On Generalizations of Dickson *k*-Fibonacci Polynomials

ENGIN OZKAN[1], HAKAN AKKUS[2]
[1]Department of Mathematics,
Faculty of Sciences,
Marmara University,
Istanbul,
TURKEY

[2]Department of Mathematics,
Graduate School of Natural and Applied Sciences,
Erzincan Binali Yıldırım University,
Erzincan,
TURKEY

Abstract: - In this study, we define a Dickson k-Fibonacci polynomial inspired by Dickson polynomials and give some terms of these polynomials. Then we present the relations between the terms of Dickson k-Fibonacci polynomials. We find Binet formulas and generating functions for these polynomials. In addition, we give some important identities like Catalan identity, Melham's identity, and Gelin-Cesaro's identity. Moreover, Catalan transformation is applied to these polynomials, and their terms are found. Finally, the Hankel transform is applied to the Catalan transform of these polynomials, and the results obtained are associated with known Fibonacci numbers.

Key-Words: - k-Fibonacci polynomials, Dickson polynomials, Binet formula, Generating function, Cassini Identity, Fibonacci sequence.

Received: June 2, 2024. Revised: October 14, 2024. Accepted: November 5, 2024. Published: December 12, 2024.

1 Introduction

Fibonacci and Lucas sequences are famous number sequences. These sequences have intrigued scientists for a long time. Fibonacci and Lucas sequences have been applied to various fields such as Algebraic Coding Theory, Physics, Phyllotaxis, Biomathematics, Computer Science, Chemistry, etc. New sequences are obtained by changing the recurrence relation and initial conditions of the generalized Fibonacci sequence. The known examples of such sequences are the Horadam, k-Pell, k-Chebyshev sequence, Gaussian Fibonacci, Oresme numbers, Perrin, Narayana sequences, etc., [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24].

With the help of the recurrence relation of the Fibonacci sequence, k-sequences have been introduced, and these sequences have an important place in number theory. In [25], they obtained many features related to the k-Fibonacci sequence. Also, they showed new properties of k-Fibonacci and k-Lucas sequences, and they found correlations between these sequences, [26].

In [27], they obtained new properties by applying different transformations to the k-Fibonacci sequence. In addition, he worked on the k-Fibonacci difference sequence, [28]. Moreover, his another study, found many new formulas on k-Fibonacci and k-Lucas sequences, [29].

In [30], they made many applications on k-Mersenne numbers. In [31], they defined hyperbolic k-Balancing and k-Lucas Balancing numbers octonions. In [32], they defined the Catalan transformation of k-Pell, k-Pell-Lucas and modified k-Pell sequences. In addition, they found many properties of this transformation.

For $n \geq 0$, Fibonacci numbers F_n and Lucas numbers L_n are defined by the recurrence relations, respectively,

$$F_{n+2} = F_{n+1} + F_n, \text{ and } L_{n+2} = L_{n+1} + L_n,$$

with the initial conditions $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$.

Binet formulas for Fibonacci numbers F_n and Lucas numbers L_n are given by the following relations, respectively,

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n.$$

Here $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation, $r^2 - r - 1 = 0$. The number α is the known golden ratio.

Fibonacci polynomials were defined with the help of the recurrence relation of the Fibonacci sequence. In addition, many studies have been done with the help of Fibonacci polynomials, [33], [34], [35].

Dickson polynomials were introduced by L. E. Dickson (1897). For integer $n \geq 2$ and α with identity in a commutative ring R, Dickson polynomials first kind $D_n(x, \alpha)$ and Dickson polynomials second kind $E_n(x, \alpha)$ are defined by recurrence relations, respectively:

$$D_n(x, \alpha) = xD_{n-1}(x, \alpha) - \alpha D_{n-2}(x, \alpha)$$

and

$$E_n(x, \alpha) = xE_{n-1}(x, \alpha) - \alpha E_{n-2}(x, \alpha)$$

with the initial conditions $D_0 = 2$, $D_1 = x$ and $E_0 = 1$, $E_1 = x$.

Dickson Polynomials arise in various areas in mathematics, such as integro-differential-difference equations, cryptography and number theory. In addition, Dickson polynomials played a major role in the proof of the so-called Schur conjecture concerning integral polynomials which induce permutations on the field F_p for infinitely many primes p. Moreover, Dickson polynomials attracted the attention of scientists, and they did a lot of work on these polynomials, [36], [37], [38].

As seen above, many generalizations of Fibonacci and Lucas sequences have been given so far. In this study, we give new generalizations inspired by the Dickson polynomial. We call these polynomials the Dickson k-Fibonacci polynomial and denote them as $D\mathcal{F}_{k,n}(x)$.

We separate the article into three parts.

In chapter 2, we define Dickson k-Fibonacci $D\mathcal{F}_{k,n}(x)$ polynomials inspired by Dickson polynomials. We introduce the characteristic equation, the Binet formulas, and some properties of these polynomials. For these polynomials, we find generating functions, sum formulas, Cassini identity, Melham's identity, D'ocagne identity, etc.
In chapter 3, we define the Catalan transformation of $D\mathcal{F}_{k,n}(x)$ polynomials, and some properties are given. In addition, Hankel transformations are applied to the Catalan transformations of $D\mathcal{F}_{k,n}(x)$ polynomials, and the results obtained are associated with classical Fibonacci numbers.

2 Dickson k-Fibonacci Polynomials

Definition 2.1. For $k \in \mathbb{R}^+$ and $n \in \mathbb{N}$, Dickson k-Fibonacci polynomials $D\mathcal{F}_{k,n}(x)$ are defined by the recurrence relation:

$$D\mathcal{F}_{k,n+2}(x) = kxD\mathcal{F}_{k,n+1}(x) + D\mathcal{F}_{k,n}(x),$$

with $D\mathcal{F}_{k,0}(x) = 0$ and $D\mathcal{F}_{k,1}(x) = 1$.

The characteristic equation of these polynomials is:

$$r^2 - kxr - 1 = 0.$$

The roots of this equation are as follows:

$$r_1 = \frac{kx+\sqrt{k^2x^2+4}}{2} \text{ and } r_2 = \frac{kx-\sqrt{k^2x^2+4}}{2}.$$

The relationship between these roots is given with

$$r_1 + r_2 = kx, r_1 - r_2 = \sqrt{k^2x^2+4},$$
$$r_1^2 + r_2^2 = k^2x^2 + 2 \text{ and } r_1 r_2 = -1.$$

Some values for $D\mathcal{F}_{k,n}(x)$ are given below:
$D\mathcal{F}_{k,0}(x) = 0$, $D\mathcal{F}_{k,1}(x) = 1$, $D\mathcal{F}_{k,2}(x) = kx$, $D\mathcal{F}_{k,3}(x) = k^2x^2 + 1$, $D\mathcal{F}_{k,4}(x) = k^3x^3 + 2kx$,

Also, the terms of these polynomials can be found with the help of the following relation. For $n \in \mathbb{N}^+$:

$$D\mathcal{F}_{k,n}(x) = \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-1-i}{i} (kx)^{n-1-2i}.$$

In the following theorem, we express the Binet formulas of the Dickson k-Fibonacci polynomials.
Theorem 2.1. Let $n \in \mathbb{N}$. $D\mathcal{F}_{k,n}(x)$ have Binet formula as follows:

$$D\mathcal{F}_{k,n}(x) = \frac{r_1^n - r_2^n}{r_1 - r_2}.$$

Proof. The Binet form of a sequence is as follows

$$D\mathcal{F}_{k,n}(x) = c{r_1}^n + d{r_2}^n.$$

Here, the scalars c and d can be obtained by substituting the initial conditions. For $n = 0$, $D\mathcal{F}_{k,0} = 0$ and for $n = 1$, $D\mathcal{F}_{k,1} = 1$. Thus, we obtain:

$$c = \frac{1}{\sqrt{k^2x^2+4}} \text{ and } d = \frac{-1}{\sqrt{k^2x^2+4}}.$$

Thus,

$$D\mathcal{F}_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}. \qquad \square$$

In the following theorem, it is seen that $D\mathcal{F}_{k,n}(x)$ has some important relations with roots of the characteristic equation for these polynomials. Please note that these relations are independent of the choice of roots.

Theorem 2.2. Let $a, b, c, d \in \mathbb{N}$ and $y = r_1$ or $y = r_2$. We obtain

i. $y^a = D\mathcal{F}_{k,a-1}(x) + yD\mathcal{F}_{k,a}(x)$,

ii. $y^a = y^b D\mathcal{F}_{k,a-b+1}(x) + y^{b-1} D\mathcal{F}_{k,a-b}(x)$,

iii. $y^{ad} = \frac{y^a D\mathcal{F}_{k,ad}(x)}{D\mathcal{F}_{k,a}(x)} - (-1)^a \frac{D\mathcal{F}_{k,a(d-1)}(x)}{D\mathcal{F}_{k,a}(x)}$,

iv. $(-1)^{ab+1} D\mathcal{F}_{k,a(b-c)}(x)$
$= y^{ac} D\mathcal{F}_{k,ab}(x) - y^{ab} D\mathcal{F}_{k,ac}(x)$.

Proof. i. For $y = r_1$, we have

$$D\mathcal{F}_{k,a-1}(x) + yD\mathcal{F}_{k,a}(x) = \left(\frac{r_1^{a-1} - r_2^{a-1}}{r_1 - r_2}\right) + r_1\left(\frac{r_1^a - r_2^a}{r_1 - r_2}\right) = \frac{r_1^{a-1}(r_1^2+1) - r_2^{a-1}(r_1 r_2 + 1)}{r_1 - r_2} = {r_1}^a.$$

For $y = r_2$, we have

$$D\mathcal{F}_{k,a-1}(x) + yD\mathcal{F}_{k,a}(x) = \left(\frac{r_1^{a-1} - r_2^{a-1}}{r_1 - r_2}\right) + r_2\left(\frac{r_1^a - r_2^a}{r_1 - r_2}\right) = \frac{r_1^a\left(r_2 + \frac{1}{r_1}\right) + r_2^a\left(-\frac{1}{r_2} - r_2\right)}{r_1 - r_2} = {r_2}^a.$$

The proofs of the others are shown similarly. □

In the next theorem, the relationship between these polynomials is examined.

Theorem 2.3. Let $s, t \in \mathbb{N}$ and $k \in \mathbb{R}^+$. We have

i. $D\mathcal{F}_{k,s+t+1}(x)$
$= D\mathcal{F}_{k,s}(x) D\mathcal{F}_{k,t}(x) + D\mathcal{F}_{k,s+1}(x) D\mathcal{F}_{k,t+1}(x)$,

ii. $(k^2x^2+4) D\mathcal{F}_{k,s}^3(x)$
$= D\mathcal{F}_{k,3s}(x) - 3(-1)^s D\mathcal{F}_{k,s}(x)$,

iii. $D\mathcal{F}_{k,s}^2(x) D\mathcal{F}_{k,s+3}(x) - D\mathcal{F}_{k,s+1}^3(x) = \frac{(-1)^s}{k^2x^2+4}$
$(D\mathcal{F}_{k,s-3}(x) - 3D\mathcal{F}_{k,s+1}(x) - 2D\mathcal{F}_{k,s+3}(x))$,

iv. $D\mathcal{F}_{k,n}(x) = (-1)^{n-1} D\mathcal{F}_{k,-n}(x)$.

Proof. ii. If the Binet formula is used, we obtain

$$(k^2x^2+4) D\mathcal{F}_{k,s}^3(x) = (k^2x^2+4)\left(\frac{r_1^s - r_2^s}{r_1 - r_2}\right)^3$$
$$= \frac{r_1^{3s} - r_2^{3s} - 3r_1^{2s} r_2^s + 3r_1^s r_2^{2s}}{r_1 - r_2}$$
$$= \frac{r_1^{3s} - r_2^{3s} - 3r_1^s r_2^s (r_1^s - r_2^s)}{(r_1 - r_2)^2}$$
$$= D\mathcal{F}_{k,3s}(x) - 3(-1)^s D\mathcal{F}_{k,s}(x).$$

The proofs of the others are shown similarly. □

In the following theorem, some known identities for these polynomials are calculated.

Theorem 2.4. Let $n, r, i, j \in \mathbb{N}$ and $k \in \mathbb{R}^+$. We obtain

i. (Cassini Identity)
$D\mathcal{F}_{k,n-1}(x) D\mathcal{F}_{k,n+1}(x) - D\mathcal{F}_{k,n}^2(x) = (-1)^{n-1}$,

ii. (Catalan Identity)
$D\mathcal{F}_{k,n+r}(x) D\mathcal{F}_{k,n-r}(x) - D\mathcal{F}_{k,n}^2(x)$
$= (-1)^{n-1} D\mathcal{F}_{k,r}^2(x)$,

iii. (D'ocagne Identity)
$D\mathcal{F}_{k,r}(x) D\mathcal{F}_{k,n+1}(x) - D\mathcal{F}_{k,n}(x) D\mathcal{F}_{k,r+1}(x)$
$= (-1)^r D\mathcal{F}_{k,n-r}(x)$,

iv. (Vajda Identity)
$D\mathcal{F}_{k,n+i}(x) D\mathcal{F}_{k,n+j}(x) - D\mathcal{F}_{k,n}(x) \mathcal{F}_{k,n+i+j}(x)$
$= (-1)^n D\mathcal{F}_{k,i}(x) D\mathcal{F}_{k,j}(x)$,

v. (Melham Identity)
$D\mathcal{F}_{k,n+1}(x) D\mathcal{F}_{k,n+2}(x) D\mathcal{F}_{k,n+6}(x) - D\mathcal{F}_{k,n}^3(x)$
$= \frac{1}{k^2x^2+4}(D\mathcal{F}_{k,3n+9}(x) - D\mathcal{F}_{k,3n}(x)$
$+3(-1)^n D\mathcal{F}_{k,n}(x) - (-1)^{n+6} D\mathcal{F}_{k,n-3}(x)$
$-(-1)^{n+2} D\mathcal{F}_{k,n+5}(x) - (-1)^{n+1} D\mathcal{F}_{k,n+7}(x))$.

Proof. If the Binet formula is used, we get

i. $D\mathcal{F}_{k,n-1}(x) D\mathcal{F}_{k,n+1}(x) - D\mathcal{F}_{k,n}^2(x)$

$$= \frac{{r_1}^{n-1} - {r_2}^{n-1}}{r_1 - r_2} \frac{{r_1}^{n+1} - {r_2}^{n+1}}{r_1 - r_2} - \frac{{r_1}^n - {r_2}^n}{r_1 - r_2} \frac{{r_1}^n - {r_2}^n}{r_1 - r_2}$$
$$= \frac{{r_1}^{2n} - {r_1}^{n+1}{r_2}^{n-1} - {r_2}^{n+1}{r_1}^{n-1} + {r_2}^{2n}}{(r_1 - r_2)^2} - \frac{{r_1}^{2n} - 2{r_1}^n{r_2}^n + {r_2}^{2n}}{(r_1 - r_2)^2}$$
$$= (-1)^{n-1}.$$

The proofs of the others are shown similarly. □

In the following theorems, we obtain the summation formula and the generating function of this polynomials.

Theorem 2.5. Let $n \in \mathbb{N}$ and $k \in \mathbb{R}^+$. We obtain

$$\sum_{s=0}^{n} D\mathcal{F}_{k,s}(x) = \frac{D\mathcal{F}_{k,n+1}(x) + D\mathcal{F}_{k,n}(x) - 1}{kx}.$$

Proof. If the Binet formula is used, we obtain

$$\sum_{s=0}^{n} D\mathcal{F}_{k,s}(x) = \frac{{r_1}^s - {r_2}^s}{r_1 - r_2}$$
$$= \frac{1}{r_1 - r_2}\left(\sum_{s=0}^{n} {r_1}^s - \sum_{s=0}^{n} {r_2}^s\right)$$
$$= \frac{1}{r_1 - r_2}\left(\frac{{r_1}^{n+1} - 1}{r_1 - 1} - \frac{{r_2}^{n+1} - 1}{r_2 - 1}\right)$$
$$= \frac{\mathcal{F}_{k,n+1}(x) + \mathcal{F}_{k,n}(x) - 1}{kx}.$$ □

Theorem 2.6. Let $, s, t \in \mathbb{N}$, $c > s$ and $k \in \mathbb{R}^+$. We obtain

i. $\sum_{j=0}^{n} \binom{n}{j} (kx)^j D\mathcal{F}_{k,j}(x) = D\mathcal{F}_{k,2n}(x)$,

ii. $\sum_{j=0}^{n} \binom{n}{j} (kx)^j D\mathcal{F}_{k,cn+s+j}(x)$
$= D\mathcal{F}_{k,cn+2n+s}(x)$,

iii. $\sum_{j=0}^{n} \binom{n}{j} (-1)^j \frac{D\mathcal{F}_{k,cn+s+j}(x)}{(kx)^j}$
$= (-1)^n \frac{D\mathcal{F}_{k,cn-n+s}(x)}{(kx)^n}$

iv. $\sum_{j=0}^{n} (\frac{t}{k})^j \left(tD\mathcal{F}_{k,j-s+2}(x) - kxD\mathcal{F}_{k,j-s+1}(x)\right)$

$$= (\frac{t}{k})^n tD\mathcal{F}_{k,n-s+2}(x) - kD\mathcal{F}_{k,-s+1}(x).$$

Proof. The following equations are obtained with the help of the characteristic equation of the Dickson k-Fibonacci polynomials:

$$r_1^2 = kxr_1 + 1 \text{ and } r_2^2 = kxr_2 + 1.$$

iv. $\sum_{j=0}^{n}(\frac{t}{k})^j \left(tD\mathcal{F}_{k,j-b+2}(x) - kxD\mathcal{F}_{k,j-b+1}(x)\right)$

$= \frac{1}{r_1-r_2}\sum_{j=0}^{n}(\frac{t}{k})^j \left(t({r_1}^{j-b+2} - {r_2}^{j-b+2}) - k({r_1}^{j-b+1} - {r_2}^{j-b+1})\right)$

$= \frac{1}{r_1-r_2}[r_1^{-b+1}(tr_1 - k)\sum_{j=0}^{n}(\frac{tr_1}{k})^j - r_2^{-b+1}(tr_2 - k)\sum_{j=0}^{n}(\frac{tr_2}{k})^j]$

$= \frac{k}{r_1-r_2}(r_1^{-b+1}\frac{(tr_1)^{n+1}-k^{n+1}}{k^{n+1}} - r_2^{-b+1}\frac{(tr_2)^{n+1}-k^{n+1}}{k^{n+1}})$

$= (\frac{t}{k})^n tD\mathcal{F}_{k,n-b+2}(x) - kD\mathcal{F}_{k,-b+1}(x).$

The proofs of the others are shown similarly. □

Theorem 2.7. Let $n \in \mathbb{N}$ and $k \in \mathbb{R}^+$. We obtain

$$\mathbb{f}(t) = \sum_{n=0}^{\infty} D\mathcal{F}_{k,n}(x)t^n = \frac{t}{1-tkx-t^2}.$$

Proof. We have

$$\begin{aligned}\mathbb{f}(t) &= \sum_{n=0}^{\infty} D\mathcal{F}_{k,n}(x)t^n = t + \sum_{n=2}^{\infty} D\mathcal{F}_{k,n}(x)t^n \\ &= t + kx\sum_{n=2}^{\infty} D\mathcal{F}_{k,n-1}(x)t^n \\ &\quad + \sum_{n=2}^{\infty} D\mathcal{F}_{k,n-2}(x)t^n \\ &= t + tkx\sum_{n=1}^{\infty} D\mathcal{F}_{k,n}(x)t^n \\ &\quad + t^2\sum_{n=0}^{\infty} D\mathcal{F}_{k,n}(x)t^n \\ &= \frac{t}{1-tkx-t^2}.\end{aligned}$$

□

3 Catalan Transform

In this chapter, we define the Catalan transformation of $D\mathcal{F}_{k,n}(x)$ polynomials, and some properties are given. In addition, Hankel transformations are applied to the Catalan transformations of $D\mathcal{F}_{k,n}(x)$ polynomials, and the results obtained are associated with classical Fibonacci numbers.

Definition 3.1. (Catalan Number) For $n \in \mathbb{N}$, the n^{th} Catalan numbers are as follows:

$$C_n = \frac{C(2n,n)}{n+1}.$$

With the help of this relation, C_n numbers are $1, 1, 2, 5, 14, 132, 429, \ldots$ [39].

3.1 Catalan Transform of the Dickson *k*-Fibonacci Polynomials

Using the Catalan transform, we define the Catalan transform of the Dickson k-Fibonacci polynomials as follows. For $n \geq 1$,

$$CD\mathcal{F}_{k,n}(x) = \sum_{i=0}^{n}\frac{i}{2n-i}\binom{2n-i}{n-i}D\mathcal{F}_{k,i}(x)$$

with $CD\mathcal{F}_{k,0}(x) = 0$.

Now we can give the Catalan transformation of the first elements of the Dickson k-Fibonacci polynomials. The $CD\mathcal{F}_{k,n}(x)$ values for the first four n natural numbers are given below:

- $CD\mathcal{F}_{k,0}(x) = 0,$
- $CD\mathcal{F}_{k,1}(x) = 1,$
- $CD\mathcal{F}_{k,2}(x) = kx + 1,$
- $CD\mathcal{F}_{k,3}(x) = k^2x^2 + 2kx + 3,$
- $CD\mathcal{F}_{k,4}(x) = k^3x^3 + 3k^2x^2 + 7kx + 8,$
- $CD\mathcal{F}_{k,5}(x) = k^4x^4 + 4k^3x^3 + 12k^2x^2 + 22kx + 24.$

Definition 3.2. Let the terms of a sequence be $A = \{v_1, v_2, v_3, \ldots\}$. In [40], the Hankel transform H_{n} of the terms of this sequence was defined as follows:

$$H_{\mathrm{n}} = \begin{vmatrix} v_1 & v_2 & v_3 & v_4 & \ldots \\ v_2 & v_3 & v_4 & v_5 & \ldots \\ v_3 & v_4 & v_5 & v_6 & \ldots \\ v_4 & v_5 & v_6 & v_7 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix}.$$

Let's apply Hankel's work to the Catalan Dickson k-Fibonacci polynomials. We get;

- $HCD\mathcal{F}_1 = det[CD\mathcal{F}_{k,1}(x)] = det[1] = 1,$
- $HCD\mathcal{F}_2 = det\begin{bmatrix} CD\mathcal{F}_{k,1}(x) & CD\mathcal{F}_{k,2}(x) \\ CD\mathcal{F}_{k,2}(x) & CD\mathcal{F}_{k,3}(x) \end{bmatrix}$

$$= det\begin{bmatrix} 1 & kx+1 \\ kx+1 & k^2x^2+2kx+3 \end{bmatrix} = 2,$$

- $HCD\mathcal{F}_3$

$$= det\begin{bmatrix} CD\mathcal{F}_{k,1}(x) & CD\mathcal{F}_{k,2}(x) & CD\mathcal{F}_{k,3}(x) \\ CD\mathcal{F}_{k,2}(x) & CD\mathcal{F}_{k,3}(x) & CD\mathcal{F}_{k,4}(x) \\ CD\mathcal{F}_{k,3}(x) & CD\mathcal{F}_{k,4}(x) & CD\mathcal{F}_{k,5}(x) \end{bmatrix} = 5$$

- $HCD\mathcal{F}_4 = 13,$
- $HCD\mathcal{F}_5 = 34.$

In the next theorem, a very interesting property is obtained.

Main Theorem 3.1. Applying the Hankel transform to the Catalan transform of Dickson k-Fibonacci polynomials, the following property is obtained:

$$HCD\mathcal{F}_n = F_{2n-1}.$$

Here, $n \in \mathbb{N}$ and F_n is classical Fibonacci sequence.

Proof. $HCD\mathcal{F}_n \neq 0$ and let us write it as $HCD\mathcal{F}_n = detA_n detB_n$. Here, the properties of matrices A_n and B_n are as follows.

A_n is the matrix with the principal diagonal $\{1, 1, 1, \ldots\}$, the nxn type lower triangular matrix and the first column

$\{CD\mathcal{F}_1(x), CD\mathcal{F}_2(x), CD\mathcal{F}_3(x), \dots\}$. B_n is the matrix with the principal diagonal $\{1, 2, \frac{5}{2}, \frac{13}{5}, \frac{34}{13}, \frac{89}{34}, \dots\}$, the nxn type upper triangular matrix and the first row $\{CD\mathcal{F}_1(x), CD\mathcal{F}_2(x), CD\mathcal{F}_3(x), \dots\}$. So,

$$HCD\mathcal{F}_n = det\begin{pmatrix} CD\mathcal{F}_1 = 1 & 0 & 0 & 0 & 0 \\ CD\mathcal{F}_2 & 1 & 0 & 0 & 0 \\ CD\mathcal{F}_3 & \dots & 1 & 0 & 0 \\ CD\mathcal{F}_4 & \dots & \cdots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

$$det\begin{pmatrix} CD\mathcal{F}_1 = 1 & CD\mathcal{F}_2 & CD\mathcal{F}_3 & CD\mathcal{F}_4 & \dots \\ 0 & 2 & \ddots & \vdots & \dots \\ 0 & 0 & \frac{5}{2} & 0 & \dots \\ 0 & 0 & 0 & \frac{13}{5} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Thus,

$$\begin{aligned} HCD\mathcal{F}_n &= detA_n detB_n = detB_n \\ &= b_{11}b_{22}b_{33}b_{44}\dots b_{nn} \\ &= \{1, 2, 5, 13, 34, 89, \dots\} \\ &= F_{2n-1}. \end{aligned}$$ □

4 Conclusion

In this paper, we defined the new Dickson k-Fibonacci polynomials. Then, we obtained the main features of these polynomials. Also, we examined the relationships between the terms of these polynomials. In addition, the Catalan transformation of the Dickson k-Fibonacci polynomials was defined, and the terms of this transformation were found. Moreover, we applied the Hankel transform to the Catalan transform, and we found an interesting feature. If this study is examined, such features can be found in other sequences such as Fermat, and Mersenne sequences.

References:

[1] James, V., Sivakumar, B., Generating Functions For a New Class of Recursive Polynomials, *Journal of Advances in Mathematics and Computer Science*, Vol.39, No.5, 2024, pp. 15-28. DOI: https://doi.org/10.9734/jamcs/2024/v39i51887

[2] Soykan, Y., On Generalized Fibonacci Polynomials: Horadam Polynomials, *Earthline Journal of Mathematical Sciences*, Vol.11 No.1, 2023, pp. 23-114. DOI: https://doi.org/10.34198/ejms.11123.23114.

[3] Soykan, Y., Generalized Oresme Numbers, *Earthline Journal of Mathematical Sciences*, Vol.7, No.2, 2021, pp. 333-367. DOI: https://doi.org/10.34198/ejms.7221.333367.

[4] De la Sen, M., Some Boundedness, Convergence, and Oscillatory Results of Generalized Fibonacci- Type and Fibonacci Ratio Recurrences. International Journal of Mathematics and Mathematical Sciences, Vol.2024, No.1, 2024, pp. 6439971. DOI: https://doi.org/10.1155/2024/6439971.

[5] Godase, A. D., Some new properties of hyperbolic k-Fibonacci and k-Lucas octonions, Notes on Number Theory and Discrete Mathematics, Vol.30, No.1, 2024, pp. 100-110. DOI: https://doi.org/10.7546/nntdm.2024.30.1.100-110.

[6] Değirmen, N., Duyar, C., A new perspective on Fibonacci and Lucas numbers, *Filomat*, Vol.37, No.28, 2024, pp. 9561-9574. DOI: https://doi.org/10.2298/FIL2328561D.

[7] Özkan, E. Kuloğlu, B., On the new Narayana polynomials, the Gauss Narayana numbers and their polynomials, *Asian-European Journal of Mathematics*, Vol.14, No.6, 2021, pp. 2150100. DOI: https://doi.org/10.1142/S179355712150100X.

[8] Catarino, P., On some identities and generating functions for k-Pell numbers, *International Journal of Mathematical Analysis*, Vol. 7, No.38, 2013, pp. 1877-1884. DOI: https://dx.doi.org/10.12988/ijma.2013.35131.

[9] Bravo, E. F., Common Values of Padovan and Perrin Sequences, *Mediterranean Journal of Mathematics*, Vol.20, No.5, 2023, pp. 268. DOI: https://doi.org/10.1007/s00009-023-02467-2.

[10] Belbachir, H., Krim, F., On hyper (r, q)-Fibonacci polynomials, *Mathematica Slovaca*, Vol.74, No.1, 2024, pp. 5-26. DOI: https://doi.org/10.1515/ms-2024-0002

[11] Özkan, E., Akkuş, H., On k-Chebsyhev Sequence, *Wseas Transactions on Mathematics*, Vol. 20, 2023, pp. 503-507. DOI: https://doi.org/10.37394/23206.2023.22.56.

[12] Falcon, S., On the k-Lucas numbers, *International Journal of Contemporary Mathematical Sciences*, Vol.6, No.21, 2011, pp. 1039-1050, [Online]. https://www.m-hikari.com/ijcms-2011/21-24-

2011/falconIJCMS21-24-2011.pdf (Accessed Date: September 15, 2024).

[13] Hong, S., Quaternion-Type Catalan Transforms of the p-Fibonacci and p-Lucas Numbers, *Journal of Mathematics*, 2023, pp. 1-11. DOI: https://doi.org/10.1155/2023/2439110.

[14] Özkan E., Taştan M., On a new family of gauss k-Lucas numbers and their polynomials, *Asian-European Journal of Mathematics*, Vol.14, No.6, 2021, pp. 2150101. DOI: https://doi.org/10.1142/S1793557121501011

[15] Soykan, Y., A study on generalized fibonacci polynomials: Sum formulas, *International Journal of Advances in Applied Mathematics and Mechanics*, Vol.10, No.1, 2022, pp. 39-118, [Online]. http://www.ijaamm.com/uploads/2/1/4/8/21481830/v10n1p4_39-118.pdf (Accessed Date: September 15, 2024).

[16] Bród, D., Szynal-Liana, A., Włoch, I., On a New Generalization of Pell Hybrid Numbers, In Annales Mathematicae Silesianae, Vol.38, No.2, 2024, pp. 221-240. DOI: https://doi.org/10.2478/amsil-2024-0011.

[17] Kirklar, E., Maden, A. D., Yilmaz, F., Some results on one type of graph family with some special number sequences, AKCE *International Journal of Graphs and Combinatorics*, Vol.17, No.3, 2020, pp. 749-754. DOI: https://doi.org/10.1016/j.akcej.2019.09.005.

[18] Özimamoğlu, H., On hyper complex numbers with higher order Pell numbers components, The Journal of Analysis, Vol.31, No.4, 2023, pp. 2443-2457. DOI: https://doi.org/10.1007/s41478-023-00579-2.

[19] Soykan, Y., Generalized Horadam-Leonardo Numbers and Polynomials, Asian Journal of Advanced Research and Reports, Vol.17, No.8, 2023, pp. 128-169. DOI: https://doi.org/10.9734/AJARR/2023/v17i8511.

[20] Aktaş, S., Soykan, Y., On the Norms of Toeplitz Matrices with the Generalized Oresme Numbers, Asian Journal of Advanced Research and Reports, Vol.17, No.10, 2023, pp. 41-57. DOI: https://doi.org/10.9734/ajarr/2023/v17i10532.

[21] Yılmaz, F., Ertaş, E., Yamaç, S. Determinants of circulant matrices with Gaussian nickel Fibonacci numbers, Filomat, Vol.37, No.25, 2022, pp. 8683-8692. DOI: https://doi.org/10.2298/FIL2325683Y.

[22] Patra, A., Kaabar, M. K., Catalan Transform of k-Balancing Sequences, International Journal of Mathematics and Mathematical Sciences, Vol.2021, No.1, 2021, pp. 9987314. DOI: https://doi.org/10.1155/2021/9987314.

[23] Yılmaz, N., Aktaş, İ., Special transforms of the generalized bivariate Fibonacci and Lucas polynomials, Hacettepe Journal of Mathematics and Statistics, Vol.52, No.3, 2023, pp. 640-651. DOI: https://doi.org/10.15672/hujms.1110311.

[24] Taher, H., Dash, S. K., Common Terms of k-Pell and Tribonacci Numbers, *European Journal of Pure and Applied Mathematics*, Vol.17, No.1, 2024, pp. 135-146. DOI: https://doi.org/10.29020/nybg.ejpam.v17i1.4989.

[25] El-Mikkawy, M., Sogabe, T., A new family of k-Fibonacci numbers, *Applied Mathematics and Computation*, Vol.215, No.12, 2015, pp. 4456-4461. DOI: https://doi.org/10.1016/j.amc.2009.12.069.

[26] Godase, A. D., Dhakne, M. B., On the properties of k-Fibonacci and k-Lucas numbers, *International Journal of Advances in Applied Mathematics and Mechanics*, Vol.2, No.1, 2014, pp. 100-106, [Online]. http://www.ijaamm.com/uploads/2/1/4/8/21481830/v2n1p12.pdf (Accessed Date: September 15, 2024).

[27] Falcon, S., and Plaza, A., On the Fibonacci k-numbers, *Chaos Solitons Fractals*, Vol.32, No.5, 2007, pp. 1615–1624. DOI: https://doi.org/10.1016/j.chaos.2006.09.022.

[28] Falcon, S., The k–Fibonacci difference sequences, *Chaos, Solitons & Fractals*, Vol.87, 2016, pp. 153-157. DOI: https://doi.org/10.1016/j.chaos.2016.03.038.

[29] Falcon, S., Some new formulas on the k-Fibonacci numbers, *Journal of Advances In Mathematics*, Vol.14, No.1, 2018, pp. 7439-7445. DOI: https://doi.org/10.24297/jam.v14i1.6918.

[30] Kumari M., Tanti J., Prasad K., On Some New Families of k-Mersenne and Generalized k-Gaussian Mersenne Numbers and their Polynomials, Contributions to Discrete Mathematics, Vol.18, No.2, 2024, pp. 244-260. DOI: https://doi.org/10.48550/arXiv.2111.09592.

[31] Prasad, K., Kumari, M., Tanti, J. Octonions and hyperbolic octonions with the k-balancing and k-Lucas balancing numbers, *The Journal*

of Analysis, Vol.32, 2024, pp. 1-16. DOI: https://doi.org/10.1007/s41478-023-00716-x.

[32] Tastan, M., Özkan, E., Catalan transform of the k-Pell, k-Pell-Lucas and modified k-Pell sequence, *Notes on Number Theory and Discrete Mathematics*, vol.27, No.1, 2021, pp. 198-207. DOI: https://doi.org/10.7546/nntdm.2021.27.1.198-207.

[33] Alatawi, M. S., Khan, W. A., Kızılateş, C., Ryoo, C. S., Some Properties of Generalized Apostol-Type Frobenius–Euler–Fibonacci Polynomials, *Mathematics*, Vol.12, No.6, 2024, pp. 800. DOI: https://doi.org/10.3390/math12060800.

[34] Alahmadi, A., Klurman, O., Luca, F., Shoaib, H., On the arguments of the roots of the generalized Fibonacci polynomial, *Lithuanian Mathematical Journal*, Vol.63, No.3, 2023, pp. 249-253. DOI: https://doi.org/10.1007/s10986-023-09604-0.

[35] Yamani, H. A., Supersymmetric Fibonacci polynomials, *Analysis and Mathematical Physics*, Vol.11, No.2, 2021, pp. 65. DOI: 10.1007/s13324-021-00496-2.

[36] Panj-Mini, H., Moghaddam, B. P., Hashemizadeh, E., A class of computational approaches for simulating fractional functional differential equations via Dickson polynomials, *Chaos, Solitons & Fractals*, Vol.152, 2021, pp. 111407. DOI: https://doi.org/10.1016/j.chaos.2021.111407.

[37] Wang, Q., Yucas, J. L., Dickson polynomials over finite fields, *Finite Fields and Their Applications*, Vol.18, No.4, 2012, pp. 814-831. DOI: https://doi.org/10.1016/j.ffa.2012.02.001.

[38] Chen, S. B., Soradi-Zeid, S., Alipour, M., Chu, Y. M., Gomez-Aguilar, J. F., Jahanshahi, H., Optimal control of nonlinear time-delay fractional differential equations with Dickson polynomials, *Chaos, Solitons & Fractals*, Vol.29, No.4 pp. 2150079. DOI: https://doi.org/10.1142/S0218348X21500791.

[39] Barry, P., A Catalan transform and related transformations on integer sequences, *Journal of Integer Sequences*, Vol.8, No.4, 2005, pp. 1-24, [Online]. https://cs.uwaterloo.ca/journals/JIS/VOL8/Barry/barry84.pdf (Accessed Date: September 15, 2024).

[40] Layman, J. W., The Hankel transform and some of its properties, *Journal of Integer Sequences*, Vol.4, No.1, 2001, pp. 1-11, [Online]. https://cs.uwaterloo.ca/journals/JIS/VOL4/LAYMAN/hankel.pdf (Accessed Date: September 15, 2024).

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

- Engin Ozkan carried out the introduction and the main result of the article.
- Hakan Akkus has improved Chapter 2 and Chapter 3.

All authors read and approved the final manuscript.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

This study did not receive any funding in any form.

Conflict of Interest

The authors have no conflict of interest to declare.