Matrix Transforms of α-absolutely A^{λ}-summable Sequences

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Abstract: - Let *A* be a matrix with real or complex entries, λ - a monotonically increasing strictly positive sequence, i.e., the speed of convergence, and α - a positive real number. In this paper, the notions of λ-reversibility of *A*, A^{λ}-boundedness, A^{λ}-summability and α-absolute A^{λ}-summability of sequences are recalled. Also necessary and sufficient conditions for a matrix *M* (with real or complex entries) to map the set of all α-absolutely A^{λ}-summable sequences into the set of all β-absolutely B^{μ}-summable sequences, or into the set of all B^{μ}-bounded or B^{μ}-summable sequences, if *A* is a normal or λ-reversible matrix, *B* – a lower triangular matrix, μ - another speed of convergence and $0<\alpha\le\beta\le 1$. As an application, we consider the case when *A* is the Zweier matrix $Z_{1/2}$.

Key-Words: - Matrix transforms, λ-reversibility of matrices, boundedness with speed, convergence with speed, zero-convergence with speed, summability with speed, α-absolute summability with speed, Zweier matrix.

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1 Introduction

Let X, Y be two sequence spaces and $M=(m_{nk})$ be an arbitrary matrix with real or complex entries. Throughout this paper we assume that indices and summation indices run from 0 to ∞ unless otherwise specified. If for each $x=(x_k)\in X$ the series:

$$M_n x := \sum_k m_{nk} x_k$$

converge and the sequence $Mx=(M_n x)$ belongs to Y, we say that *M* transforms *X* into *Y*. By (X,Y) we denote the set of all matrices, which transform *X* into *Y*. Let *ω* be the set of all real or complex-valued sequences. Further, we need the following well-known sub-spaces of *ω*: *c* - the space of all convergent sequences, c_0 - the space of all sequences converging to zero, l_∞ - the space of all bounded sequences, and

$$l_\alpha := \left\{ x=(x_k) : \sum_n |x_n|^{\alpha} < \infty \right\}, \alpha > 0.$$

For estimation and comparison of speeds of convergence of sequences are used different methods, see, for example, [1], [2], [3], [4], [5], [6], [7]. We use the method, introduced in [1] and [6]. Let $\lambda := (\lambda_k)$ be a positive (i.e.; $\lambda_k > 0$ for every *k*) monotonically increasing sequence. Following [6] (see also [1]), a convergent sequence $x=(x_k)$ with

$$\lim_k x_k := s \text{ and } v_k := \lambda_k (x_k - s) \tag{1}$$

is called bounded with the speed *λ* (shortly, λ-bounded) if $v_k = O(1)$ (or $(v_k)\in l_\infty$), and convergent with the speed *λ* (shortly, *λ*-convergent) if the finite limit:

$$\lim_k v_k := b$$

exists (or $(v_k)\in c$). Following [8], a convergent sequence $x=(x_k)$ with the finite limit *s* is called *α*-absolutely convergent with speed *λ* (or shortly, *α*-absolutely *λ*-convergent), if $(v_k)\in l_\alpha$. Besides, for “1-absolutely” we simply write “absolutely”. We denote the set of all *λ*-bounded sequences by l_∞^{λ}, the set of all *λ*-convergent sequences by c^{λ}, and the set of all α-absolutely λ-convergent sequences by l_α^λ. Moreover, let:

$$c_0^\lambda := \left\{ x = (x_k) : x \in c^\lambda \text{ and } \lim_k \lambda_k (x_k - s) = 0 \right\}.$$

It is not difficult to see that:

$$l_\alpha^\lambda \subset c_0^\lambda \subset c^\lambda \subset l_\infty^\lambda \subset c.$$

In addition to it, for unbounded sequence λ these inclusions are strict. For $\lambda_k = O(1)$, we get $b = 0$, and hence:

$$c^\lambda = l_\infty^\lambda = c.$$

Therefore, the most important case is $\lambda_k \neq O(1)$, because in this case relation (1) allows to evaluate the quality of convergence of converging sequences. Indeed, let x^1 and x^2 be two convergent sequences with the finite limit s. If $(v_k) \in c$ (or $v_k = O(1)$) for $x = x^1$, and $(v_k) \notin c$ (or $v_k \neq O(1)$) for $x = x^2$, then the sequence x^1 converges "better" (more precisely, faster) than sequence x^2. Thus λ, in the case $\lambda_k \neq O(1)$, measures the speed of convergence of the observed sequences.

Further we need the sequences:

$$e := (1,1,\ldots),\ e^k := (1,0,\ldots,1,0,\ldots),$$

where 1 is in the position number k. We note that

$$e, e^k \in c^\lambda, \quad e, e^k \in l_\alpha^\lambda.$$

Let $A = (a_{nk})$ be an arbitrary matrix with real or complex entries. Following [1] and [6], a sequence $x = (x_k)$ is said to be A^λ-bounded (A^λ-summable), if $Ax \in l_\infty^\lambda$ ($Ax \in c^\lambda$, respectively).

Definition 1. We say that a convergent sequence $x = (x_k)$ is α-absolutely A^λ-summable, if $Ax \in l_\alpha^\lambda$.

The set of all A^λ-bounded sequences we denote by $\left(l_\infty\right)_A^\lambda$, the set of all A^λ-summable sequences by c_A^λ, and the set of all α-absolutely A^λ-summable sequences by $\left(l_\alpha\right)_A^\lambda$. Let,

$$\left(c_0\right)_A^\lambda := \left\{ x \in c_A^\lambda : Ax \in c_0^\lambda \right\},$$

and c_A be the summability domain of A, i.e., the set of sequences x (with real or complex entries), for which the finite limit $\lim_n A_n x$ exists. It is easy to see that

$$\left(l_\alpha\right)_A^\lambda \subset \left(c_0\right)_A^\lambda \subset c_A^\lambda \subset \left(l_\infty\right)_A^\lambda \subset c_A,$$

and if λ is a bounded sequence, then

$$\left(l_\infty\right)_A^\lambda = c_A^\lambda = \left(c_0\right)_A^\lambda = c_A.$$

Matrix transformations, and boundedness and convergence with speed are widely used in approximation theory to transform non-convergent sequences into convergent ones, or to transform convergent sequences into “better” convergent sequences, [5], [9], [10], [11]. An interesting task would also be to accelerate converging processes in wavelet and spline approximation, such as, for example in works [12], [13], [14], [15] and [16]. Besides, in [1] matrix transformations and boundedness with speed are used for the estimation of the order of approximation of Fourier expansions in Banach spaces by one author of the present paper.

In general, the problems of improvement of the quality of convergence of sequences by matrix transformations have been studied by several authors for example, [1], [17], [18], [19], [20], [21], [22], [23] and [24]. Moreover, in [23] and [24], the λ-convergence and the λ-boundedness in abstract spaces, considering instead of a matrix with real or complex entries a matrix, whose elements are bounded linear operators from a Banach space X into a Banach space Y, have been studied. We note that the results connected with convergence, absolute convergence, α-absolute convergence, and boundedness with speed can be used in several applications. For example, in theoretical physics, such results can be used for accelerating the slowly convergent processes, a good overview of such investigations can be found, for example, from the sources [25] and [26].

Let further $B = (b_{nk})$ a lower triangular matrix with real or complex entries, and $A = (a_{nk})$ a normal or λ-reversible matrix. We recall that A is said to be normal, if it is lower triangular and $a_{nn} \neq 0$ for every n, and λ-reversible, if the infinite system of equations $z_n = A_n x$ has a unique solution for every sequence $(z_n) \in c^\lambda$. Matrices A and B are called M-consistent on a sequence space X, if the transformation $y = Mx$ exist and

$$\lim_n B_n(Mx) = \lim_n A_n x$$

for each $x \in X$, [1]. If $M = (\delta_{nk})$ (where $\delta_{nk} = 1$ if $n = k$ and $\delta_{nk} = 0$ if $n \neq k$), that is, $Mx = x$ for each $x \in X$, then M-consistency of A and B on X is reduced to the ordinary consistency of A and B on X, [1].

Let $\mu := (\mu_k)$ be another speed of convergence, i.e., a monotonically increasing positive sequence. Matrix transforms from *X* into *Y*, where *X* is one of the sets c^λ, c_0^λ, l_∞^λ, l_1^λ, c_A^λ, $(c_0)_A^\lambda$, $(l_\infty)_A^\lambda$ or $(l_1)_A^\lambda$, and *Y* one of the sets c^μ, c_0^μ, l_∞^μ, l_α^μ $(\alpha > 1)$, c_B^μ, $(c_0)_B^\mu$, $(l_\infty)_B^\mu$ or $(l_1)_B^\mu$ have been studied in [6], and by the authors of the present paper in several works, for example we mention only [1] and [8].

The present paper is the continuation of [8]. We remember that in [8] are given the characterization for the sets $(l_\alpha^\lambda, l_\infty^\mu)$, $(l_\alpha^\lambda, c^\mu)$, $(l_\alpha^\lambda, c_0^\mu)$ if $0 < \alpha \le 1$, and for the set $(l_\alpha^\lambda, l_\beta^\mu)$ if $0 < \alpha \le \beta \le 1$. In this work we prove necessary and sufficient conditions for $M \in ((l_\alpha)_A^\lambda, (l_\infty)_B^\mu)$, $M \in ((l_\alpha)_A^\lambda, c_B^\mu)$, $M \in ((l_\alpha)_A^\lambda, (c_0)_B^\mu)$ $(0 < \alpha \le 1)$, and for $M \in ((l_\alpha)_A^\lambda, (l_\beta)_B^\mu)$ $(0 < \alpha \le \beta \le 1)$, if *A* is a normal or λ-reversible matrix and *B* – a triangular matrix. Also, we prove necessary and sufficient conditions for *A* and *B* to be *M*-consistent on $(l_\varepsilon)_A^\lambda$ if *A* is a λ-reversible matrix. As an application of the main results, we consider the case if *A* is the Zweier matrix $Z_{1/2}$.

2 Auxiliary Results

In this section we present the known results from the summability theory, which are necessary for the proof of the main results of the present paper. In [8] are presented the following lemmas (see also the references therein).

Lemma 1. *A matrix* $A = (a_{nk}) \in (l_\alpha, l_\infty)$ *for* $0 < \alpha \le 1$ *if and only if*

$$a_{nk} = O(1). \tag{2}$$

Lemma 2. *A matrix* $A = (a_{nk}) \in (l_\alpha, c)$ *for* $0 < \alpha \le 1$ *if and only if condition* (2) *is satisfied, and there exists finite limits* $\lim_m a_{nk} := a_k$. (3)

Moreover,

$$\lim_n A_n x = \sum_k a_k x_k$$

for every $x = (x_k) \in l_\alpha$.

Lemma 3. *A matrix* $A = (a_{nk}) \in (l_\alpha, c_0)$ *for* $0 < \alpha \le 1$ *if and only if condition* (2) *and condition* (3) *with* $a_k = 0$ *are satisfied.*

Lemma 4. *A matrix* $A = (a_{nk}) \in (l_\alpha, l_\beta)$ *for* $0 < \alpha \le \beta \le 1$ *if and only if*

$$\sum_n |a_{nk}|^\beta = O(1).$$

Moreover, if $0 < \alpha < \beta \le 1$, *then* $A \in (l_\alpha, l_\beta)$ *if and only if* $A \in (l_\beta, l_\beta)$.

Let $A = (a_{nk})$ be a λ-reversible matrix. For $x \in (l_\alpha)_A^\lambda$, let

$$\phi := \lim_n A_n x,\ d_n := \lambda_n (A_n x - \phi),\ d := \lim_n d_n,$$

and $\eta := (\eta_k)$, $\eta^j := (\eta_{kj})$, for each fixed *j*, are the solutions of the system $y = Ax$ corresponding to $y = (\delta_{nn})$ and $y = y^j = (\delta_{nj})$. As $(l_\alpha)_A^\lambda \subset c_A^\lambda$ and $\lim_n d_n = 0$ for all $x \in (l_\alpha)_A^\lambda$, then from Corollary 9.1 of [1] we get the following result.

Lemma 5 *Let* $A = (a_{nk})$ *be a* λ*-reversible matrix. Every coordinate* x_k *of a sequence* $x = (x_k) \in (l_\alpha)_A^\lambda$ *can be represented in the form*

$$x_k = \phi \eta_k + \sum_n \frac{\eta_{kn}}{\lambda_n} d_n, \quad \sum_n \left| \frac{\eta_{kn}}{\lambda_n} \right| < \infty \tag{4}$$

for every fixed k.

As an application of the main results, we consider the case if *A* is the Zweier matrix $Z_{1/2}$, defined by $A = (a_{nk})$, where, [27], $a_{00} = 1/2$ and

$$a_{nk} = \begin{cases} 1/2, \text{if } k = n-1 \text{ and } k = n, \\ 0, \text{ if } k < n-1 \text{ and } k > n \end{cases}$$

for $n \ge 1$. It is easy to calculate that the inverse matrix of $Z_{1/2}$ is the lower triangular matrix $A^{-1} = (\eta_{nk})$ with

$$\eta_{nk} = 2(-1)^{n-k}. \tag{5}$$

In examples we later use also the factorable matrix $M^f = (m_{nk})$, defined by

$$m_{nk} = t_n s_k, \tag{6}$$

where (t_n) and (s_k) are sequences of numbers.

3 Matrix Transforms for λ-reversible Matrices

In this section we characterize the sets $((l_\alpha)_A^\lambda, (l_\infty)_B^\mu)$, $((l_\alpha)_A^\lambda, c_B^\mu)$, $((l_\alpha)_A^\lambda, (c_0)_B^\mu)$ and

$\left((l_\alpha)_A^\lambda,(l_\beta)_B^\mu\right)$ for the case $0<\alpha\le\beta\le1$, if A is a λ-reversible matrix and B – a triangular matrix. First, we present necessary and sufficient conditions for existence of the transformation $y=Mx$ for every $x\in\left(l_\alpha\right)_A^\lambda$.

Proposition 1. *Let $A=(a_{nk})$ be a λ-reversible matrix, $M=(m_{nk})$ an arbitrary matrix and $0<\alpha\le1$. Then the transformation $y=Mx$ exists for every $x\in\left(l_\alpha\right)_A^\lambda$ if and only if there exist finite limits*

$$\lim_j h_{jl}^n := h_{nl}, \tag{7}$$

$$\frac{\left|h_{jl}^n\right|}{\lambda_l}=O_n(1) \text{ for every fixed } n, \tag{8}$$

$$\sum_k m_{nk}\eta_k<\infty \text{ for every fixed } n, \tag{9}$$

where

$$h_{jl}^n:=\sum_{k=0}^j m_{nk}\eta_{kl}.$$

Proof. Necessity. Assume that the transformation $y=Mx$ exists for every $x\in\left(l_\alpha\right)_A^\lambda$. By Lemma 5, every coordinate x_k of a sequence $x=(x_k)\in\left(l_\alpha\right)_A^\lambda$ can be represented in the form (4) for every fixed *k*. Hence, we can write

$$\sum_{k=0}^j m_{nk}x_k=\phi\sum_{k=0}^j m_{nk}\eta_k+\sum_{l=0}^j\frac{h_{jl}^n}{\lambda_l}d_l \tag{10}$$

for every sequence $x\in\left(l_\alpha\right)_A^\lambda$. It is easy to see that $\eta\in\left(l_\alpha\right)_A^\lambda$, since $e\in l_\alpha^\lambda$ and A is λ-reversible. Consequently condition (9) holds.

Using (10), we obtain that the matrix $H_\lambda^n:=\left(h_{jl}^n/\lambda_l\right)$ for every *n* transforms this sequence $(d_l)\in l_\alpha$ into *c*. We show that H_λ^n transforms every sequence $(d_l)\in l_\alpha$ into *c*. Indeed, for every sequence $(d_l)\in l_\alpha$, the sequence $(d_l/\lambda_l)\in l_\alpha$. But, for $(d_l/\lambda_l)\in l_\alpha$, there exists a convergent sequence $z=(z_l)$ with $\phi=\lim_{l\to\infty}z_l$, such that $d_l/\lambda_l=z_l-\phi$. Due to λ-reversibility of *A* for every convergent sequence $z=(z_l)$ with $\phi=\lim_{l\to\infty}z_l$ there exists a convergent sequence *x*, such that $z_l=A_lx$. Thus, we have proved that, for every sequence $(d_l)\in l_\alpha$ there exists a sequence $(x_k)\in\left(l_\alpha\right)_A^\lambda$ such that $d_l=\lambda_l(A_lx-\phi)$. Hence $H_\lambda^n\in(l_\alpha,c)$. Therefore, by Lemma 2, conditions (7) and (8) are satisfied.

Sufficiency. Let all conditions of the present proposition be satisfied. Then conditions (7) and (8) imply, by Lemma 2, that $H_\lambda^n\in(l_\alpha,c)$. Consequently, from (10), we can conclude, by (9) that the transformation $y=Mx$ exists for every $x\in\left(l_\alpha\right)_A^\lambda$.

As the validity of (7) and (8) implies the validity of

$$\frac{\left|h_{nl}\right|}{\lambda_l}=O_n(1) \text{ for every fixed } n, \tag{11}$$

then from Proposition 1 we immediately obtain the following result.

Corollary 1. Let $A=(a_{nk})$ *be a λ-reversible matrix and $M=(m_{nk})$ an arbitrary matrix* If the transformation $y=Mx$ exists for every $x\in\left(l_\alpha\right)_A^\lambda$, then condition (11) holds.

To formulate the main result of this section, we use the matrix $G=(g_{nk})=BM;$ i.e.,

$$g_{nk}:=\sum_{l=0}^n b_{nl}m_{lk},$$

and lower triangular matrices $\Gamma^n:=\left(\gamma_{nl}^j\right)$, where

$$\gamma_{nl}^j:=\sum_{k=0}^j g_{nk}\eta_{kl}.$$

If the matrix transformation $y=Mx$ exists for every $x\in\left(l_\alpha\right)_A^\lambda$, then the finite limits

$$\gamma_{nl}:=\lim_j\gamma_{nl}^j$$

exist.

Theorem 1. *Let $A=(a_{nk})$ be a λ-reversible matrix, $B=(b_{nk})$ a triangular matrix, $M=(m_{nk})$ an arbitrary matrix, and $0<\alpha\le\beta\le1$. Then $M\in\left(\left(l_\alpha\right)_A^\lambda,\left(l_\beta\right)_B^\mu\right)$ if and only if conditions* (7) – (9) *are satisfied, there exist finite limits:*

$$\lim_j\gamma_{nl}:=\gamma_l, \tag{12}$$

$$\frac{\left|\gamma_{nl}\right|}{\lambda_l}=O(1), \tag{13}$$

$$\frac{1}{\lambda_l^\beta}\sum_n\mu_n^\beta\left|\gamma_{nl}-\gamma_l\right|^\beta=O(1), \tag{14}$$

$$\eta \in \left(l_\beta\right)_G^\mu . \tag{15}$$

Proof. Necessity. Suppose that $M \in \left(\left(l_\alpha\right)_A^\lambda, \left(l_\beta\right)_B^\mu\right)$. Then the transformation $y = Mx$ exists for every $x \in \left(l_\alpha\right)_A^\lambda$. Hence conditions (7) - (9) hold by Proposition 1, and

$$B_n y = G_n x \tag{16}$$

for every $x \in \left(l_\alpha\right)_A^\lambda$, because the change of the order of summation is allowed by the lower triangularity of *B*. This implies that $\left(l_\alpha\right)_A^\lambda \subset \left(l_\beta\right)_G^\mu$. Hence condition (15) is satisfied because $\eta \in \left(l_\alpha\right)_A^\lambda$. As every element x_k of a sequence $x = (x_k) \in \left(l_\alpha\right)_A^\lambda$ may be represented in the form (4) by Lemma 5, we can write

$$\sum_{k=0}^{j} g_{nk} x_k = \phi \sum_{k=0}^{j} g_{nk} \eta_k + \sum_{l=0}^{j} \frac{\gamma_{nl}^j}{\lambda_l} d_l \tag{17}$$

for every $x \in \left(l_\alpha\right)_A^\lambda$. From (17) it follows, by (15), that $\Gamma_\lambda^n := \left(\gamma_{nl}^j / \lambda_l\right) \in (l_\alpha, c)$ for every *n* because *A* is λ-reversible (see the proof of necessity of Proposition 1). Moreover, from conditions (7) - (9), we can conclude that the series $G_n x$ are convergent for every $x \in \left(l_\alpha\right)_A^\lambda$. Therefore there exist the finite limits γ_{nl}, and from (17) we obtain

$$G_n x = \phi G_n \eta + \sum_{l=0}^{j} \frac{\gamma_{nl}}{\lambda_l} d_l \tag{18}$$

for every $x \in \left(l_\alpha\right)_A^\lambda$ by Lemma 2. From (18) we see, with the help of (15) that $\Gamma_\lambda := \left(\gamma_{nl} / \lambda_l\right) \in (l_\alpha, c)$. Consequently, by Lemma 2 we obtain that conditions (12) and (13) hold, and

$$\lim_n G_n x = \phi\gamma + \sum_{l=0}^{j} \frac{\gamma_l}{\lambda_l} d_l \tag{19}$$

for every $x \in \left(l_\alpha\right)_A^\lambda$, where

$$\gamma := \lim_n G_n \eta.$$

Therefore, we can write:

$$\mu_n (G_n x - \lim_n G_n x) = \phi \mu_n (G_n \eta - \gamma) + \mu_n \sum_l \frac{\gamma_{nl} - \gamma_l}{\lambda_l} d_l \tag{20}$$

for every $x \in \left(l_\alpha\right)_A^\lambda$. With the help of (15), it follows from (20) that the matrix

$$\Gamma_{\lambda,\mu} := \left(\mu_n (\gamma_{nl} - \gamma_l) / \lambda_l\right) \in (l_\alpha, l_\beta).$$

Hence, using Lemma 4, we conclude that condition (14) holds.

Sufficiency. Assume that all conditions of Theorem 1 are satisfied. Then the transformation $y = Mx$ exists for every $x \in \left(l_\alpha\right)_A^\lambda$ by Proposition 1. This implies that relations (16) and (17) hold for every $x \in \left(l_\alpha\right)_A^\lambda$ (see the proof of necessity of the present theorem). Using (12) and (13), we conclude, with the help of Lemma 2, that $\Gamma_\lambda \in (l_\alpha, c)$, one can take the limit under the summation sign in the last summand of (17). Then, from (17), we obtain by (15), the validity of (18) for every $x \in \left(l_\alpha\right)_A^\lambda$. Conditions (12), (13) and (15) imply that (19) holds for every $x \in \left(l_\alpha\right)_A^\lambda$, due to Lemma 2. Then relation (20) also holds for every $x \in \left(l_\alpha\right)_A^\lambda$. Moreover, $\Gamma_{\lambda,\mu} \in (l_\alpha, l_\beta)$ by Lemma 4. Therefore $M \in \left(\left(l_\alpha\right)_A^\lambda, \left(l_\beta\right)_B^\mu\right)$ by (15).

From Theorem 1 we, with the help of Lemma 4, immediately obtain the following result.

Corollary 2. *Let $A = (a_{nk})$ be a λ-reversible matrix, $B = (b_{nk})$ a triangular matrix, $M = (m_{nk})$ an arbitrary matrix, and $0 < \alpha < \beta \le 1$. Then $M \in \left(\left(l_\alpha\right)_A^\lambda, \left(l_\beta\right)_B^\mu\right)$ if and only if $M \in \left(\left(l_\beta\right)_A^\lambda, \left(l_\beta\right)_B^\mu\right)$.*

Now we consider necessary and sufficient conditions for *A* and *B* to be *M*-consistent on $\left(l_\alpha\right)_A^\lambda$.

Theorem 2. *Let $A = (a_{nk})$ be a λ-reversible matrix, $B = (b_{nk})$ a triangular matrix, $M = (m_{nk})$ an arbitrary matrix, and $0 < \alpha \le 1$.* Then *A* and *B* are *M*-consistent on $\left(l_\alpha\right)_A^\lambda$ *if and only if conditions* (7)-(9), (12) with $\gamma_l = 0$, *and* (13) *are satisfied and* $\gamma = 1$.

Proof. Necessity. Assume that *A* and *B* are *M*-consistent on $\left(l_\alpha\right)_A^\lambda$. Then

$$\lim_n G_n x = \lim_n A_n x \tag{21}$$

for every $x \in \left(l_\alpha\right)_A^\lambda$ by (16), and $M \in \left(\left(l_\alpha\right)_A^\lambda, c_B\right)$.

Therefore conditions (7) - (9) are satisfied, relation (18) holds for every $x\in\left(l_\alpha\right)_A^\lambda$ and the series $G_n\eta$ is convergent for each *n*. As $A\eta=e$ and $e\in l_\alpha^\lambda$, then

$$\lim_n G_n\eta=\lim_n A_n\eta=1$$

by (21); i.e., $\gamma=1$. From (18) we conclude $\Gamma_\lambda\in(l_\alpha,c_0)$. Hence, by Lemma 3, we obtain that condition (12), with $\gamma_l=0$, and condition (13) hold.

Sufficiency. Assume that all the conditions of the present theorem are satisfied. Then relation (18) holds for every $x\in\left(l_\alpha\right)_A^\lambda$ (see the proof of Theorem 1). Hence *A* and *B* are *M*-consistent on $\left(l_\alpha\right)_A^\lambda$, because $\Gamma_\lambda\in(l_\alpha,c_0)$ by Lemma 3.

Theorem 3. *Let* $A=(a_{nk})$ *be a* λ*-reversible matrix,* $B=(b_{nk})$ *a triangular matrix,* $M=(m_{nk})$ *an arbitrary matrix, and* $0<\alpha\le 1$. *Then* $M\in\left(\left(l_\alpha\right)_A^\lambda,c_B^\mu\right)$ *if and only if conditions* (7) – (9), (12), (13) *are satisfied,* $\eta\in c_G^\mu$. *and*

$$\frac{\left|\mu_n(\gamma_{nl}-\gamma_l)\right|}{\lambda_l}=O(1), \tag{22}$$

There exists finite limits:

$$\lim_n \mu_n(\gamma_{nl}-\gamma_l):=r_l, \tag{23}$$

Theorem 4. *Let* $A=(a_{nk})$ *be a* λ*-reversible matrix,* $B=(b_{nk})$ *a triangular matrix,* $M=(m_{nk})$ *an arbitrary matrix, and* $0<\alpha\le 1$. *Then* $M\in\left(\left(l_\alpha\right)_A^\lambda,(c_0)_B^\mu\right)$ *if and only if conditions* (7) – (9), (12), (13), (22) and (23) *with* $r_l=0$ *are satisfied, and* $\eta\in(c_0)_G^\mu$.

Theorem 5. *Let* $A=(a_{nk})$ *be a* λ*-reversible matrix,* $B=(b_{nk})$ *a triangular matrix,* $M=(m_{nk})$ *an arbitrary matrix, and* $0<\alpha\le 1$. *Then* $M\in\left(\left(l_\alpha\right)_A^\lambda,(l_\infty)_B^\mu\right)$ *if and only if conditions* (7) – (9), (12), (13), (22) *are satisfied, and* $\eta\in(l_\infty)_G^\mu$.

As the proofs of Theorems 3 - 5 are similar to the proof of Theorem 1, then we only give a short outline of the proofs. In these results always every element x_k of a sequence $x=(x_k)\in\left(l_\alpha\right)_A^\lambda$ may be presented in the form (4) by Lemma 5. Hence for all cases conditions (7) – (9) are necessary and sufficient for existence of $y=Mx$, and relations (16) - (20) hold with $\Gamma_\lambda^n\in(l_\alpha,c)$ and $\Gamma_\lambda\in(l_\alpha,c)$. But the role of the matrix $\Gamma_{\lambda,\mu}$ μ is different: in the proof of Theorem 3, $\Gamma_{\lambda,\mu}\in(l_\alpha,c)$, in the proof of Theorem 4, $\Gamma_{\lambda,\mu}\in(l_\alpha,c_0)$, and in the proof of Theorem 5, $\Gamma_{\lambda,\mu}\in(l_\alpha,l_\infty)$. Therefore, instead of Lemma 4, for completing the proof of Theorem 3 it is necessary to use Lemma 2, for completing the proof of Theorem 4 - Lemma 3, and for completing the proof of Theorem 5 - Lemma 1.

Corollary 3. *Condition* (13) *can be replaced by condition*

$$\frac{|\gamma_l|}{\lambda_l}=O(1) \tag{24}$$

In Theorems 1 - 5.

Proof. It is easy to see that condition (24) follows from (12) and (13). Vice versa, conditions (12), (24), and condition (14) (or condition (22)) imply the validity of (13). Indeed, from condition (14) (or from (22)) we obtain that

$$\frac{|\gamma_{nl}-\gamma_l|}{\lambda_l}=O(1), \tag{25}$$

since (μ_n) is bounded from below. As,

$$\frac{\gamma_{nl}}{\lambda_l}=\frac{\gamma_{nl}-\gamma_l}{\lambda_l}+\frac{\gamma_l}{\lambda_l},$$

then condition (13) is satisfied by (24) and (25).

4 The Case of Normal Matrices

In this section we characterize the sets $\left((l_\alpha)_A^\lambda,(l_\infty)_B^\mu\right)$, $\left((l_\alpha)_A^\lambda,c_B^\mu\right)$, $\left((l_\alpha)_A^\lambda,(c_0)_B^\mu\right)$ and $\left((l_\alpha)_A^\lambda,(l_\beta)_B^\mu\right)$ for the case $0<\alpha\le\beta\le 1$, if *A* is a normal matrix. As in previous section, let *M* be an arbitrary matrix and *B* a triangular matrix with real or complex entries. Let $A^{-1}:=\left(\eta_{nk}\right)$ be the inverse matrix of *A*. It is known, [1], that between A^{-1} and η_k the relationship:

$$\eta_k=\sum_{l=0}^k\eta_{kl} \tag{26}$$

holds (in this case A^{-1} is also normal). Therefore, in the present case $H^n:=\left(h_{jl}^n\right)$ and $\Gamma^n:=\left(\gamma_{nl}^j\right)$ are triangle matrices with

$$h_{jl}^n=\sum_{k=l}^{j} m_{nk}\eta_{kl},\ \ \gamma_{nl}^j:=\sum_{k=l}^{j} g_{nk}\eta_{kl},\ l\le j.$$

We get the following analogy of Theorem 1 for normal matrix *A*.

Theorem 6. *Let* $0<\alpha\le\beta\le 1$. *Then* $M\in\left((l_\alpha)_A^\lambda,(l_\beta)_B^\mu\right)$ *if and only if conditions* (7), (8), (12) – (14) *are satisfied, and*

$$\lim_j \sum_{l=0}^{j} h_{jl}^n \text{ exists and finite,} \tag{27}$$

$$(\rho_n)\in l_\beta^\mu,\ \ \rho_n:=\lim_j \sum_{l=0}^{j}\gamma_{nl}^j. \tag{28}$$

Proof. Using relation (26) we obtain

$$\sum_{l=0}^{j} m_{nl}\eta_l=\sum_{l=0}^{j} h_{jl}^n,\ \ \sum_{l=0}^{j} g_{nl}\eta_l=\sum_{l=0}^{j}\gamma_{nl}^j.$$

Hence condition (27) is equivalent to (9), and condition (28) to (15). Thus, $M\in\left((l_\alpha)_A^\lambda,(l_\beta)_B^\mu\right)$ by Theorem 1.

Similarly to Theorem 6 for a normal matrix *A* it is possible to obtain analogies of Theorems 3 -5. As the proofs of these theorems are similar to the proof of Theorem 6, we omit them.

Theorem 7. *Let* $0<\alpha\le 1$. *Then* $M\in\left((l_\alpha)_A^\lambda, c_B^\mu\right)$ *if and only if conditions* (7), (8), (12), (13), (22), (23), (27) *are satisfied, and*

$$(\rho_n)\in c^\mu,\ \rho_n:=\lim_j \sum_{l=0}^{j}\gamma_{nl}^j. \tag{29}$$

Theorem 8. *Let* $0<\alpha\le 1$. *Then* $M\in\left((l_\alpha)_A^\lambda,(c_0)_B^\mu\right)$ *if and only if conditions* (7), (8), (12), (13), (22), (23) *with* $r_l=0$, *and* (27) *are satisfied, and*

$$(\rho_n)\in c_0^\mu,\ \ \rho_n:=\lim_j \sum_{l=0}^{j}\gamma_{nl}^j. \tag{30}$$

Theorem 9. *Let* $0<\alpha\le 1$. *Then* $M\in\left((l_\alpha)_A^\lambda,(l_\infty)_B^\mu\right)$ *if and only if conditions* (7), (8), (12), (13), (22), (27) *are satisfied, and*

$$(\rho_n)\in l_\infty^\mu,\ \ \rho_n:=\lim_j \sum_{l=0}^{j}\gamma_{nl}^j. \tag{31}$$

Let now $A=(a_{nk})$ be a normal matrix satisfying the property $a_{n0}=1$ for every *n*. Then $\eta_k=\delta_{k0}$ by Lemma 7.3 of [1]. Hence, with the help of (26) we obtain

$$\sum_{l=0}^{j} h_{jl}^n=m_{n0} \text{ and } \sum_{l=0}^{j}\gamma_{nl}^j=g_{n0}.$$

Therefore, we immediately get from Theorems 6-9 the following corollary.

Corollary 4. *Let* $A=(a_{nk})$ *be a normal matrix satisfying the property* $a_{n0}=1$ *for every n. Then condition* (27) *is redundant in Theorems* 6-9, and *condition* (28) *can be replaced by condition* $e^0\in(l_\beta)_G^\mu$ *in Theorem* 6, *condition* (29) *by condition* $e^0\in c_G^\mu$ *in Theorem* 7, *condition* (30) *by condition* $e^0\in(c_0)_G^\mu$ *in Theorem* 8, *and condition* (31) by *condition* $e^0\in(l_\infty)_G^\mu$ *in Theorem* 9.

In the following we apply Theorems 6-9 for the case if *A* is the Zweier matrix $Z_{1/2}$. Using (5) we obtain that

$$h_{jl}^n=2\sum_{k=l}^{j}(-1)^{k-l}m_{nk},\ \gamma_{nl}^j=2\sum_{k=l}^{j}(-1)^{k-l}g_{nk},\ l\le j, \tag{32}$$

and

$$h_{nl}=2\sum_{k=l}^{\infty}(-1)^{k-l}m_{nk},\ \gamma_{nl}=2\sum_{k=l}^{\infty}(-1)^{k-l}g_{nk}, \tag{33}$$

if the series in the right-hand side of last equalities are convergent. Moreover,

$$\sum_{l=0}^{j} h_{jl}^n=\begin{cases}2(m_{n0}+m_{n2}+\dots+m_{nj}), & \text{if } j \text{ is even number,}\\ 2(m_{n0}+m_{n2}+\dots+m_{n,j-1}), & \text{if } j \text{ is odd number,}\end{cases}$$

$$\sum_{l=0}^{j}\gamma_{nl}^j=\begin{cases}2(g_{n0}+g_{n2}+\dots+g_{nj}), & \text{if } j \text{ is even number,}\\ 2(g_{n0}+g_{n2}+\dots+g_{n,j-1}), & \text{if } j \text{ is odd number.}\end{cases}$$

Therefore, for the validity of condition (7) it is necessary and sufficient that the series

$$\sum_k (-1)^k m_{nk} \text{ are convergent for every } n, \tag{34}$$

and for the validity of condition (27) it is necessary and sufficient that the series

$$\sum_k m_{n,2k} \text{ are convergent for every } n. \tag{35}$$

Hence from (32) and (34) follows that $\left|h_{jl}^n\right|=O_n(1)$ for every fixed *n*, and this implies the validity of condition (6), since λ is bounded from below. Thus, from (34) and (35) follows the validity of all conditions of Proposition 1. Therefore (34) and (35) imply also the existence of finite limits γ_{nl} and

$$\rho_n = \sum_k g_{n,2k}. \qquad (36)$$

Consequently, we obtain the following corollaries from Theorems 6-9.

Corollary 5. *Let* $0<\alpha\le\beta\le 1$. *Then* $M\in\left(\left(l_\alpha\right)^\lambda_{Z_{1/2}},\left(l_\beta\right)^\mu_B\right)$ *if and only if conditions* (34), (35), *and conditions* (12) – (14), (28) *are satisfied, where* γ_{nl} *and* ρ_n *are determined correspondingly by* (33) *and* (36).

Corollary 6. *Let* $0<\alpha\le 1$. *Then* $M\in\left(\left(l_\alpha\right)^\lambda_{Z_{1/2}},c^\mu_B\right)$ *if and only if conditions* (34), (35), *and conditions* (12), (13), (22), (23), (29) *are satisfied, where* γ_{nl} *and* ρ_n *are determined correspondingly by* (33) *and* (36).

Corollary 7. *Let* $0<\alpha\le 1$. *Then* $M\in\left(\left(l_\alpha\right)^\lambda_{Z_{1/2}},(c_0)^\mu_B\right)$ *if and only if conditions* (34), (35), *and conditions* (12), (13), (22), (23) *with* $r_l=0$ *, and* (30) *are satisfied, where* γ_{nl} *and* ρ_n *are determined correspondingly by* (33) *and* (36).

Corollary 8. *Let* $0<\alpha\le 1$. *Then* $M\in\left(\left(l_\alpha\right)^\lambda_{Z_{1/2}},(l_\infty)^\mu_B\right)$ *if and only if conditions* (34), (35), *and conditions* (12), (13), (22), (31) *are satisfied, where* γ_{nl} *and* ρ_n *are determined correspondingly by* (33) *and* (36).

As an application of Corollaries 5 – 8, we present some examples, where *M* is the factorable matrix M^f defined by (6).

Example 1. Let $0<\alpha\le\beta\le 1$. We prove that $M^f\in\left(\left(l_\alpha\right)^\lambda_{Z_{1/2}},\left(l_\beta\right)^\mu_B\right)$ for an arbitrary speed λ if and only

if the series $S_1 := \sum_k (-1)^k s_k$ is convergent, (37)

the series $S_2 := \sum_k s_{2k}$ is convergent, (38)

there exist finite limit $\lim_n B_n t := K,\ t:=(t_n)$, (39)

$$\sum_n \left|\mu_n\left(B_n t - K\right)\right|^\beta < \infty. \qquad (40)$$

For the proof of this assertion, it is sufficient to show that conditions (37) – (40) are equivalent to all conditions of Corollary 5. First, we see those conditions (37) and (38) are correspondingly equivalent to conditions (34) and (35), since:

$$\sum_k (-1)^k m_{nk} = S_1 t_n$$

and

$$\sum_k m_{n,2k} = S_2 t_n.$$

As

$$g_{nk} = s_k B_n t,$$

$$\gamma^j_{nl} = 2B_n t \sum_{k=l}^{j} (-1)^{k-l} s_k,$$

then

$$\gamma_{nl} = 2S_l B_n t,$$

where the series

$$S_l := \sum_{k=l}^{\infty} (-1)^{k-l} s_k$$

is convergent for every *l* by (37). Hence

$$\gamma_l = \lim_{n\to\infty}\gamma_{nl} = 2KS_l,$$

so condition (39) is equivalent to (12). As λ is bounded from below and $|S_l| = O(1)$ by (37), then condition (13) is also holds by (39).

Condition (40) is equivalent to (28). Indeed, using (36) and (39), we obtain

$$\rho_n = \sum_k g_{n,2k} = S_2 B_n t \ \text{ and } \lim_n \rho_n == S_2 K.$$

Therefore, we can write

$$\mu_n\left(\rho_n - \lim_{x\to\infty}\rho_n\right) = S_2\mu_n\left(B_n t - K\right) \qquad (41)$$

and

$$\sum_n \left|\mu_n\left(\rho_n - \lim_{x\to\infty}\rho_n\right)\right|^\beta = |S_2|^\beta \sum_n \left|\mu_n\left(B_n t - K\right)\right|^\beta.$$

This implies that condition (40) is equivalent to (28).

As

$$\gamma_{nl} - \gamma_l = 2S_l(B_n t - K), \qquad (42)$$

then

$$\frac{1}{\lambda_l^\beta}\sum_n \mu_n^\beta \left|\gamma_{nl}-\gamma_l\right|^\beta = \left(\frac{|2S_l|}{\lambda_l}\right)^\beta \sum_n \left|\mu_n\left(B_n t - K\right)\right|^\beta.$$

Hence condition (14) holds by (40), since:

$$\left(\frac{|2S_l|}{\lambda_l}\right)^\beta = O(1)$$

by (37) and lower boundedness of λ. Thus, conditions (37) – (40) are equivalent to all conditions of Corollary 5.

Example 2. Let $0<\alpha\le 1$. Then $M^f \in \left(\left(l_\alpha\right)^{\lambda}_{Z_{1/2}}, c^{\mu}_B\right)$ for an arbitrary speed λ if and only if conditions (37)–(39) hold and there exist finite limit:

$$\lim_j \mu_n\left(B_n t-K\right):=R. \quad (43)$$

For the proof of this assertion, it is sufficient to show that conditions (37) – (39) and (43) are equivalent to all conditions of Corollary 6. First, conditions (37) – (39) are equivalent to conditions (34), (35), (12) and (13) (see Exercise 1). Using (42), we get that condition (43) is equivalent to condition (23). As:

$$\frac{\left|2S_l\right|}{\lambda_l}=O(1)$$

by (37) and lower boundedness of λ, then condition (22) holds by (43). Finally, condition (29) holds by (38) and (41).

Example 3. Let $0<\alpha\le 1$. Then $M^f \in \left(\left(l_\alpha\right)^{\lambda}_{Z_{1/2}}, \left(c_0\right)^{\mu}_B\right)$ for an arbitrary speed λ if and only if conditions (37) – (39), and condition (43) with $R=0$ are satisfied. Indeed, similarly, to Exercise 2 it is possible to show that conditions (37) – (39), and condition (43) with $R=0$ are equivalent to all conditions of Corollary 7.

Example 4. Let $0<\alpha\le 1$. Then $M^f \in \left(\left(l_\alpha\right)^{\lambda}_{Z_{1/2}}, \left(l_\infty\right)^{\mu}_B\right)$ for an arbitrary speed λ if and only if conditions (37) – (39) are satisfied, and

$$\mu_n\left(B_n t-K\right)=O(1). \quad (44)$$

For the proof of this assertion, it is sufficient to show that conditions (37) – (39) and (44) are equivalent to all conditions of Corollary 8. Similarly, to Exercises 1 and 2 we obtain those conditions (37) – (39) are equivalent to conditions (34), (35), (12) and (13), and condition (44) is equivalent to condition (22). Then condition (31) holds due to (38) and (44).

In the next examples we assume that $\sum_k s_k$ is an absolutely convergent sequence, $B=(b_{nl})$ a lower triangular matrix defined by:

$$b_{nl}=\frac{(l+1)^c}{(n+1)^d},\ l\le n,\ c\in \mathrm{R},\ d>0,$$

and $t=(t_l)$, $\mu=(\mu_n)$ are sequences defined by

$$t_l=(l+1)^q,\ q\in \mathrm{R} \ \text{ and } \ \mu_n=(n+1)^p\ p>0.$$

Example 5. Let $0<\alpha\le 1$ and $c+q\ge -1$. We prove that the following assertions hold:

I. $M^f \in\left(\left(l_\alpha\right)^{\lambda}_{Z_{1/2}}, \left(l_\infty\right)^{\mu}_B\right)$ for an arbitrary speed λ, if $-1<c+q\le d-p-1$, or $-1\le c+q<d-p-1$,

II. $M^f \in\left(\left(l_\alpha\right)^{\lambda}_{Z_{1/2}}, \left(c_0\right)^{\mu}_B\right)$ for an arbitrary speed λ, if $c+q<d-p-1$.

III. $M^f \in\left(\left(l_\alpha\right)^{\lambda}_{Z_{1/2}}, \left(l_\beta\right)^{\mu}_B\right)$ for an arbitrary speed λ, and for $\alpha\le\beta\le 1$, if $c+q<-1+d-p-1/\beta$.

For the proof of these assertions, it is sufficient to show that corresponding conditions from Example 1 and from Examples 3 and 4 hold. First, it is easy to see those conditions (37) and (38) are satisfied for all assertions I – III. For $c+q>-1$ we obtain:

$$B_n t=\frac{1}{(n+1)^d}\sum_{l=0}^{\infty}(n+1)^{c+q}=O(1)(n+1)^{c+q+1-d}.$$

Hence $K=\lim_n B_n t=0$ for $c+q<d-1$. If $c+q=-1$, then

$$B_n t=\frac{1}{(n+1)^d}\sum_{l=0}^{n}(l+1)^{c+q}=O(1)\frac{\ln(n+2)}{(n+1)^d},$$

and again $K=0$, since $d>0$. Thus, condition (39) holds for all assertions I – III. Further,

$$T_n:=\mu_n\left(B_n t-K\right)=O(1)(n+1)^{p+c+q+1-d},$$

if $c+q>-1$, and

$$T_n=O(1)\frac{\ln(n+2)}{(n+1)^{d-p}},$$

if $c+q=-1$. Thus $T_n=O(1)$ in assertion I, and $R=\lim_n T_n=0$ in assertion II hold, i.e., condition (44) for assertion I, and condition (43) with $R=0$ for assertion II are satisfied. Hence assertions I and II are proved.

Further, we can write

$$V_n:=\sum_n\left|\mu_n\left(B_n t-K\right)\right|^{\beta}=\sum_n\left|(n+1)^{p+c+q+1-d}\right|^{\beta}$$

for $c+q>-1$, and

$$V_n=\sum_n\left|\frac{\ln(n+1)}{(n+1)^{d-p}}\right|^{\beta}$$

For $c+q=-1$. Hence for $c+q>-1$, $V_n=O(1)$, if $\left(p+c+q+1-d\right)\beta<-1$, or

$c+q<-1+d-p-1/\beta$. For $c+q=-1$ it is possible to show with the help of the logarithmic convergence criterion that $V_n=O(1)$, if $(d-p)\beta>1$, or $d-p-1/\beta>0$. Summarizing, $V_n=O(1)$, if $c+q<-1+d-p-1/\beta$, i.e., condition (40) for assertion III holds. Hence assertion III is proved.

There exists a collection $\{\alpha,\beta,c,d,p,q\}$ satisfying all conditions of Example 5, for example, $\alpha=1/4$, $\beta=1/2$, $c=1/2$, $d=6$, $p=1$ and $q=1/2$.

Example 6. Let $0<\alpha\le 1$ and $c+q<-1$. We prove that the following assertions hold:

I. $M^f\in\left(\left(l_\alpha\right)^\lambda_{Z_{1/2}},\left(l_\infty\right)^\mu_B\right)$ and $M^f\in\left(\left(l_\alpha\right)^\lambda_{Z_{1/2}},c^\mu_B\right)$ for an arbitrary speed λ, if $p\le d$.

II. $M^f\in\left(\left(l_\alpha\right)^\lambda_{Z_{1/2}},\left(c_0\right)^\mu_B\right)$ for an arbitrary speed λ, if $p<d$.

III. $M^f\in\left(\left(l_\alpha\right)^\lambda_{Z_{1/2}},\left(l_\beta\right)^\mu_B\right)$ for an arbitrary speed λ, and for $\alpha\le\beta\le 1$, if $p<d-1/\beta$.

For the proof of these assertions, it is sufficient to show that corresponding conditions from Examples 1 - 4 hold. First, it is easy to see those conditions (37) and (38) are satisfied for all assertions I – III. In this case $K=\lim_n B_n t=0$, since $d>0$ and the series

$$\sum_l (l+1)^{c+q}<\infty, \tag{45}$$

i.e., condition /3)) holds for all assertions I – III.

Then

$$T_n=\frac{1}{(n+1)^{d-p}}\sum_{l=0}^{\infty}(l+1)^{c+q}.$$

Hence

$$R=\lim_n T_n=\sum_n (n+1)^{c+q}\ne`0$$

for $p=d$, and $R=0$ for $p<d$. Therefore conditions (43) and (44) hold for assertion I, and condition (43) with $R=0$ holds for assertion II. Hence assertions I and II are proved.

Further,

$$V_n=\sum_n\left|(n+1)^{p-d}\sum_{l=0}^{n}(l+1)^{c+q}\right|^\beta$$

$$=O(1)\sum_n (n+1)^{(p-d)\beta}=O(1),$$

if $(p-d)\beta<-1$, or $p<d-1/\beta$. So condition (4) for assertion III holds, and thus, assertion III is proved.

There exists a collection $\{\alpha,\beta,c,d,p,q\}$ satisfying all conditions of Example 5, for example, $\alpha=1/4$, $\beta=1/2$, $c=-3$, $d=4$, $p=1$ and $q=1$.

5 Conclusion

In this paper, we continued the investigations started in [8], where we studied the matrix transforms from the set of all α-absolutely λ-convergent sequences, where λ is a strictly positive monotonically increasing sequence, i.e., the speed of convergence, and $0<\alpha\le 1$, into the set of all μ-bounded or μ-convergent or β-absolutely μ-convergent sequences, where μ is another speed of convergence and $0<\alpha\le\beta\le 1$.

Now we found necessary and sufficient conditions for a matrix *M* (with real or complex entries) to map the α-absolute λ-convergence domain of a λ-reversible or a normal matrix *A* (with real or complex entries) into the μ-boundedness, or the μ-convergence or the β-absolute μ-convergence domain of a triangular matrix *B* (with real or complex entries), if $0<\alpha\le\beta\le 1$. As an application we consider the case, where *A* is the Zweier matrix $Z_{1/2}$. For this case we present some examples for the case, then *M* is the factorable matrix.

In the future, we intend to extend our results to the case where we consider matrix transformations in abstract spaces, for example, in the ultrametric space. Also, we wish to study the possibilities of applying the findings of the present paper and [8] for solving real-life problems.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed to the present research, at all stages from the formulation of the problem to the final findings and solution.

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Conflict of Interest

The authors have no conflicts of interest to declare.