The Classical Harmonic Oscillator on $\mathbb{R}$ Perturbed by a Certain Scalar Potential

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Abstract: We investigate the perturbation $\mathcal{A} = H + V$, where $H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right)$ represents the harmonic oscillator in $\mathbb{R}$, and V is a specific scalar potential. Let λ_k denote the k^{th} eigenvalue of the operator H. The eigenvalues of the perturbed operator L are given by $\lambda_k + \mu_k$ where μ_k accounts for the perturbative effects of the potential V. The primary result of this study is to provide an asymptotic expansion of μ_k and to establish a connection between the coefficients of this expansion and a particular transform of the potential V.

Key–Words: Pseudo-differential operator, Spectrum, Harmonic oscillator, Perturbation theory, Eigenvalue asymptotics, Averaging method

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1 Introduction

In this paper, we investigate the spectral properties of the perturbed harmonic oscillator in $\mathbb{R}$, a fundamental system in quantum mechanics and spectral theory. The unperturbed harmonic oscillator H is defined by the differential operator:

$$H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right). \tag{1}$$

which is self-adjoint with a compact resolvent. Its spectrum is well-known and consists of the simple eigenvalues $\left\{\lambda_k = k + \frac{1}{2}\right\}_{k\in\mathbb{N}}$, corresponding to the discrete energy levels of the system.
We introduce a perturbation by considering an even scalar potential $V \in C^\infty(\mathbb{R}, \mathbb{R})$ which satisfies the following decay condition for all $x \in \mathbb{R}, k \in \mathbb{N}$,

$$\left|V^{(k)}(x)\right| \leq c_n \left(1 + x^2\right)^{\frac{-s}{2}}, s \in]1, +\infty[. \tag{2}$$

The perturbed operator $\mathcal{A} = H + V$ remains self-adjoint with a compact resolvent, [1], and its spectrum consists of the perturbed eigenvalues $\{\lambda_k + \mu_k\}_k$ where μ_k represents the corrections to the unperturbed eigenvalues due to the potential V. The study of spectral perturbations of the harmonic oscillator is a classical problem with applications in various fields, such as quantum mechanics, semiclassical analysis, and mathematical physics. While many techniques have been developed to analyze such perturbations, including semiclassical methods and pseudo-differential operator theory, we employ here the averaging method. This approach leverages the periodic structure of the harmonic oscillator and provides a direct means to analyze the asymptotic behavior of the perturbed eigenvalues as k tends to infinity. The primary goal of this paper is to determine the asymptotic behavior of the eigenvalue corrections μ_k as $k \to \infty$ and to establish a relationship between the coefficients of this asymptotic expansion and a specific transform of the potential V. This result provides new insights into the spectral structure of perturbed oscillators and offers a fresh perspective on the long-term behavior of the spectrum. Moreover, our approach presents certain advantages over existing methods by providing an explicit link between the perturbation and the spectral corrections, making the analysis more transparent and easier to generalize to other systems, such as anharmonic oscillators. Pseudo-differential operators are generalizations of differential operators, defined by symbols that describe their behavior in the spatial and spectral domains, enabling the analysis of complex problems in partial differential equations. They are widely used in physics, engineering, and signal processing. Weyl quantization associates physical observables in quantum mechanics with operators on a Hilbert space, establishing a link between classical and quantum mechanics. For example, a pseudo-differential operator can model wave diffusion in inhomogeneous media, allowing

the study of wave propagation effects.
We aim to study the asymptotic behavior of μ_k as k tends to infinity. Let us state now the main results proved in this paper.

Theorem 1. *The asymptotic behavior of μ_k is:*

$$\mu_k = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} V(\sqrt{2\lambda_k}\cos t)dt + O\left(\lambda_k^{-(1-\eta)}\right), k \to +\infty \tag{3}$$

with $\eta \in \left]0, \frac{1}{2}\right[$.

To assess the validity of Theorem 1, we present the following theorem:

Theorem 2. *For $s > 1$ we have*

$$\mu_k = \frac{1}{\pi\sqrt{2\lambda_k}}\int_{\mathbb{R}} V(x)dx + O(\lambda_k^{-\frac{(1+\alpha)}{2}})$$

$0 < \alpha < \min(s-1, 1-2\eta), \quad \eta \in \left]0, \frac{1}{2}\right[$.

Remarks

$R.1$ If the potential V is neither even nor odd, we keep the even part of V in the integral of (3)
$R.2$ The integral of (3) can be viewed as an $\widetilde{V}$ transform of V :

$$\widetilde{V}(x) = \frac{2}{\pi}\int_0^x \frac{V(y)}{\sqrt{x^2-y^2}}dy,$$

by a change of variable we can write:

$$\widetilde{V}(x) = \frac{1}{\pi}\int_0^{x^2} \frac{V(\sqrt{y})}{\sqrt{y}\sqrt{x^2-y}}dy, \tag{4}$$

$\widetilde{V}$ is none other than Abel's transformate applied to x^2 of the function $y \to (y)^{-\frac{1}{2}}V(\sqrt{y})$. Once the $\widetilde{V}$ function is determined from a distribution of its values on $\mathbb{R}$, we can recover V (inverse problem) of (4) by reducing it to an Abel integral equation.
$R.3$ In (3) we get the best approximation for η near zero.
$R.4$ The theorem 1 can be extended to the case of the operator:

$$(-1)^h\frac{d^{2h}}{dx^{2h}} + x^{2k}, \tag{5}$$

where $h, k \in \mathbb{N}^*$. We hope to elaborate on this in the future.

In [2], the study investigates the harmonic oscillator on $\mathbb{R}$ perturbed by a scalar potential B, which has the following asymptotic form:

$$B(x) \sim |x|^\alpha \sum_m a_m \cos\omega_m x,$$

Where $\alpha > 0$, a_m and ω_m are real numbers. Our research addresses a more extensive class of scalar potentials. Our approach utilizes the averaging method developed in [3] and [4], which involves replacing V in $\mathcal{A} = H + V$ with the average

$$\overline{V} = \frac{1}{2\pi}\int_0^{2\pi} e^{-itH}Ve^{itH}dt.$$

Then it turns out that the spectrum of $\overline{\mathcal{A}} = H + \overline{V}$ is very close to that of $\mathcal{A}$, $\overline{V}$ and $\mathcal{A}$ are almost unitarily equivalent, and $\left[H, \overline{V}\right] = 0$. We first study the spectrum of $\overline{V}$, and then we move on to that of $\mathcal{A}$. For an overall view of this kind of problem, refer to [4]. The rest of this article is organized as follows. In Section 2, this section presents supplementary information regarding certain properties of Weyl pseudo-differential operators and their functional calculus. In section 3, we study the spectrum of $\mathcal{A}$ and $\overline{V}$ and specify in what sense they are close. Section 4 is devoted to the study of the asymptotic behavior of μ_k. In Section 5, we prove Theorem 2

2 Weyl pseudo-differential operator and functional calculus

Let $\rho \in [0,1]$ and $m \in \mathbb{R}$. We consider the weight function, [5]

$$(x,\xi) \longrightarrow \left(1+x^2+\xi^2\right)^{\frac{m}{2}}, \qquad (x,\xi) \in \mathbb{R}^2.$$

We denote by $\Gamma_\rho^m(\mathbb{R}\times\mathbb{R})$ the space of symbols associated with the temperate weight function, precisely:

$$\begin{aligned}\Gamma_\rho^m \quad &= \quad \{a \in C^\infty\left(\mathbb{R}^2\right) : \forall \alpha, \beta \in \mathbb{N}^2, \exists\, c_{\alpha,\beta} > 0 \\ &\qquad / \left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \\ &\leq \quad c_{\alpha,\beta}(1+x^2+\xi^2)^{\frac{m-\rho(\alpha+\beta)}{2}}\}.\end{aligned}$$

We will use the standard Weyl quantization of symbols. Specifically, if $a \in \Gamma_\rho^m$, then for $u \in S(\mathbb{R})$ the operator associated is defined by :

$$\begin{aligned}&op^w(a)\,u(x) \\ =\; &\tfrac{1}{(2\pi)^2}\int_{\mathbb{R}\times\mathbb{R}} e^{i(x-y)\xi}a\left(\frac{x+y}{2},\xi\right)u(y)\,dyd\xi\end{aligned}.$$

We denote by G_ρ^m the operator class whose symbol belongs to $a \in \Gamma_\rho^m$. For example, $H \in G_1^2$ and $V \in G_0^0$.
Let us now introduce the notion of the asymptotic expansion of symbols.

Proposition 3. *(see, [6])*
i) If $A \in G_1^{m_1}$ and $B \in G_0^{m_2}$ then the operator $AB \in G_0^{m_1+m_2}$. Its Weyl symbol admits an asymptotic development:

$$c = \sum_{j=0}^{+\infty} c_j, \quad c_j \in \Gamma_0^{m_1+m_2-j},$$

where

$$c_j = \frac{1}{2^j} \sum_{\alpha+\beta=j} \frac{(-1)^{|\beta|}}{\alpha!\beta!} (\partial_\xi^\alpha \partial_x^\beta a)(\partial_x^\alpha \partial_\xi^\beta b).$$

ii) The commutator $[A, B] \in G_0^{m_1+m_2-1}$.
iii) If $(B_i)_{i\in\{1,\cdots,n\}}$ is the family of operators such as $B_i \in G_0^{m_i}$. Then the operator

$$B_1 B_2 \cdots B_n H^{-\frac{m_1+\cdots m_n}{2}}$$

is bounded.

Theorem 4. *(Calderon-Vaillancourt Theorem)*
If $a \in \Gamma_0^0$ then the operator $op^w(a)$ is bounded on $L^2(\mathbb{R})$.

In the following, we will need the functional calculus of the operator H. The functional calculus for pseudo-differential operators (OPD) has been studied in cases where the functions belong to the Hörmander class $S_1^r, r \in \mathbb{R}$ (see, [7]). In our work, we utilize the properties of a function f that satisfies, for all $r \in \mathbb{R}, k \in \mathbb{N}$ and $\rho \in [\frac{1}{2}, 1]$, the following condition:

$$|f^{(k)}(x)| \leq C_k(1 + |x|)^{r-\rho k}.$$

This means that we are dealing with the case of the operator H plus a function belonging to the class S_ρ^r.

Proposition 5. *$f(H)$ is a (ΨDO) included in $G_{1-2(1-\rho)}^{2r}$ and its weyl symbol admit the following development*

$$\sigma_{f(H)} = \sum_{j\geq 0} \sigma_{f(H),2j}$$

$$\sigma_{f(H),2j} = \sum_{k=2}^{3j} \frac{d_{jk}}{k!} f^{(k)}(\sigma_H), \quad \forall j \geq 1$$

where

$$d_{j,k} \in \Gamma_1^{2k-4j}, \quad \sigma_{f(H),2j} \in \Gamma_{1-2(1-\rho)}^{2r-j(6\rho-2)}, \qquad (6)$$

in particular

$$\sigma_{f(H),0} = f(\sigma_H).$$

Proof: For studying $f(H)$, we follow the same strategy in [7], using the Mellin transformation, the latter consists of the following steps:
(1) We prove by induction that $(H-\lambda)^{-1}$,$\lambda \in C$, is a (ΨDO) and its Weyl symbol admits the development $b_\lambda = \sum_{j=0}^{+\infty} b_{j,\lambda}$ where

$$\begin{cases} b_{0,\lambda} = (\sigma_H - \lambda)^{-1}, \\ b_{2j+1,\lambda} = 0, \\ b_{2j,\lambda} = \sum_{k=2}^{3j} (-1)^k d_{j,k}.b_{0,\lambda}^{k+1}, \quad d_{j,k} \in \Gamma_1^{2k-4j}. \end{cases}$$

(2) Studying the operator H^s using the Cauchy's integral formula

$$H^s = \frac{1}{2\pi i} \int_\Delta \lambda^s (H - \lambda)^{-1} d\lambda.$$

Δ is the same domain defined in the article, [7]
H^s is a (ΨDO), and its Weyl symbol is $\sigma_s = \sum_{j=0}^{+\infty} \sigma_{s,2j}$ with $\sigma_{s,0} = \sigma_H^s$ and

$$\sigma_{s,2j} = \sum_{k=2}^{3j} d_{j,k}.\frac{s(s-1)\cdots(s-k+1)}{k!} \sigma_H^{s-k},$$

$$\sigma_{s,2j} \in \Gamma_1^{2s-4j}.$$

(3) Studying $f(H)$ using the representation formula

$$f(H) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} M[f](s) H^{-s} ds,$$

$\sigma \in [0, -r[$, $r < 0$ and $M[f]$ is the Mellin transformation of f. □

3 The relation between the spectrum of $\mathcal{A}$ and $\overline{\mathcal{A}}$

In this section, we will apply the averaging method. To begin with, let's observe the Hamiltonian flow related to the symbol of the operator H

$$\sigma_H(x, \xi) = \frac{1}{2}(x^2 + \xi^2), \quad x, \xi \in \mathbb{R}, \qquad (7)$$

is a group with a parameter whose elements are square matrix of size 2.

$$\chi_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \qquad (8)$$

Observe that this flow is periodic with a period of 2π. To initiate the averaging method, we introduce the following operators.

$$W(t) = e^{-itH} V e^{itH}, \tag{9}$$

$$\overline{V} = \frac{1}{2\pi} \int_0^{2\pi} W(t) dt, \tag{10}$$

$$\overline{\overline{V}} = \frac{1}{2\pi i} \int_0^{2\pi} \int_0^t [W(t), W(r)] dr dt. \tag{11}$$

Since H commute with $\overline{V}$, the spectrum of $\overline{\mathcal{A}}$ is $\{\lambda_k + \overline{\mu}_k\}$, where $\bar{\mu}_k$ is the k^{th} eigenvalue of $\overline{V}$. To compare μ_k and $\bar{\mu}_k$ we will need the following lemmas

Lemma 6. $\left[H, \overline{V}\right] = 0$

Proof: After we derive $W(t)$, we obtain

$$\frac{dW(t)}{dt} = \frac{1}{i} \left[H, W(t)\right]. \tag{12}$$

Now we have

$$\left[H, \overline{V}\right] = \frac{i}{2\pi} \int_0^{2\pi} \frac{dW(t)}{dt} dt = \frac{i}{2\pi} (W(2\pi) - W(0)), \tag{13}$$

since $e^{2\pi i H} = -id_{L^2(\mathbb{R})}$, we get $W(2\pi) = W(0)$. Finally, we have $\left[H, \overline{V}\right] = 0$. □

Lemma 7.

$$i/\overline{V} \in G_0^{-1}, \quad ii/\overline{\overline{V}} \in G_0^{-2+2\eta},$$

where $\eta \in \left]0, \frac{1}{2}\right[$.

Proof:
i/The Weyl symbol of the operator $W(t)$ is

$$\sigma_{W(t)} = V o \chi_t, \tag{14}$$

where χ_t is the flow described in (8).
This result is due to the fact that, on the one hand, e^{itH} belongs to the metaplectic group, and on the other hand, Weyl's quantization is invariant under this group, ([8], [9]). The Weyl's symbol of $\overline{V}$ is obtained by integrating the symbol of $W(t)$ uniformly with respect to t.

$$\sigma_{\overline{V}}(x, \xi) = \frac{1}{2\pi} \int_0^{2\pi} V(x \cos t + \xi \sin t) dt. \tag{15}$$

By using (2), we get the following estimate, for $\alpha, \beta \in \mathbb{N}$ and $x, \xi \in \mathbb{R}$:

$$\begin{aligned} & \left| \partial_x^\alpha \partial_\xi^\beta \sigma_{\overline{V}}(x, \xi) \right| \\ \leq \;& C_{\alpha,\beta} \textstyle\int_0^{2\pi} \left[1 + (x \cos t + \xi \sin t)^2\right]^{\frac{-s}{2}} dt. \\ \leq \;& C_{\alpha,\beta} \textstyle\int_0^{2\pi} \left[1 + (x^2 + \xi^2) \cos^2 t\right]^{\frac{-s}{2}} dt. \end{aligned} \tag{16}$$

Now we apply the change of variable $y = \cos t$, we get $\left| \partial_x^\alpha \partial_\xi^\beta \sigma_{\bar{V}}(x, \xi) \right| \leq C_{\alpha,\beta} \int_0^1 \frac{\left[1 + (x^2 + \xi^2) u^2\right]^{\frac{-s}{2}}}{\sqrt{1 - u^2}} du$. We split this integral into two parts, then

$$\left| \partial_x^\alpha \partial_\xi^\beta \sigma_{\bar{V}}(x, \xi) \right| \leq I_1 + I_2,$$

with

$$I_1 = \int_0^{\frac{1}{2}} \frac{1}{(1 + (x^2 + \xi^2) u^2)^{\frac{s}{2}}} \times \frac{1}{\sqrt{1 - u^2}} du,$$

and

$$I_2 = \int_{\frac{1}{2}}^1 \frac{1}{(1 + (x^2 + \xi^2) u^2)^{\frac{s}{2}}} \times \frac{1}{\sqrt{1 - u^2}} du.$$

we put $h = \sqrt{x^2 + \xi^2}$, so, we have:

$$I_1 \leq c \int_0^{\frac{1}{2}} \frac{1}{(1 + h^2 u^2)^{\frac{s}{2}}} du.$$

After applying the change of variables, $u = \frac{y}{y+1}$ and $v = y\sqrt{1 + h}$ we obtain

$$I_1 \leq \frac{c}{1 + h}.$$

On the other side we have

$$I_2 \leq \frac{c}{(1 + h)^s} \int_{\frac{1}{2}}^1 \frac{1}{\sqrt{1 - u^2}} du \leq \frac{c}{(1 + h)^s},$$

Finally, we obtain the following estimate:

$$\left| \partial_x^\alpha \partial_\xi^\beta \sigma_{\bar{V}}(x, \xi) \right| \leq C_{\alpha,\beta} \left(1 + x^2 + \xi^2\right)^{-\frac{1}{2}}.$$

ii/ According to the previous calculations, the operator $B(t) = \displaystyle\int_0^t W(r) dr$ belongs to G_0^{-1}, its Weyl's symbol $\sigma_{B(t)}$ check :

$$\left| \partial_x^\alpha \partial_\xi^\beta \sigma_{B(t)}(x, \xi) \right| \leq C_{\alpha,\beta} (1 + x^2 + \xi^2)^{\frac{-1}{2}}, \tag{17}$$

uniformly with respect to t.
Let us begin by clarifying the class of the operator. $\displaystyle\int_0^{2\pi} W(t) B(t) dt$. At this point, we are focusing on the operator. $W(t)B(t)$, its Weyl symbol c_t is given in [5], by

$$\begin{aligned} c_t(x, \xi) \;=\; & \tfrac{1}{\pi^2} \int e^{-2i(r\rho - \omega\tau)} \sigma_{W(t)}(x + \omega, \xi + \rho) \\ & \times \sigma_{B(t)}(x + r, \xi + \tau) d\rho d\omega d\tau dr. \end{aligned} \tag{18}$$

We split the oscillator integral c_t into two parts $c_t^{(1)}$ and $c_t^{(2)}$, then we use the cutoff functions.

$$\omega_{1,\varepsilon}(x,\xi,\omega,\tau,r,\rho)=\chi\left[\frac{\omega^2+\rho^2+r^2+\tau^2}{\varepsilon(1+x^2+\xi^2)^{\frac{\eta}{2}}}\right] \text{ and } \omega_{2,\varepsilon}=1-\omega_{1,\varepsilon},$$

where $\chi \in C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ in $[-1,1]$, $\chi \equiv 0$ in $\mathbb{R}\setminus\left]-2,2\right[$, $R=\omega^2+\rho^2+r^2+\tau^2$, $\varepsilon>0$ and $\eta \in \left]0,\frac{1}{2}\right[$. Let's consider

$$\begin{aligned} d_j(x,\xi,\omega,\tau,r,\rho) &= \omega_{j,\varepsilon}(x,\xi,\omega,\tau,r,\rho) \\ &\quad \times\sigma_{W(t)}(x+\omega,\xi+\rho) \\ &\quad \times\sigma_{B(t)}(x+r,\xi+\tau), \end{aligned} \tag{19}$$

$c_t^{(1)}$ (resp $c_t^{(2)}$) the integral obtained in (18) by replacing the amplitude by d_1 (resp d_2)

Study of $c_t^{(2)}$

On the support of d_2 we have $R \geq \varepsilon(1+x^2+\xi^2)^{\frac{\eta}{2}}$. We make an integration by parts using the operator.

$$M=\frac{1}{2iR}(-\rho\partial_r-r\partial_\rho+\tau\partial_\omega+\omega\partial_\tau).$$

We have for all $k \in N$

$$c_t^{(2)}=\frac{1}{\pi^2}\int e^{-2i(r\rho-\omega\tau)}\left({}^tM\right)^k d_2\, d\rho\, d\omega\, d\tau\, dr.$$

Then we get for all $k>0$

$$\left|c_t^{(2)}\right| \leq C_k(1+x^2+\xi^2)^{\frac{-\eta k}{4}},$$

Uniformly with respect to $t \in [0,2\pi]$.

Study of $c_t^{(1)}$

On the support of d_1, we have

$$\begin{aligned} c_t^{(1)}(x,\xi) &= \frac{1}{\pi^2}\int_{R\leq 2\varepsilon(1+x^2+\xi^2)^{\frac{\eta}{2}}} e^{-2i(r\rho-\omega\tau)} \\ &\quad \times \sigma_{W(t)}(x+\omega,\xi+\rho) \\ &\quad \times \sigma_{B(t)}(x+r,\xi+\tau)\omega_{1,\varepsilon}d\rho d\omega d\tau dr, \end{aligned} \tag{20}$$

$$\begin{aligned} \int_0^\pi \left|c_t^{(1)}\right|dt &\leq c\int_{R\leq 2\varepsilon(1+x^2+\xi^2)^{\frac{\eta}{2}}} d\rho d\omega d\tau dr \\ &\quad \times \int_0^\pi \left|\sigma_{W(t)}(x+\omega,\xi+\rho)\right|dt \\ &\quad \times \int_0^\pi \left|\sigma_{B(t)}(x+r,\xi+\tau)\right|dt, \end{aligned} \tag{21}$$

On the support of d_1, for ε small enough and since $\eta \in \left]0,\frac{1}{2}\right[$, there are positive constants c, c', C, C' such that

$$\begin{cases} c(1+x^2+\xi^2)^{\frac{1}{2}} \leq (1+(x+\omega)^2+(\rho+\xi)^2)^{\frac{1}{2}}, \\ (1+(x+\omega)^2+(\rho+\xi)^2)^{\frac{1}{2}} \leq C(1+x^2+\xi^2)^{\frac{1}{2}}, \\ c'(1+x^2+\xi^2)^{\frac{1}{2}} \leq (1+(x+r)^2+(\tau+\xi)^2)^{\frac{1}{2}}, \\ (1+(x+r)^2+(\tau+\xi)^2)^{\frac{1}{2}} \leq C'(1+x^2+\xi^2)^{\frac{1}{2}}. \end{cases}$$

Therefore

$$\begin{aligned} &\int_0^\pi \left|c_t^{(1)}\right| dt \\ &\leq C\left(1+x^2+\xi^2\right)^{-1}\int_{R\leq 2\varepsilon(1+x^2+\xi^2)^{\frac{\eta}{2}}} d\rho d\omega d\tau dr. \end{aligned} \tag{22}$$

Finally

$$\int_0^\pi \left|c_t^{(1)}\right| dt \leq c\left(1+x^2+\xi^2\right)^{-1+\eta}. \tag{23}$$

In the end, by denoting σ as the Weyl symbol of the operator $\int_0^\pi W(t)B(t)dt$, we have

$$\begin{aligned} |\sigma| &\leq \int_0^\pi \left|c_t^{(1)}\right|dt+\int_0^\pi \left|c_t^{(2)}\right|dt \\ &\leq C\left[(1+x^2+\xi^2)^{\frac{-\eta k}{4}}+(1+x^2+\xi^2)^{-1+\eta}\right] \\ &\leq C(1+x^2+\xi^2)^{\frac{-2+2\eta}{2}}. \end{aligned}$$

Finally, we deduce that $\overline{\overline{V}} \in G_0^{-2+2\eta}$. □

Lemma 8. *There exists a skew-symmetric operator $P \in G_0^{-1}$ such that the operator $(e^P\mathcal{A}e^{-P}-\overline{\mathcal{A}})H^{1-\eta}$ is bounded.*

Proof: Take the following antisymmetrical operator, P:

$$P=P_1+P_2, \tag{24}$$

where

$$P_1=\frac{i}{2\pi}\int_0^{2\pi}(2\pi-t)W(t)dt,$$

$$P_2=\frac{-1}{4\pi}\int_0^{2\pi}(2\pi-t)\int_0^t [W(t),W(r)]drdt.$$

Using the same calculations as those in Lemma 7, we obtain: $P_1 \in G_0^{-1}$ and $P_2 \in G_0^{-2+2\eta}$. Finally, $P \in G_0^{-1}$.

Before beginning the proof, we will need the following relations:

$$\begin{aligned} [P_1,H] &= \frac{i}{2\pi}\int_0^{2\pi}(2\pi-t)\frac{dW(t)}{dt}dt \\ &= \overline{V}-V. \end{aligned} \tag{25}$$

and $[P_2,H]$

$$\begin{aligned} &= \frac{-1}{4\pi}\int_0^{2\pi}(2\pi-t)\int_0^t [[W(t),W(r)],H]drdt \\ &= \frac{i}{4\pi}\int_0^{2\pi}(2\pi-t) \end{aligned}$$

$$\times \int_0^t \left(\left[W(t), W'(r)\right] + \left[W'(t), W(r)\right]\right) dr dt.$$

We set

$$F(t) = \frac{1}{2\pi}\int_0^t W(r)dr.$$

On the one hand :

$$\begin{array}{l} \dfrac{i}{4\pi}\displaystyle\int_0^{2\pi}(2\pi - t)\int_0^t \left[W(t), W'(r)\right] dr dt \\ = \dfrac{i}{4\pi}\displaystyle\int_0^{2\pi}(2\pi - t)\left[W(t), \int_0^t W'(r)dr\right]dt \\ = \dfrac{-i}{4\pi}\displaystyle\int_0^{2\pi}(2\pi - t)\left[W(t), V\right]dt \\ = \frac{-1}{2}\left[P_1, V\right]. \end{array}.$$

on the other hand :

$$\begin{array}{l} \dfrac{i}{4\pi}\displaystyle\int_0^{2\pi}(2\pi - t)\int_0^t \left[W'(t), W(r)\right] dr dt \\ = \dfrac{i}{2}\displaystyle\int_0^{2\pi}(2\pi - t)\left[W'(t), F(t)\right]dt \\ = \dfrac{i}{2}\displaystyle\int_0^{2\pi}(2\pi - t)\frac{d}{dt}([W(t), F(t)])dt \\ = \dfrac{i}{2}\left(\left[(2\pi - t)\left[W(t), F(t)\right]\right]_0^{2\pi}\right) + \\ \left(\int_0^{2\pi}\left[W(t), F(t)\right]dt\right) \\ = -\overline{\overline{V}}. \end{array}.$$

Finally, we have :

$$[P_2, H] = -\overline{\overline{V}} - \frac{1}{2}\left[P_1, V\right]. \tag{26}$$

We notice $AdP.\mathcal{A} = [P, \mathcal{A}]$. The differential equation

$$\left\{\begin{array}{l} \frac{dX}{dt} = [P, X] \\ X(0) = \mathcal{A}, \end{array}\right. \tag{27}$$

has a unique solution

$$X(t) = e^{tADP}.\mathcal{A} = e^{tP}\mathcal{A}e^{-tP}.$$

We deduce, taking into account (25) and (26) that :

$$\begin{array}{rl} & e^P\mathcal{A}\,e^{-P} - \overline{\mathcal{A}} \\ = & -\overline{\overline{V}} + \frac{1}{2}\left[P_2, V\right] \\ & +\frac{1}{2}\left[P, \overline{V}\right] + \frac{1}{4}\left[P, [P_1, V]\right] \\ & +\frac{1}{2}\left[P, [P_2, V]\right] - \frac{1}{2}\left[P, \overline{\overline{V}}\right] \\ & +\displaystyle\sum_{n\geq 0}\frac{(AdP)^n}{(n+3)!}\left[P, [P, [P, \mathcal{A}]]\right]. \end{array} \tag{28}$$

We now apply Proposition 3, since $V \in \sum_0^0$, $\overline{V} \in G_0^{-1}$, $P_1, P \in G_0^{-1}$ and $P_2, \overline{\overline{V}} \in G_0^{-2+2\eta}$, we get :

$$\left\{\begin{array}{cl} \left\|\overline{\overline{V}}.H^{1-\eta}\right\| & \leq C, \\ \left\|\left[P_2, V\right]H^{1-\eta}\right\| & \leq C, \\ \left\|\left[P, \overline{V}\right]H\right\| & \leq C, \\ \left\|\left[P, [P_1, V]\right]H\right\| & \leq C, \\ \left\|\left[P, [P_2, V]\right]H^{\frac{3}{2}-\eta}\right\| & \leq C, \\ \left\|\left[P, \overline{\overline{V}}\right]H^{\frac{3}{2}-\eta}\right\| & \leq C, \\ \left\|\frac{(AdP)^n}{(n+3)!}\left[P, [P, [P, \mathcal{A}]]\right]H\right\| & \leq C\left\|P\right\|^n. \end{array}\right. \tag{29}$$

For the last inequality, we used the following identity:

$$(AdP)^n.W = \sum_{i=0}^n (-1)^{n-i}C_n^i P^i W P^{n-i}.$$

From (28) and (29) we deduce that :

$$(e^P\mathcal{A}e^{-P} - \overline{\mathcal{A}})H^{1-\eta}.$$

is bounded.

We can now compare μ_k and $\overline{\mu}_k$. From lemma 8 we deduce that there exists a constant $c > 0$ such that

$$-cH^{-(1-\eta)} \leq e^P\mathcal{A}e^{-P} - \overline{\mathcal{A}} \leq cH^{-(1-\eta)}$$

The min-max theorem, [10], implies that:

$$\mu_k = \overline{\mu}_k + O(\lambda_k^{-(1-\eta)}), \tag{30}$$

where $\eta \in \left]0, \frac{1}{2}\right[$.

$\square$

4 The asymptotic behavior of μ_k

We begin by studying the asymptotic behavior of $\overline{\mu}_k$, as a result of using (30) we deduce that of μ_k. Let us first recall that $\overline{\mu}_k$ is the k^{th} own value of $\overline{V}$. In polar coordinates, the identity (15) that presents the symbol of Weyl of $\overline{V}$ is written:

$$\sigma_{\overline{V}}(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} V\left(r\left(\cos(t - \theta)\right)\right)dt.$$

From the parity of V we get

$$\sigma_{\overline{V}} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} V\left(r\cos(t)\right)dt = f(\sqrt{\sigma_H}).$$

where $f(x) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} V\left(\sqrt{2x}\cos t\right)dt$. A direct calculation shows that: $|f(x)| \leq c(1 + |x|)^{-\frac{1}{2}}$ and $\left|f^{(k)}(x)\right| \leq c_k(1 + |x|)^{-\frac{1}{2}-\frac{k}{2}}$, so f is in the class of

Hörmander $S^{\frac{-1}{2}}_{\frac{1}{2}}$. By applying the proposition 5, we get

$$f(H) \in G_0^{-1},$$

and

$$\overline{V} - f(H) \in G_0^{-2}. \tag{31}$$

By combining the equation (31) and the proposition 3-iii) we deduce that

$$(\overline{V} - f(H))H$$

is bounded.
Come back to the Proof of Theorem 1. Therefore, there exists a constant $c > 0$ such that

$$-cH^{-1} \le \overline{V} - f(H) \le cH^{-1}$$

According to the min-max theorem, [10], we get:

$$\overline{\mu}_k = f(\lambda_k) + O(\lambda_k^{-1}). \tag{32}$$

By combining the equations (32) and (30) we deduce:

$$\mu_k = f(\lambda_k) + O(\lambda_k^{-(1-\eta)}),$$

Finaly, we have

$$\mu_k = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} V\left(\sqrt{2\lambda_k}\cos t\right) dt + O(\lambda_k^{-(1-\eta)}),$$

where $\eta \in \left]0, \frac{1}{2}\right[$.

5 Refinement of Theorem 2

We now proceed to prove the Theorem 2. Performing the change of variables $y = \cos t$, we obtain that

$$\mu_k = \frac{2}{\pi}\int_0^1 \frac{V\left(\sqrt{2\lambda_k}y\right)}{\sqrt{1-y^2}} dy + O(\lambda_k^{-(1-\eta)}),$$

we put

$$\beta_k = \int_0^1 \frac{V\left(\sqrt{2\lambda_k}y\right)}{\sqrt{1-y^2}} dy. \tag{33}$$

For the moment, we start by studying the asymptotic behavior of β_k. By a direct calculation, there is a function $\theta \in C([0,1], \mathbb{R})$ such as

$$\frac{1}{\sqrt{1-y^2}} = 1 + \frac{y^2\theta(y)}{\sqrt{1-y^2}},$$

so we have

$$\begin{aligned}\beta_k &= \int_0^1 V(\sqrt{2\lambda_k}y)dy + \int_0^1 \frac{V\sqrt{2\lambda_k}y)y^2\theta(y)}{(\sqrt{1-y^2})}dy \\ &= \beta_{k,1} + \beta_{k,2}.\end{aligned} \tag{34}$$

firstly

$$\begin{aligned}\beta_{k,2} &= \int_0^{\frac{1}{2}} \frac{V\sqrt{2\lambda_k}y)y^2\theta(y)}{(\sqrt{1-y^2})}dy \\ &\quad + \int_{\frac{1}{2}}^1 \frac{V\sqrt{2\lambda_k}y)y^2\theta(y)}{(\sqrt{1-y^2})}dy.\end{aligned} \tag{35}$$

Using (5), we have

$$\left|\int_{\frac{1}{2}}^1 \frac{V\sqrt{2\lambda_k}y)y^2\theta(y)}{(\sqrt{1-y^2})}dy\right| \le \frac{c}{(1+\lambda_k)^{\frac{s}{2}}}\int_{\frac{1}{2}}^1 \frac{y^2\theta(y)}{\sqrt{1-y^2}}dy,$$

consequently

$$\int_{\frac{1}{2}}^1 \frac{V\sqrt{2\lambda_k}y)y^2\theta(y)}{(\sqrt{1-y^2})}dy = O(\lambda_k^{-\frac{s}{2}}). \tag{36}$$

We have : for $0 < \alpha < 2$

$$\begin{aligned}&\left|\int_0^{\frac{1}{2}} \frac{V(\sqrt{2\lambda_k}y)y^2\theta(y)}{(\sqrt{1-y^2})}dy\right| \\ &\le c\int_0^{\frac{1}{2}} (1 + (y\sqrt{2\lambda_k})^2)^{-\frac{s}{2}} y^\alpha dy,\end{aligned} \tag{37}$$

we make a change of variable, $y\sqrt{2\lambda_k} = u$ we obtain

$$\begin{aligned}&\int_0^{\frac{1}{2}} (1 + (y\sqrt{2\lambda_k})^2)^{-\frac{s}{2}} y^\alpha dy \\ &= 2^{\frac{-(\alpha+1)}{2}} \lambda_k^{-\frac{1}{2}(1+\alpha)} \int_0^{\frac{\sqrt{2}}{2}\lambda_k^{\frac{1}{2}}} \frac{u^\alpha}{(1+u^2)^{\frac{s}{2}}}du \\ &\le C\lambda_k^{-\frac{1}{2}(1+\alpha)} \int_0^{+\infty} \frac{u^\alpha}{(1+u^2)^{\frac{s}{2}}}du,\end{aligned}$$

we take $s > 1$ and $0 < \alpha < min(2, s-1)$ we get

$$\int_0^{\frac{1}{2}} \frac{V(\sqrt{2\lambda_k}y)y^2\theta(y)}{(\sqrt{1-y^2})}dy = O(\lambda_k^{-\frac{1}{2}(1+\alpha)}),$$

we apply the change of variable $y\sqrt{2\lambda_k} = x$ for $\beta_{k,1}$, we obtain

$$\begin{aligned}\beta_{k,1} &= \frac{1}{\sqrt{2\lambda_k}}\int_0^{\sqrt{2\lambda_k}} V(x)dx \\ &= \frac{1}{\sqrt{2\lambda_k}}\int_0^{+\infty} V(x)dx - \frac{1}{\sqrt{2\lambda_k}}\int_{\sqrt{2\lambda_k}}^{+\infty} V(x)dx,\end{aligned}$$

using (2) and since $s > 1$, we have

$$\frac{1}{\sqrt{2\lambda_k}}\int_{\sqrt{2\lambda_k}}^{+\infty} V(x)dx = O(\lambda_k^{-\frac{s}{2}}), \tag{38}$$

then

$$\beta_{k,1} = \frac{1}{\sqrt{2\lambda_k}}\int_0^{+\infty} V(x)dx + O(\lambda_k^{-\frac{s}{2}}). \tag{39}$$

Finally, we conclude

$$\beta_k = \frac{1}{2\sqrt{2\lambda_k}} \int_{\mathbb{R}} V(x)dx + O(\lambda_k^{-\frac{(1+\alpha)}{2}}), \tag{40}$$

so

$$\mu_k = \frac{1}{\pi\sqrt{2\lambda_k}} \int_{\mathbb{R}} V(x)dx + O(\lambda_k^{-\frac{(1+\alpha)}{2}}),$$

with $0 < \alpha < \min(s-1, 1-2\eta), \quad \eta \in \left]0, \frac{1}{2}\right[$.

6 Conclusion

We addressed the spectral problem of the perturbed harmonic oscillator, a well-known system in spectral theory due to its importance in various physical applications. By applying the averaging method, we successfully derived the asymptotic expansion of the eigenvalue corrections μ_k and demonstrated how the coefficients of this expansion relate to a transform of the perturbing potential V. This approach highlights the strengths of the averaging method in handling periodic systems like the harmonic oscillator, offering advantages in terms of simplicity and precision. Our findings provide new insights into the spectral behavior of such perturbed systems, and the methodology presented can be extended to more complex cases. In particular, future work will focus on applying this technique to anharmonic oscillators, where nonlinearity introduces additional challenges. This extension could open up new avenues for applications in quantum mechanics, wave propagation, and other areas of mathematical physics.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

The authors wrote, reviewed, and edited the content as needed and have not utilised artificial intelligence (AI) tools. The authors take full responsibility for the content of the publication.

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