincides with the unit circle, in other words the unit circle is the *natural boundary* of $B(z)$.

3 Blaschke Products with Natural Boundary

We have studied in [10], the Blaschke product $B(z)$ with the zeros

$$a_{n,k} = \left(1 - \left(\frac{1}{3}\right)^n\right) e^{\frac{k\pi i}{2^{n-1}}}, \tag{8}$$

for $n = 1, 2, ...$ and $k = 1, 2, ..., 2^n$.

Let us notice that

$$\sum_{n=1}^{\infty}\sum_{k=1}^{2^n}(1 - |a_{n,k}|) = \sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n = 2,$$

hence the Blaschke condition for $B(z)$ is fulfilled.

We also notice that for every n the zeros $a_{n,k}$ are located on a circle centered at the origin and of radius $r_n = 1 - \left(\frac{1}{3}\right)^n$. There are 2^n such zeros equally spaced. When $n \to \infty$, these radii tend to 1 and $a_{n,k}$ will reach almost every point of the unit circle. Thus $B(z)$ has as natural boundary the unit circle.

Obviously, there are infinitely many similar ways to choose the zeros $a_{n,k}$ such that the Blaschke condition is fulfilled and $(a_{n,k})$ accumulates to every point of the unit circle, hence there is an infinite family of Blaschke products with natural boundary the unit circle.

If we take a subproduct $B_{\alpha,\beta}(z)$ of $B(z)$ obtained by keeping only the zeros $a_{n,k}$ located in a sector

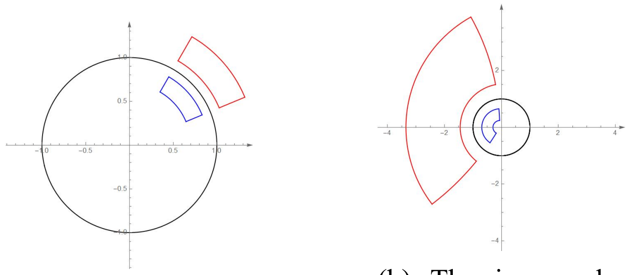
$$E = \{z | z = \rho e^{i\theta}, 0 \le \rho < 1, 0 \le \alpha \le \theta \le \beta < 2\pi\},$$

we get a similar Blaschke product having the arc E as boundary, in the sense that for $e^{i\theta} \in E$, there is no limit $z \to e^{i\theta}$ from $B_{\alpha,\beta}(z)$. However, by Theorem 2, $B_{\alpha,\beta}(z)$ has an analytic continuation to $\overline{\mathbb{C}} \setminus E$.

We can construct infinite Blaschke products having the set E of the essential singular points any *generalized Cantor subset*, [11], of the unit circle. Such a subset is obtained in the following way. Suppose that an infinite sequence (α_n), $0 < \alpha_n < 1$ is given. Let us remove from the interval $(0, 2\pi)$ a closed interval $I_{1,1}$ of length α_1. What remains are two open intervals $J_{1,1}$ and $J_{1,2}$ of lengths $l_{1,1}$ and $l_{1,2}$ Next, remove from each one of these last intervals the closed intervals of length $\alpha_2 l_{1,1}$ and respectively $\alpha_2 l_{1,2}$. What remains are four open intervals $J_{2,1}, J_{2,2}, J_{2,3}$ and $J_{2,4}$ of lengths equal respectively to $l_{2,1}, l_{2,2}, l_{2,3}$ and $l_{2,4}$. Now, we remove from each one of these intervals a closed interval of length equal respectively to $\alpha_3 l_{2,1}, ..., \alpha_3 l_{2,4}$ and we continue this process indefinitely.

To every interval $J_{n,k}$, $k = 1, 2, ..., 2^n$, corresponds on the unit circle an arc

$$\gamma_{n,k} = \{\zeta = e^{i\theta} \mid \theta \in J_{n,k}\}.$$



(a) Mirror images with respect to the unit circle

(b) The images by $B(z)$ and $\tilde{B}(z)$ of mirror images are mirror images

Fig. 3: The images by $B(z)$ and by $\overline{B(\frac{1}{\bar{z}})}$ of symmetric figures with respect to the unit circle are symmetric with respect to the unit circle

Let $r_n = 1 - \left(\frac{1}{2}\right)^{2n}$ and let $B(z)$ be the Blaschke product with the zeros

$$a_{n,k} = r_n e^{i\alpha_{n,k}}, \quad \alpha_{n,k} \in \gamma_{n,k}.$$

Thus, every zero of $B(z)$ belongs to some circle centered at the origin and of radius r_n and located in the sector determined by γ_n. We have

$$\sum_{n=1}^{\infty}\sum_{k=1}^{2^n}(1 - |a_{n,k}|) = \sum_{n=1}^{\infty}\frac{2^n}{2^{2n}} = 1,$$

hence the Blaschke condition for $B(z)$ is fulfilled.

Continuation of $B(z)$ can be performed in $\overline{\mathbb{C}} \setminus E$, where E is a *perfect set*, [11], on the unit circle. We obtain a meromorphic function in $\overline{\mathbb{C}} \setminus E$, having the zeros $a_{n,k}$ and the poles $1/\overline{a_{n,k}}$. This is a function which coincide with its mirror in the unit circle, in the sense that it maps symmetric figures with respect to the unit circle into symmetric figures with respect to the unit circle.

Let us notice that $B\left(\frac{z-i}{z+i}\right)$ has the natural boundary the real axis. Fig. 3a and Fig. 3b portray mirror images of figures by $B(z)$ and $\tilde{B}(z)$.

4 Dirichlet Series with Natural Boundaries

A *general Dirichlet series* is an expression of the form:

$$\zeta_{A,\Lambda}(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \tag{9}$$

where $A = (a_n)$ is an arbitrary sequence of complex numbers, $a_n \neq 0$ infinitely many times and

$$\Lambda = \{0 = \lambda_1 \le \lambda_2 \le ...\}$$

is a non-decreasing sequence of non-negative real numbers such that $\lim_{n\to\infty}\lambda_n = +\infty$.

If $\lambda_n = \ln n$, then $e^{-\lambda_n s} = \frac{1}{n^s}$ and the series (9) is an *ordinary Dirichlet series*. If $\lambda_n = n-1$, we obtain a power series in e^{-s}. Reciprocally, any power series

$$f(z) = \sum_{n=0}^{\infty}(z-z_0)^n \tag{10}$$

can be converted into a Dirichlet series

$$g(s) = \sum_{n=0}^{\infty} a_n e^{-ns} \tag{11}$$

by the substitution $z - z_0 = e^{-s}$. The Hadamard formula

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \tag{12}$$

for the radius of convergence R of the series (10) implies that the series (11) converges for $\Re s > \ln(\frac{1}{R})$ and it diverges for $\Re s < \ln(\frac{1}{R})$. A similar formula to (12) is known for general Dirichlet series, [12], namely, if (9) is not convergent for $s = 0$, then:

$$\sigma_c = \limsup_{n\to\infty} \ln\left|\sum_{k=1}^{n} a_k\right|^{\frac{1}{\lambda_n}} > 0 \tag{13}$$

If (9) converges for $s = 0$, then

$$\sigma_c = \limsup_{n\to\infty} \frac{1}{\lambda_{n+1}} \ln\left|\zeta_{A,\Lambda}(0) - \sum_{k=1}^{n} a_k\right| < 0 \tag{14}$$

The number σ_c is called the *abscissa of convergence* of the Dirichlet series (9) since $\zeta_{A,\Lambda}(\sigma+it)$ converges for $\sigma > \sigma_c$ and it diverges for $\sigma < \sigma_c$. The function $\zeta_{A,\Lambda}(s)$ is an analytic function in the half-plane $\Re s > \sigma_c$. To every series (9) a Dirichlet series $\zeta_{A,e^{\Lambda}}(s)$ can be associated, in which the sequence (λ_n) is replaced by (e^{λ_n}). It is known that if $\zeta_{A,e^{\Lambda}}(s)$ has only isolated singular points on the imaginary axis, then the series (9) can be continued analytically to ameromorphic function in the whole complex plane. Otherwise, the line $\Re s = \sigma_c$ is the natural boundary of $\zeta_{A,\Lambda}(s)$. Examples of Dirichlet series with natural boundaries can be constructed by using power series with natural boundaries as in the formula (11).

Theorem 8. To every power series (10) with natural boundary having the radius of convergence R corresponds a Dirichlet series (11) having as natural boundary the line $\Re s = \ln\frac{1}{R}$.

Proof. Indeed, the abscissa of convergence of the series (11) is $\ln\frac{1}{R}$. Suppose that the series can be continued analytically to some s with $\sigma = \Re s < \ln(\frac{1}{R})$. Then $f(z) = g(z_0 + e^{-s})$ is an analytic continuation of $f(z)$ to some $z = z_0 + e^{-\sigma+it}$ where $e^{-\sigma} > R$, which is absurd, since $|z - z_0| = R$ is the natural boundary of the series (10). □

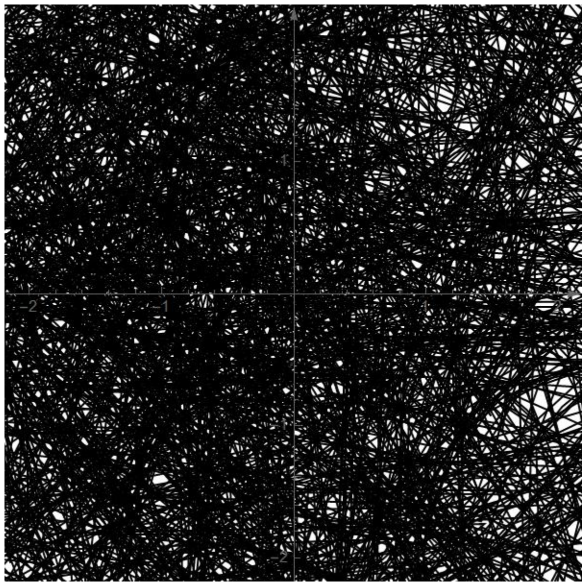


Fig. 4: An illustration of the denseness theorem for a Hadamard type of power series. This is the image of the circle $|z| = 0.9999$.

Example. To the Hadamard series $\sum_{n=0}^{\infty} z^{2^n}$ with the natural boundary the unit circle corresponds the Dirichlet series $\sum_{n=0}^{\infty} e^{-2^n s}$ having the natural boundary the imaginary axis.

Fig. 4 portrays the denseness property for this Dirichlet series related to the line $\Re s = 0.001$.

5 Denseness Theorems for Functions with Natural Boundaries

Theorem 9. If $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is a lacunary series having as natural boundary the circle $|z| = r$, then every neighborhood of a point $z_0 = re^{i\alpha}$, where $\alpha \in \mathbb{R}$, contains infinitely many fundamental domains of $f(z)$.

Proof. Suppose that there is a neighborhood V of z_0 which contains only a finite number of fundamental domains of $f(z)$. Then some of these domains must have a part of the boundary on the circle $|z| = r$. Let Ω be such a domain. The function $f(z)$ maps conformally the domain Ω onto the complex plane with a slit L such that the boundary $\partial\Omega$ of Ω is carried by $f(z)$ onto L. Then for every common point

re^{it}, $t \in \mathbb{R}$ of $\partial\Omega$ and the circle $|z| = r$ we have that $\lim_{z \to re^{it}} f(z)$ exists and it belongs to L. Thus an arc of the circle $|z| = r$ is carried by $f(z)$ into L and the symmetry of $f(z)$ with respect to this arc represents a continuation of this function outside the disk $|z| < r$, which contradicts the hypothesis that $|z| = r$ is the natural boundary of $f(z)$ and the theorem is proved. $\square$

The prototype lacunary series is the Hadamard series $h(z) = \Sigma_{n=1}^{\infty} z^{2^n}$. For any integer $m > 2$ the series $\Sigma_{n=1}^{\infty} z^{m^n}$, $\Sigma_{n=1}^{\infty} z^{\tau(n)m^n}$, where $\tau(n)$ is a positive arithmetic function, are also lacunary series. We will call them lacunary series of Hadamard type. Some other types of lacunary series are at hand, as for example $\Sigma_{n=1}^{\infty} a_n z^{m^n}$, where

$$\limsup_{n \to \infty} |a_n|^{\frac{1}{m^n}} = \frac{1}{R} \neq 1.$$

Besides the fact that all these series have the circle $|z| = R$ as natural boundary, they enjoy many other common properties. One of them is that expressed by Theorem 9. Based on this theorem, we are now able to say more about the fundamental domains of a lacunary series $f(z)$.

Theorem 10. Let $f(z)$ be a lacunary series having the natural boundary the circle $|z| = R$. Then $f'(z)$ has infinitely many zeros accumulating to every point of $|z| = R$. They are the knots of an infinite tree graph with the lives forming a dense set on the circle $|z| = R$. The branches of the tree bound domains where $f(z)$ is injective. If $f(z)$ has real coefficients, these zeros are two by two complex conjugate.

Proof: Indeed, since the boundary of every fundamental domain contains at least one branch point, where $f'(z)$ cancels, and by Theorem 9, there are infinitely many fundamental domains, the number of zeros z_n of $f'(z)$ must be infinite. They cannot have any cluster point in the disk $|z| < R$, since then, by the permanence of functional equations, $f'(z)$ would be a constant, which is absurd. They accumulate to every point of the circle $|z| = R$. The tree graph is formed by connecting them with segments of line such that no finite set of these segments bound a domain. We include every zero of $f'(z)$ in such a graph. The zeros of $f'(z)$ can be counted following any counting algorithm of the knots of a tree. For example, we designate one of the knots as the *root* z_0 of the tree. Then we count counterclockwise starting from positive real half-axis the knots directly connected to z_0. This is the first layer of knots. Then we do the same with the knots directly connected with the first layer knots, which represent the second layer, and so on. The branches of the graph end all in points of the circle $|z| = R$ such that every neighborhood of such a point contains infinitely many lives, therefore these lives form a dense set on the circle. Since there are no branch points of the function between the branches of the tree, the domains bounded by adjacent branches are univalence domains of the function. They are not necessarily fundamental domains.

If $f(z)$ has real coefficients, then so does $f'(z)$. In addition, $f'(z_0) = 0$ if and only if

$$f'(\overline{z}_0) = \overline{f'(z_0)} = 0.$$

Moreover, for any two zeros z_1 and z_2 of $f'(z)$, the segment

$$z(t) = (1-t)z_1 + tz_2, \quad 0 \le t \le 1,$$

is obviously the symmetric with respect to the real axis of the segment

$$\overline{z}(t) = (1-t)\overline{z}_1 + t\overline{z}_2.$$

We can construct the tree graph such that its branches are two by two symmetric with respect to the real axis. Fig. 5 below exhibits such a graph for the Hadamard series. $\square$

Let us notice that for the Hadamard series we have

$$h'(z) = 1 + 2z + 4z^3 + \ldots \text{ and } h'(0) = 1.$$

Thus $h(z)$ is injective in a small disk D of radius r, centered at the origin. Moreover, $h(z)$ satisfies the functional equation $h(z) = h(z^2) + z$. This implies that $\lim_{x \to 1} h(x) = +\infty$. We also have that for z real, $h(z)$ is real, which shows that the image by $h(z)$ of the interval $[0, 1)$ is the interval $[0, +\infty)$. The mapping is one to one, since $0 \le x_1 < x_2$ implies $h(x_1) < h(x_2)$.

We can normalize every lacunary series such that it satisfies this property. Suppose that $f(z)$ is an arbitrary lacunary series such that $f(0) = 0$ and $f'(0) = 1$, thus $f(z)$ is injective in a small disk of radius r centered at the origin. For $0 < x < r$, we must have $f(-x) < 0$, since otherwise the injectiveness of $f(z)$ in that disk would be violated. Let x_0 be the smallest in absolute value negative number such that $f'(x_0) = 0$. Then $f(x)$ is injective in the interval (x_0, R) and it maps this interval one to one onto the interval $(f(x_0), +\infty)$. The point x_0 is a branch point for $f(z)$ and we will designate it as *the root* of the previously described tree.

Let us notice that the same functional equation implies that $\lim_{x \to -1} h(x) = +\infty$, hence there must be a point x_0, $-1 < x_0 < 0$ such that $h'(x_0) = 0$. The point x_0 is a branch point of $h(z)$. On the other hand, a little computation shows that $h'(x)$ is a strictly increasing function, hence x_0 is the only branch point of $h(z)$ on the real diameter of the unit circle. Moreover, $h(z)$ maps one to one the interval $(x_0, 1)$ and the interval $(-1, x_0)$ onto the interval $(h(x_0), +\infty)$ and two Jordan arcs γ_1 and γ_2 connecting x_0 with points ζ_1 and ζ_2 of the unit circle onto $(-\infty, h(x_0))$. The arcs γ_1 and γ_2 are symmetric with respect to the real axis.

The functional equation can be seen also as

$$h(z^2) = h(z^4) + z^2,$$

hence

$$h(z) = h(z^4) + z + z^2,$$

and in general

$$h(z) = h(z^{2^n}) + z + z^2 + ... + z^{2^{n-1}}.$$

This last equation implies that

$$\begin{aligned}&\lim_{r\to 1} h\left(re^{\frac{2k\pi i}{2^n}}\right)\\&= \lim_{r\to 1}\left[h(r^{2^n}e^{2k\pi i}) + re^{\frac{2k\pi i}{2^n}} + ... + r^{2^{n-1}}e^{k\pi i}\right]\\&= h(1) + e^{\frac{2k\pi i}{2^n}} + ... + e^{k\pi i}\\&= \infty,\end{aligned}$$

for $k = 1, 2, ..., 2^n$, which shows that for almost every point ζ on the unit circle we have $\lim_{z\to\zeta} h(z) = \infty$, where z tends radially to ζ. The continuation over the real axis from ζ shows that the pre-image by $h(z)$ of the real axis has at least one component ending in ζ. Actually, the pre-image of the real axis has infinitely many components ending in ζ since, by the Big Picard Theorem, in every neighborhood of ζ there must be infinitely many points at which $h(z)$ takes the same real value. On some of these components $\lim_{z\to\zeta} h(z)$ is $+\infty$ and on others it is $-\infty$. They are all orthogonal curves to the pre-image of every circle centered at the origin that they meet, and obviously, to the unit circle. When a point moves on the pre-image of a circle centered at the origin between two consecutive components of the pre-image of the real axis ending in the same point ζ, its image describes half of that circle between points with the argument $k\pi$, $k = 0, 1$, hence on adjacent components of the pre-image of the real axis ending in ζ the limit $+\infty$ and $-\infty$ of $h(z)$ when $z \to \zeta$ must alternate.

To find the fundamental domains of $h(z)$ we can proceed in the following way. Consider the adjacent component of the pre-image of the real axis to the real diameter of the circle located in the upper half-disk which ends in $z = 1$ and $\lim_{z\to 1} h(z) = +\infty$ when z belongs to that component. There is a point w_1 on that component for which $h(w_1) = h(x_0)$. Let us do continuation by $h(z)$ over the image of the segment from x_0 to z_1, where $h'(z_1) = 0$, starting from w_1. This is a Jordan arc connecting w_1 with z_1 whose image by $h(z)$ is the same as the image of the segment from x_0 to z_1. Let us denote by Ω_1 the domain bounded by the two components of the pre-image of the real axis, the respective segment and this continuation arc. It can be easily seen that Ω_1 is a fundamental domain of $h(z)$, which is mapped conformally by $h(z)$ onto the complex plane with a slit alongside the interval $(h(x_0), +\infty)$ of the real axis followed by a slit from $h(x_0)$ to $h(z_1)$. The component of the pre-image of the interval $(-\infty, h(x_0))$ included in Ω_1 has the ends in $z = 1$ and $z = w_1$, hence $\lim_{z\to 1} h(z) = -\infty$, where z belongs to this component. The point w_1 is a branch point of $h(z)$ on which the two components of the pre-image of the real axis intersect each other orthogonally. The value of $h(z)$ at this branch point is real. There are infinitely many fundamental domains of $h(z)$ in the upper half disk accumulating to $z = 1$ having the same image as Ω_1, hence infinitely many branch points of $h(z)$ in which $h(z)$ has the real value $h(x_0)$.

Let us deal now with the component of the pre-image of the real axis adjacent to the real diameter of the unit circle located in the upper half disk which ends in $z = -1$ and such that $\lim_{z\to -1} h(z) = +\infty$ when z belongs to that component. There is a point w_2 on that component such that $h(w_2) = h(x_0)$. Let us do continuation by $h(z)$ over the image of the segment from x_0 to z_2,where $h'(z_2) = 0$. We obtain another Jordan arc connecting x_0 to w_2 such that the image by $h(z)$ of this arc is a slit from $h(x_0)$ to $h(z_2)$ and the domain Ω_2 bounded this arc and the two components of the pre-image of the real axis is a fundamental domain of $h(z)$ which is mapped conformally by $h(z)$ onto the complex plane with a slit alongside the interval $(h(x_0), +\infty)$ of the real axis followed by a slit from $h(x_0)$ to $h(z_2)$. There are infinitely many fundamental domains of $h(z)$ in the upper half disk accumulating to $z = -1$ having the same image as Ω_2, hence another infinity of branch points of $h(z)$ in which $h(z)$ has the real value $h(x_0)$.

We denote by Ω_3 and Ω_4 the symmetric domains of Ω_2 and Ω_1 with respect to the real axis. Obviously, they are also fundamental domains of $h(z)$. With the second layer of knots, we define similarly a second layer of fundamental domains, and so on.

A similar construction can be performed for any normalized lacunary series $f(z)$. When $f(z)$ has non real coefficients, there is no symmetry with respect to the real axis, yet the other features are the same. The geometry of conformal mapping by any lacunary series is completed by the following two theorems.

Theorem 11. To every lacunary series $f(z)$,

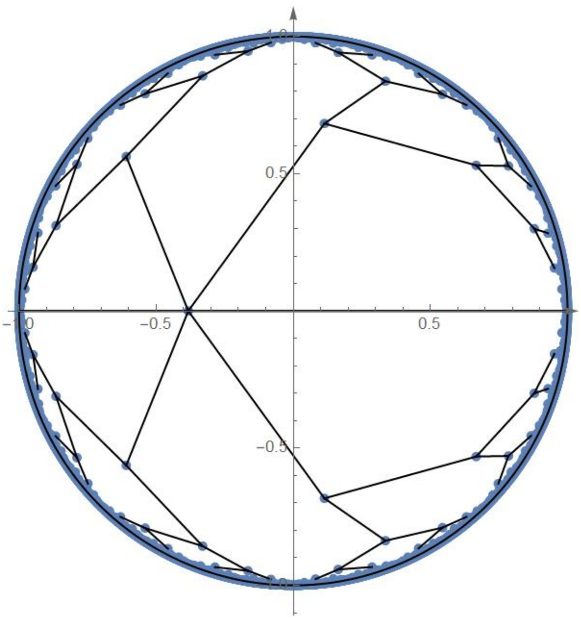


Fig. 5: An idea of infinite tree defining the fundamental domains of $h(z)$.

having the circle $|z| = R$ as natural boundary, corresponds an analytic function $g(z)$ which exists in the exterior of the that circle, having the same natural boundary and the zeros $\frac{R^2}{\overline{a_n}}$, where a_n are the zeros of $f(z)$. If the lacunary series has real coefficients the symmetric with respect to the circle $|z| = R$ of the fundamental domains of $f(z)$ are fundamental domains of $g(z)$.

Proof. Indeed, the function $g(z) = \overline{f\left(\frac{R^2}{\overline{z}}\right)}$ for $|z| > R$ satisfies all these properties. It exists obviously only for $|z| > R$, since only there we have $\left|\frac{R^2}{\overline{z}}\right| < R$, where $f\left(\frac{R^2}{\overline{z}}\right)$ exists. Moreover, $f(a_n) = 0$ if and only if $g\left(\frac{R^2}{\overline{a}_n}\right) = \overline{f(a_n)} = 0$. That $g(z)$ is an analytic function appears clearly when we replace z by $\frac{1}{\overline{z}}$ in the series of $f(z)$ and then conjugate every term. We obtain a Laurent series which converges uniformly on compact subsets of $|z| > R$, and therefore it is an analytic function. Moreover, the transformation $z \to \frac{R^2}{\overline{z}}$ carries the tree defining the fundamental domains of $f(z)$ into a tree included in $\{z \mid |z| > R\}$. The arcs of the two trees are symmetric with respect to the circle $|z| = R$, therefore they bound symmetric fundamental domains.

We notice that for $|z| > R$ we have $|g(z)| < R$ and $\lim_{z\to\infty} g(z) = 0$. Also $g\left(\frac{R^2}{x_0}\right) = f(x_0)$ and if $f'(x_0) = 0$, then $g'\left(\frac{R^2}{x_0}\right) = 0$, hence the root of the tree defined by $g(z)$ is $\frac{R^2}{x_0}$, where x_0 is the root of the tree defined by $f(z)$. □

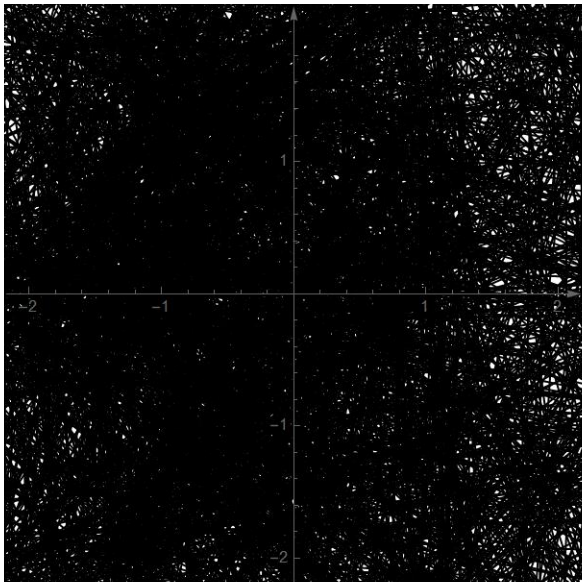


Fig. 6: Illustration of the density property for the Dirichlet function $\psi(s) = \Sigma_{n=0}^{\infty} e^{-2^n s}$ of the line $s = 0.001 + it$, $-100 \le t \le 100$

Let $f(z) = \Sigma_{n=1}^{\infty} a_n z^{\lambda_n}$ be a lacunary series having the radius of convergence R and the natural boundary the circle $|z| = R$ and let us define the Dirichlet series $\psi(s) = \Sigma_{n=1}^{\infty} a_n e^{-\lambda_n s}$ obtained from $f(z)$ by the substitution $z = e^{-s}$. Then $\psi(s)$ converges for

$$|e^{-s}| = |e^{-\sigma-it}| = e^{-\sigma} < R,$$

i.e. for $\sigma > \ln\frac{1}{R}$ and diverges for $\sigma < \ln\frac{1}{R}$. Moreover, the convergence line $s = \ln\frac{1}{R} + it$ is the natural boundary of $\psi(s)$. For the Hadamard type of series $f(z)$, we have $R = 1$ and therefore the natural boundary of the corresponding Dirichlet series $\psi(s)$ is the imaginary axis. For $h(z) = \Sigma_{n=0}^{\infty} z^{2^n}$, we have $\psi(s) = \Sigma_{n=0}^{\infty} e^{-2^n s}$, for which the abscissa of convergence is $\sigma_c = 0$. Fig. 6 illustrates the denseness property of the image by $\psi(s)$ of the line $s = 0.001 + it$.

Theorem 12 (Denseness Theorem). Let $w = f(z)$ be an analytic function with the natural boundary $|z| = r$ and let (w_m) be a sequence of points spread throughout the plane such that two neighboring points are at a distance less than ϵ one of each other for a given arbitrarily small ϵ. Then for every m_0 there is ρ, $0 < \rho < r$ such that the image by $f(z)$ of the circle $C_\rho : |z| = \rho$ passes at the distance less than ϵ of every w_m for $m \le m_0$.

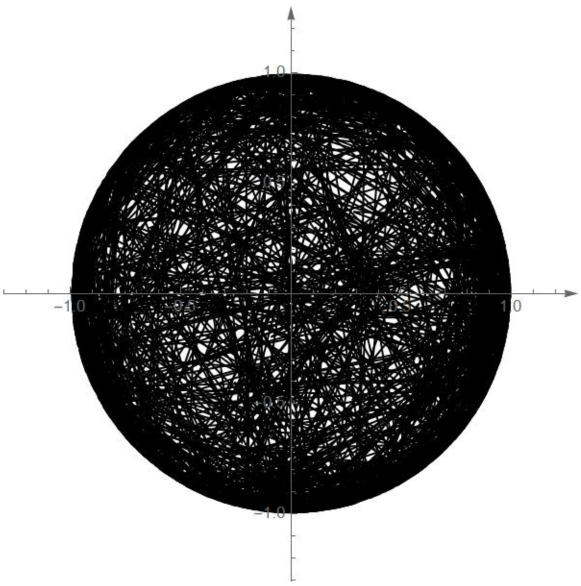


Fig. 7: An illustration of the denseness property for a Blaschke product with natural boundary. This is the image of the circle $|z| = 0.9999$ by $B(z)$.

Proof. We give a constructive proof accompanied by the illustration Fig. 4 of the case when $f(z)$ is the Hadamard function and Fig. 7 of the case when $f(z)$ is a Blaschke product $B(z)$ given by relation (1) with zeros given by relation (8).

Suppose that a mesh obtained by drawing vertical and horizontal lines with distance equal to $\frac{\epsilon}{3}$ between adjacent lines is given. The coordinate axes are supposed to belong to this mesh. Let us denote by D_m the eyes of the mesh counted following a spiral starting from the origin. They are open connected sets. We can suppose, with no restriction, that $w_m \in D_m$. Obviously, the distance between neighboring w_m is less then ϵ.

The pre-images of D_m by $f(z)$ is formed by domains $\Omega_{n,m}$, one in every fundamental domain Ω_n of $f(z)$. By Theorem 5, the domains Ω_n accumulate to every point of the circle $|z| = r$ and, obviously, so do the domains $\Omega_{n,m}$. Since $f(z)$ is injective in a neighborhood of the origin, for small values of ρ the circle C_ρ has the image only in the first four domains D_m. When ρ increases past of the second knot of the tree C_ρ will intersect a new fundamental domain Ω_2, hence $\Omega_{2,1}, ..., \Omega_{2,4}$ and in Ω_1 new domains $\Omega_{1,5}, .., \Omega_{1,16}$. Thus the image of C_ρ will pas through $D_1, D_2, ..., D_{16}$. Continuing this way, we reach D_{m_0} and the image of C_ρ passes through every D_m, $m \leq m_0$. Obviously, it is at a distance less than ϵ of every w_m, $m \leq m_0$. $\square$

6 Conclusions

The analytic functions with natural boundaries have been treated up to now as some pathological cases. No detailed study has been devoted to such functions. The purpose of this article was to highlight the fact that these functions are not so unusual as it may seem and they deserve a better place in the mainstream study of complex analytic functions. Moreover, they possess a remarkable property, which is that of denseness of the image of a line close enough to the natural boundary. This simulates chaos behavior in a context different of that which has been well studied in the literature. We are hinting to a possible application in quantum physics.

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