New fixed point theorems in complete rectangular M-metric spaces

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Abstract: In this article we extend the Banach contraction principle known in the framework of rectangular metric spaces ($\theta-$contraction) to the more general rectangular M-metric spaces. We also investigate the existence and uniqueness of fixed point for mappings satisfying $\theta-$contraction in rectangular M-metric spaces. Moreover, we provide some examples to highlight the obtained improvements. Finally, as an application, we investigate the existence and uniqueness of a solution of a non-linear integral equation of Fredholm type.

Key-Words: - Fixed point, M-metric spaces, generalized $\theta-$contraction

Received: June 4, 2024. Revised: October 16, 2024. Accepted: November 7, 2024. Published: December 12, 2024.

2000 MSC (2000 *Mathematics Subject Classification*): Primary 47H10; Secondary 54H25

1 Introduction

The fixed point theory is one of the most powerful tools in many fields such as nonlinear analysis, operator theory, differential equations, integral equations, theory of fractals, engineering, computer sciences, mathematical modelling, econometrics, optimization problems, game theory, etc., [1]. In 1922 [2] first established a theorem concerning a contraction mapping. Due to its wide range of applications in mathematical research, several authors have obtained many interesting extensions and generalizations of the Banach contraction principle, either by weakening the conditions of contraction mapping or by changing the abstract structure. So, many different metric-type spaces have been considered, among which quasi metric, partial metric, rectangular metric, b-metric, Super metric, fuzzy metric spaces and many other their combinations, [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16].

A partial metric space is one of the most influential generalizations of ordinary metric space. It was first introduced in [17], replacing the equality $m(a, a) = 0$ in the definition of metric with the inequality $m(a, a) \leq m(a, b)$ for all a, b, therefore the self distance of any point of the space may not be zero. Further, in [17], it was showed that the Banach contraction principle is valid in partial metric spaces and can be applied in program verification. In 2014, the partial metric space was generalized to M-metric space, [4], and the rectangular metric spaces and partial metric spaces were extended to partial rectangular metric spaces, [14].

On the other hand, [5], gave a generalization of the notion of metric spaces, which are called Branciari distance spaces, by replacing triangle inequality with trapezoidal inequality, and he gave an extension of the Banach contraction principle in these spaces. In 2018, [13], introduced the rectangular M-metric space and obtained some theorems related to these spaces.

In this paper we introduce the notion of generalized θ-contraction to extend both previous notions in rectangular metric spaces. Moreover, we provide some examples to illustrate the obtained results and we derive some useful corollaries of these results.

2 Preliminaries

Definition 2.1. [5] Let $\mathcal{X}$ be a non-empty set and $m : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ be a mapping such that, for all $a, b \in \mathcal{X}$ and for all distinct points $c, d \in \mathcal{X} \setminus \{a, b\}$, one has

(i) $m(a, b) = 0 \Leftrightarrow a = b$;

(ii) $m(a, b) = m(b, a)$;

(iii) $m(a, b) \leq m(a, c) + m(c, d) + m(d, b)$.

Then $(\mathcal{X}, m)$ is called *rectangular metric space*.

Note that every metric space is a rectangular metric space.

Later, the partial rectangular metric space was introduced as follows.

Definition 2.2. [14] Let $\mathcal{X}$ be a non-empty set and $m : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ be a mapping such that, for all $a, b \in \mathcal{X}$ and for all distinct points $c, d \in \mathcal{X} \setminus \{a, b\}$, one has

(i) $a = b \Leftrightarrow m(a, b) = m(a, a) = m(b, b)$;

(ii) $m(a, b) = m(b, a)$;

(iii) $m(a, b) \leq m(a, c)+m(c, d)+m(d, b)-m(c, c)-m(d, d)$.

Then $(\mathcal{X}, m)$ is called *partial rectangular metric space*.

Remark 2.3. [14] In a partial rectangular metric space $(\mathcal{X}, m)$ if $a, b \in \mathcal{X}$ and $m(a, b) = 0$, then $a = b$ but the converse may not be true.

In 2014, [4], generalized the partial metric space to the M-metric space and obtained certain theorems related to M-metric spaces.

Let us denote

$$\begin{aligned} m_{a,b} &= \min\{m(a, a), m(b, b)\}, \\ M_{a,b} &= \max\{m(a, a), m(b, b)\}. \end{aligned}$$

Definition 2.4. [4] Let $\mathcal{X}$ be a non-empty set and $m : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ be a mapping such that, for all $a, b \in \mathcal{X}$ and for all distinct points $c, d \in \mathcal{X} \setminus \{a, b\}$, one has

(i) $a = b \Leftrightarrow m(a, b) = m(a, a) = m(b, b)$;

(ii) $m(a, b) = m(b, a)$;

(ii) $m_{ab} \leq m(a, b)$;

(iv) $m(a, b)-m_{a,b} \leq m(a, c)-m_{a,c}+m(c, b)-m_{c,b}$.

Then $(\mathcal{X}, m)$ is called *M-metric space*.

In 2018, [13], introduced the rectangular M-metric space as follows. In the sequel we will use the following notations:

$$\begin{aligned} m_{r_{a,b}} &= \min\{m_r(a, a), m_r(b, b)\}, \\ M_{r_{a,b}} &= \max\{m_r(a, a), m_r(b, b)\}. \end{aligned} \tag{1}$$

Definition 2.5. [13] Let $\mathcal{X}$ be a non-empty set and $m_r : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ be a mapping such that, for all $a, b \in \mathcal{X}$ and for all distinct points $c, d \in \mathcal{X} \setminus \{a, b\}$, one has

(i) $a = b \Leftrightarrow m_r(a, b) = m_r(a, a) = m_r(b, b)$;

(ii) $m_r(a, b) = m_r(b, a)$;

(ii) $m_{r_{a,b}} \leq m_r(a, b)$;

(iv) $m_r(a, b)-m_{r_{a,b}} \leq m_r(a, c)-m_{r_{a,c}}+m_r(c, d)-m_{r_{c,d}} + m_r(d, b) - m_{r_{d,b}}$
(rectangular M-inequality).

Then $(\mathcal{X}, m_r)$ is called *rectangular M-metric space*.

Example. [13] Let $(\mathcal{X}, m_r)$ be a rectangular M-metric space and $m_r^\omega(a, b) : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a function defined as

$$m_r^\omega(a, b) = m_r(a, b) - 2m_{r_{a,b}} + M_{r_{a,b}},$$

for all $a, b \in \mathcal{X}$. Then, m_r^ω is a rectangular metric and the pair $(\mathcal{X}, m_r^\omega)$ is a rectangular metric space.

Example. [13] Let $(\mathcal{X}, m_r)$ be a rectangular M-metric space and $m_r^s(a, b) : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a function defined as

$$m_r^s(a, b) = m_r(a, b) - m_{r_{a,b}},$$

for all $a, b \in \mathcal{X}$ such that, if $m_r^s(a, b) = 0$, then $a = b$. Then, m_r^s is a rectangular metric and the pair $(\mathcal{X}, m_r^s)$ is a rectangular metric space.

Remark 2.6. The connections among the spaces defined above are described in [13]:

- metric space $\Rightarrow$ rectangular metric space $\Rightarrow$ partial rectangular metric space $\Rightarrow$ rectangular M-metric space
- metric space $\Rightarrow$ M-metric space $\Rightarrow$ rectangular M-metric space.

Definition 2.7. [13] Let $(\mathcal{X}, m_r)$ be a rectangular M-metric space. Then

(i) A sequence $\{a_n\}_{n\in\mathbb{N}}$ in $\mathcal{X}$ converges to a point a if and only if

$$\lim_{n\to+\infty} (m_r(a_n, a) - m_{r_{a_n,a}}) = 0. \tag{2}$$

(ii) A sequence $\{a_n\}_{n\in\mathbb{N}}$ in $\mathcal{X}$ is said to be m_r-Cauchy sequence if and only if

$$\begin{aligned} &\lim_{n,m\to+\infty} (m_r(a_n, a_m) - m_{r_{a_n,a_m}}) \\ &\lim_{n,m\to+\infty} (M_{r_{a_n,a}} - m_{r_{a_n,a_m}}) \end{aligned} \tag{3}$$

exist and are finite.

(iii) A rectangular M-metric space is said to be m_r-complete if every m_r- Cauchy sequence $\{a_n\}$ converges to a point a such that

$$\begin{aligned} &\lim_{n\to+\infty} (m_r(a_n, a) - m_{r_{a_n,a}}) = 0, \\ &\lim_{n\to+\infty} (M_{r_{a_n,a}} - m_{r_{a_n,a}}) = 0. \end{aligned} \tag{4}$$

Lemma 2.8. [13] *Let $(\mathcal{X}, m_r)$ be a rectangular M-metric space. Then,*

(1) $\{a_n\}$ is a m_r-Cauchy sequence in $(\mathcal{X}, m_r)$ if and only if $\{a_n\}$ is a Cauchy sequence in $(\mathcal{X}, m_r^\omega)$ (resp. $(\mathcal{X}, m_r^s)$).

(2) $(\mathcal{X}, m_r)$ *is* m_r*-complete if and only if* $(\mathcal{X}, m_r^{\omega})$ *(resp.* $(\mathcal{X}, m_r^{s})$*) is complete.*

Lemma 2.9. [13] *Assume that* $a_n \to a$ *as* $n \to \infty$ *in a rectangular* M*- metric space* $(\mathcal{X}, m_r)$*. Then,*

$$\lim_{n\to+\infty}(m_r(a_n, b) - m_{r_{a_n,b}}) = m_r(a, b) - m_{r_{a,b}},\quad \forall\, b \in \mathcal{X}.$$

Lemma 2.10. [13] *Assume that* $a_n \to a$*,* $b_n \to b$*, as* $n \to \infty$*, in a rectangular* M*-metric space* $(\mathcal{X}, m_r)$*. Then,*

$$\lim_{n\to+\infty}(m_r(a_n, b_n) - m_{r_{a_n,b_n}}) = m_r(a, b) - m_{r_{a,b}}.$$

Lemma 2.11. [13] *Let* a_n *be a sequence in a rectangular* M*-metric space* $(\mathcal{X}, m_r)$ *and there exists* $k \in]0, 1[$ *such that*

$$m_r(a_{n+1}, a_n) \le k\, m_r(a_n, a_{n-1}),\ for\ all\ n \in \mathbb{N}.$$

Then,

(A) $\lim_{n\to\infty} m_r(a_n, a_{n-1}) = 0$*,*

(B) $\lim_{n\to\infty} m_r(a_n, a_n) = 0$

(C) $\lim_{n,m\to\infty} m_{r_{a_n,a_m}} = 0$

(D) $\{a_n\}$ *is a* m_r*-Cauchy sequence.*

Definition 2.12. [5] Let $(\mathcal{X}, m_r)$ be a rectangular metric space. m_r is said to be complete if every Cauchy sequence $\{a_n\}_n$ in $\mathcal{X}$ converges to $a \in \mathcal{X}$.

The following definitions was given by [8], [18].

Definition 2.13. Let Θ_G be the family of all functions $\theta :]0, +\infty[\to]1, +\infty[$ satisfying:

(θ_1) θ is increasing;

(θ_2) for each sequence $(\nu_n) \subset]0, +\infty[$,

$$\lim_{n\to\infty} \nu_n = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} \theta(\nu_n) = 1;$$

(θ_3) there exists $\alpha \in]0, 1[$ and $l \in]0, +\infty]$ such that $\lim_{t\to 0} \frac{\theta(\nu) - 1}{\nu^{\alpha}} = l.$

Definition 2.14. Let Θ_c be the family of all functions $\theta :]0, +\infty[\to]1, +\infty[$ satisfying:

(θ_1) θ is increasing;

(θ_2) for each sequence $(\nu_n) \subset]0, +\infty[$,

$$\lim_{n\to\infty} \nu_n = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} \theta(\nu_n) = 1;$$

(θ_3) θ is continuous.

In [8], the authors introduced the following concept of θ_G-contraction and proved a fixed point theorem that generalizes the classical Banach contraction mapping principle. They proved that any θ_G-contraction has a unique fixed point.

Definition 2.15. [18] Let $(\mathcal{X}, \varrho)$ be a rectangular metric space and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a mapping. $\mathcal{T}$ is said to be a θ_G-contraction if there exists $\theta \in \Theta_G$ and $k \in]0, 1[$ such that, for any $u, v \in \mathcal{X}$,

$$\varrho(\mathcal{T}u, \mathcal{T}v) > 0 \Rightarrow \theta(\varrho(\mathcal{T}u, \mathcal{T}v)) \le [\theta(M(u, v))]^k,$$

where
$M(u, v) = \max\{\varrho(u, v), \varrho(u, \mathcal{T}u), \varrho(v, \mathcal{T}v)\}$.

Theorem 2.16. [18] *Let* $(\mathcal{X}, \varrho)$ *be a complete metric space and let* $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ *be a* θ_G*-contraction. Then* $\mathcal{T}$ *has a unique fixed point.*

Remark 2.17. These two sets Θ_G and Θ_C are different.

Example. Define $\theta : (0, +\infty) \to (1, +\infty)$ by

$$\theta(t) = \begin{cases} \sqrt{t} + 1, & \text{if } t \in \left(0, \frac{1}{2}\right] \\ e^{\sqrt{t}} & \text{if } t \in \left(\frac{1}{2}, +\infty\right) \end{cases}$$

Then $\theta \in \Theta_G$ but, for any $t > 0$,

$$\lim_{t\to\frac{1}{2}^-} \theta(t) = \sqrt{\frac{1}{2}} + 1$$

and

$$\lim_{t\to\frac{1}{2}^+} \theta(t) = e^{\sqrt{\frac{1}{2}}}.$$

Since $\sqrt{\frac{1}{2}} + 1 \neq e^{\sqrt{\frac{1}{2}}}$ so, θ does not satisfy the condition (θ_3) of the definition 2.14, then $\theta \notin \Theta_C$.

Example. Define $\theta : (0, +\infty) \to (1, +\infty)$ by

$$\theta(t) = e^{e^{\frac{-1}{t^p}}},\ p > 0.$$

Then $\theta \in \Theta_C$, but, for any $r > 0$,

$$\lim_{t\to 0^+} \frac{\theta(t) - 1}{t^r} = \lim_{t\to 0^+} \frac{e^{e^{\frac{-1}{t^p}}} - 1}{t^r} = \lim_{t\to 0^+} \frac{e^{\frac{-1}{t^p}}}{t^r} = \lim_{t\to 0^+} \frac{\frac{1}{t^r}}{\frac{1}{e^{\frac{1}{t^p}}}} = 0.$$

So, θ does not satisfy the condition (θ_3) of the definition 2.13, then $\theta \notin \Theta_G$.

3 Main Results

The following definition is a new version of the θ-contraction for a rectangular M-metric space.

Definition 3.1. Let $(\mathcal{X}, m_r)$ be a rectangular M-metric space and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a mapping.
$\mathcal{T}$ is said to be a $\theta - G$-contraction on $\mathcal{X}$ if there exist $\theta \in \Theta_G$ and $0 < k < 1$ such that, for any $u, v \in \mathcal{X}$,

$$m_r(\mathcal{T}u, \mathcal{T}v) > 0 \Rightarrow \theta(m_r(\mathcal{T}u, \mathcal{T}v)) \leq [\theta(m_r(u, v))]^k . \tag{5}$$

Theorem 3.2. *Let $(\mathcal{X}, m_r)$ be a complete rectangular M-metric space and let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a continuous $\theta - G$-contraction. Consider the sequence $\{a_n\}_{n\in\mathbb{N}}$ defined by $a_{n+1} = \mathcal{T}a_n,\ n = 0, 1, \dots$. Then $\mathcal{T}$ has a unique fixed point $a \in \mathcal{X}$ and, for every $a_0 \in \mathcal{X}$, the sequence $\{\mathcal{T}^n(a_0)\}_{n\in\mathbb{N}}$ is convergent to a.*

Proof. Suppose that there exists $n_0 \in \mathbb{N}$ such that $a_{n_0} = \mathcal{T}a_{n_0}$. Then a_{n_0} is a fixed point of $\mathcal{T}$ and the proof is finished. Hence, we assume that $a_n \neq \mathcal{T}a_n$, i.e. $m_r(a_{n+1}, a_n) > 0$ for all $n \in \mathbb{N}$. We have

$$a_n \neq a_m,\ \forall m, n \in \mathbb{N},\ m \neq n. \tag{6}$$

Indeed, suppose that $a_n = a_m$ for some $n \neq m$. Put $n = m + h$ with $h > 0$, so we have

$$a_{n+1} = \mathcal{T}a_n = \mathcal{T}a_m = a_{m+1}.$$

Denote $m_{r_n} = m_r(a_n, a_{n+1}) - m_{r_{a_n, a_{n+1}}}$. Then (3.1) implies that

$$\begin{aligned}
&\theta(m_{r_m}) = \theta\left(m_r(a_m, a_{m+1}) - m_{r_{a_m, a_{m+1}}}\right)\\
&= \theta\left(m_r(a_n, a_{n+1}) - m_{r_{a_n, a_{n+1}}}\right)\\
&= \theta\left(m_r(\mathcal{T}a_{n-1}, \mathcal{T}a_n) - m_{r_{\mathcal{T}a_{n-1}, \mathcal{T}a_n}}\right)\\
&= \theta\left(m_r(\mathcal{T}a_{m+h-1}, \mathcal{T}a_{m+h}) - m_{r_{\mathcal{T}a_{m+h-1}, \mathcal{T}a_{m+h}}}\right)\\
&\leq \left[\theta\left(m_r(a_{m+h-1}, a_{m+h}) - m_{r_{a_{m+h-1}, a_{m+h}}}\right)\right]^k\\
&\leq \left[\theta\left(m_r(a_{m+h-2}, a_{m+h-1}) - m_{r_{a_{m+h-2}, a_{m+h-1}}}\right)\right]^{k^2}\\
&\leq \left[\theta\left(m_r(a_{m+h-3}, a_{m+h-2}) - m_{r_{a_{m+h-3}, a_{m+h-2}}}\right)\right]^{k^3}\\
&\leq \dots \leq \left[\theta\left(m_r(a_m, a_{m+1}) - m_{r_{a_m, a_{m+1}}}\right)\right]^{k^h}.
\end{aligned}$$

Therefore

$$\theta\left(m_r(a_n, a_{n+1}) - m_{r_{a_n, a_{n+1}}}\right) \leq \left[\theta\left(m_r(a_m, a_{m+1}) - m_{r_{a_m, a_{m+1}}}\right)\right]^{k^h}$$

Since $k \in (0, 1)$, we conclude that

$$\theta\left(m_r(a_n, a_{n+1}) - m_{r_{a_n, a_{n+1}}}\right) < \theta\left(m_r(a_m, a_{m+1}) - m_{r_{a_m, a_{m+1}}}\right).$$

which is a contradiction. Thus, in what follows, we can assume that (6) holds.

Substituting $u = a_{n-1}$ and $v = a_n$ in (5), for all $n \in \mathbb{N}$, we have

$$\theta(m_r(a_n, a_{n+1})) \leq (\theta(m_r(a_{n-1}, a_n)))^k . \tag{7}$$

Repeating this step, we conclude that

$$\begin{aligned}
\theta(m_r(a_n, a_{n+1})) &\leq (\theta(m_r(a_{n-1}, a_n)))^k\\
&\leq (\theta(m_r(a_{n-2}, a_{n-1})))^{k^2}\\
&\leq \dots \leq (\theta(m_r(a_0, a_1)))^{k^n}.
\end{aligned}$$

By the property of θ we get,

$$1 < \theta(m_r(a_n, a_{n+1})) \leq (\theta(m_r(a_0, a_1)))^{k^n}. \tag{8}$$

By letting $n \to \infty$ in inequality (8), we obtain

$$\begin{aligned}
1 &\leq \lim_{n\to\infty} \theta(m_r(a_n, a_{n+1}))\\
&\leq \lim_{n\to\infty} (\theta(m_r(a_0, a_1)))^{k^n}.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \theta(m_r(a_n, a_{n+1})) = 1. \tag{9}$$

By (θ_2) in Definition 2.13, we obtain

$$\lim_{n\to\infty} m_r(a_n, a_{n+1}) = 0. \tag{10}$$

Substituting $u = a_{n-1}$ and $v = a_{n+1}$ in (5), for all $n \in \mathbb{N}$, we have

$$\theta(m_r(a_n, a_{n+2})) \leq (\theta(m_r(a_{n-1}, a_{n+1})))^k . \tag{11}$$

Repeating this step, we conclude that

$$\begin{aligned}
\theta(m_r(a_n, a_{n+2})) &\leq (\theta(m_r(a_{n-1}, a_{n+1})))^k\\
&\leq (\theta((m_r(a_{n-2}, a_n)))^{k^2}\\
&\leq \dots \leq \theta((m_r(a_0, a_2))^{k^n} .
\end{aligned}$$

By property of θ we get,

$$1 < \theta(m_r(a_n, a_{n+2})) \leq (\theta(m_r(a_0, a_2)))^{k^n}. \tag{12}$$

By letting $n \to \infty$ in inequality (12), we obtain

$$1 \leq \lim_{n\to\infty} \theta\left(m_r\left(a_n, a_{n+2}\right)\right) \leq \lim_{n\to\infty} \left(\theta(m_r\left(a_0, a_2\right))\right)^{k^n}.$$

Therefore,

$$\lim_{n\to\infty} \theta\left(m_r\left(a_n, a_{n+2}\right)\right) = 1. \tag{13}$$

By (θ_2) of the Definition 2.13, we obtain

$$\lim_{n\to\infty} m_r\left(a_n, a_{n+2}\right) = 0. \tag{14}$$

Next, we shall prove that $\{a_n\}_{n\in\mathbb{N}}$ is a m_r-Cauchy sequence, that is,

$$\lim_{n,m\to\infty} m_r\left(a_{n,} a_m\right) - m_{r_{a_n,a_m}} = 0, \text{ for all } n, m \in \mathbb{N}.$$

By (θ_3) in Definition 2.13, there exists $\alpha \in]0, 1[$ and $l \in]0, +\infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(m_r(a_n, a_{n+1})) - 1}{(m_r(a_n, a_{n+1}))^\alpha} = l.$$

Suppose that $l < \infty$. So, there exists $n_1 \in \mathbb{N}$ such that

$$\left|\frac{\theta(m_r(a_n, a_{n+1})) - 1}{(m_r(a_n, a_{n+1}))^\alpha} - l\right| < \frac{l}{2}, \ \forall n \geq n_1.$$

Taking $\mathcal{M} = \frac{2}{l}$, we have

$$n\left[m_r(a_n, a_{n+1})\right]^\alpha < \mathcal{M} \cdot n\left[\theta(m_r(a_n, a_{n+1})) - 1\right], \ \forall n \geq n_1.$$

Suppose now that $l = \infty$. Let $\mathcal{N} > 0$ be an arbitrary positive number. So, there exists $n_2 \in \mathbb{N}$ such that

$$\frac{\theta(m_r(a_n, a_{n+1})) - 1}{(m_r(a_n, a_{n+1}))^\alpha} > \mathcal{N}, \ \forall n \geq n_2.$$

Taking $\mathcal{M} = \frac{1}{\mathcal{N}}$, we have

$$n\left[m_r(a_n, a_{n+1})\right]^\alpha < \mathcal{M} \cdot n\left[\theta(m_r(a_n, a_{n+1})) - 1\right], \ \forall n \geq n_2.$$

Thus, in both cases, there exists $\mathcal{M} > 0$ and $n_q \in \mathbb{N}$ such that

$$n\left[m_r(a_n, a_{n+1})\right]^\alpha < \mathcal{M} \cdot n\left[\theta(m_r(a_n, a_{n+1})) - 1\right], \ \forall n \geq n_q.$$

By induction, we obtain

$$\begin{aligned} n\left[m_r(a_n, a_{n+1})\right]^\alpha &< \mathcal{M} \cdot n\left[\theta(m_r(a_n, a_{n+1})) - 1\right] \\ &< \ldots < \\ &< \mathcal{M} \cdot n\left[\left(\theta(m_r(a_0, a_1))\right)^{k^n} - 1\right] \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} n\left[m_r(a_n, a_{n+1})\right]^\alpha = 0.$$

So, there exists $n_3 \in \mathbb{N}$ such that

$$m_r(a_n, a_{n+1}) \leq \frac{1}{n^{\frac{1}{\alpha}}}, \ \forall n \geq n_3.$$

By property (θ_3) in Definition 2.13, there exists $\alpha \in (0, 1)$ and $h \in]0, +\infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(m_r(a_n, a_{n+2})) - 1}{(m_r(a_n, a_{n+2}))^\alpha} = h.$$

Suppose that $h < \infty$. So, there exists $n_4 \in \mathbb{N}$ such that

$$\left|\frac{\theta(m_r(a_n, a_{n+2})) - 1}{(m_r(a_n, a_{n+2}))^\alpha} - h\right| < \frac{h}{2}, \ \forall n \geq n_4.$$

Taking $\mathcal{P} = \frac{2}{h}$, we have

$$n\left[m_r(a_n, a_{n+2})\right]^\alpha < \mathcal{P} \cdot n\left[\theta(m_r(a_n, a_{n+2})) - 1\right], \ \forall n \geq n_4.$$

Suppose now that $h = \infty$. Let $\mathcal{Q} > 0$ be an arbitrary positive number. So, there exists $n_5 \in \mathbb{N}$ such that

$$\frac{\theta(m_r(a_n, a_{n+2})) - 1}{(m_r(ax_n, a_{n+2})^\alpha} > \mathcal{Q}, \ \forall n \geq n_5.$$

So by taking $\mathcal{P} = \frac{1}{\mathcal{Q}}$, we have, for any $n \geq n_5$

$$n\left[m_r(a_n, a_{n+2})\right]^\alpha < \mathcal{P} \cdot n\left[\theta(m_r(a_n, a_{n+2})) - 1\right].$$

Thus, in all cases, there exist $\mathcal{P} > 0$ and $w \in \mathbb{N}$ such that, for any $n \geq w$,

$$n\left[m_r(a_n, a_{n+2})\right]^\alpha < \mathcal{P} \cdot n\left[\theta(m_r(a_n, a_{n+2})) - 1\right].$$

By induction, we obtain

$$\begin{aligned} n\left[m_r(a_n, a_{n+2})\right]^\alpha &< \mathcal{P} \cdot n\left[\theta(m_r(a_n, a_{n+2})) - 1\right] \\ &< \ldots < \mathcal{P} \cdot n\left[\left(\theta(m_r(a_0, a_2))\right)^{k^n} - 1\right]. \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} n\left[m_r(a_n, a_{n+2})\right]^\alpha = 0.$$

So, there exists $n_6 \in \mathbb{N}$ such that

$$m_r(a_n, a_{n+2}) \leq \frac{1}{n^{\frac{1}{\alpha}}}, \ \forall n \geq n_6.$$

Next, we show that a_n is m_r-Cauchy sequence, that is

$$\lim_{n\to\infty} (m_r(a_n, a_{n+p}) - m_{r_{a_n,a_{n+p}}}) = 0,$$

for all $p \in \mathbb{N}$.

Case 1. Firstly, let p odd, that is $p = 2m+1$ for any $m \geq 1$, $n \in \mathbb{N}$. From the condition (iv) of Definition 2.4 of the m_r-metric, we get

$$
\begin{aligned}
& m_r(a_n, a_{n+p}) - m_{r_{a_n, a_{n+p}}} \\
& = m_r(a_n, a_{n+2m+1}) - m_{r_{a_n, a_{n+2m+1}}} \\
& \leq m_r(a_n, a_{n+1}) - m_{r_{a_n, a_{n+1}}} \\
& + m_r(a_{n+1}, a_{n+2}) - m_{r_{a_{n+1}, a_{n+2}}} \\
& + m_r(a_{n+2}, a_{n+2m+1}) - m_{r_{a_{n+2}, a_{n+2m+1}}} \\
& \leq m_r(a_n, a_{n+1}) - m_{r_{a_n, a_{n+1}}} \\
& + m_r(a_{n+1}, a_{n+2}) - m_{r_{a_{n+1}, a_{n+2}}} \\
& + m_r(a_{n+2}, a_{n+3}) - m_{r_{a_{n+2}, a_{n+3}}} \\
& + m_r(a_{n+3}, a_{n+4}) - m_{r_{a_{n+3}, a_{n+4}}} \\
& + m_r(a_{n+4}, a_{n+5}) - m_{r_{a_{n+4}, a_{n+5}}} \\
& + m_r(a_{n+5}, a_{n+6}) - m_{r_{a_{n+5}, a_{n+6}}} \\
& \leq m_r(a_n, a_{n+1}) - m_{r_{a_n, a_{n+1}}} \\
& + m_r(a_{n+1}, a_{n+2}) - m_{r_{a_{n+1}, a_{n+2}}} \\
& + m_r(a_{n+2}, a_{n+3}) - m_{r_{a_{n+2}, a_{n+3}}} \\
& + m_r(a_{n+3}, a_{n+4}) - m_{r_{a_{n+3}, a_{n+4}}} \\
& \vdots \\
& + m_r(a_{n+2m-1}, a_{n+2m}) - m_{r_{a_{n+2m-1}, a_{n+2m}}} \\
& + m_r(a_{n+2m}, a_{n+2m+1}) - m_{r_{a_{n+2m}, a_{n+2m+1}}}.
\end{aligned}
$$

Then

$$
\begin{aligned}
& m_r(a_n, a_{n+p}) - m_{r_{a_n, a_{n+p}}} \\
& \leq \left[\sum_{i=n}^{p-1} m_r(a_i, a_{i+1}) - m_{r_{a_i, a_{i+1}}} \right] \\
& \leq \left[\sum_{i=n}^{\infty} m_r(a_i, a_{i+1}) - m_{r_{a_i, a_{i+1}}} \right] \\
& \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{\alpha}}}.
\end{aligned}
$$

From the convergence of the series we have

$$
\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{\alpha}}} < \infty \ \Rightarrow
$$
$$
\lim_{n \to \infty} m_r(a_n, a_{n+p}) - m_{r_{a_n, a_{n+p}}} = 0.
$$

Case 2. Firstly, let p even that is $p = 2m$ for any $m \geq 1$, $n \in \mathbb{N}$. From the condition (iv) of Definition 2.4 of the m_r-metric, we get

$$
\begin{aligned}
& m_r(a_n, a_{n+p}) - m_{r_{a_n, a_{n+p}}} \\
& \leq m_r(a_n, a_{n+2}) - m_{r_{a_n, a_{n+2}}} \\
& + m_r(a_{n+2}, a_{n+3}) - m_{r_{a_{n+2}, a_{n+3}}} \\
& + m_r(a_{n+3}, a_{n+2m}) - m_{r_{a_{n+3}, a_{n+2m}}} \\
& \leq m_r(a_n, a_{n+2}) - m_{r_{a_n, a_{n+2}}} \\
& + m_r(a_{n+2}, a_{n+3}) - m_{r_{a_{n+2}, a_{n+3}}} \\
& + m_r(a_{n+3}, a_{n+4}) - m_{r_{a_{n+3}, a_{n+4}}} \\
& + m_r(a_{n+4}, a_{n+5}) - m_{r_{a_{n+4}, a_{n+5}}} \\
& + m_r(a_{n+5}, a_{n+6}) - m_{r_{a_{n+5}, a_{n+6}}} \\
& + m_r(a_{n+2m-1}, a_{n+2m}) - m_{r_{a_{n+2m-1}, a_{n+2m}}} \\
& \leq m_r(a_n, a_{n+2}) - m_{r_{a_n, a_{n+2}}} \\
& + m_r(a_{n+2}, a_{n+3}) - m_{r_{a_{n+2}, a_{n+3}}} \\
& + m_r(a_{n+3}, a_{n+4}) - m_{r_{a_{n+3}, a_{n+4}}} \\
& + m_r(a_{n+4}, a_{n+5}) - m_{r_{a_{n+4}, a_{n+5}}} \\
& \vdots \\
& + m_r(a_{n+2m-2}, a_{n+2m-1}) - m_{r_{a_{n+2m-2}, a_{n+2m-1}}} \\
& + m_r(a_{n+2m-1}, a_{n+2m}) - m_{r_{a_{n+2m-1}, a_{n+2m}}}.
\end{aligned}
$$

Then

$$
\begin{aligned}
& m_r(a_n, a_{n+p}) - m_{r_{a_n, a_{n+p}}} \\
& \leq m_r(a_n, a_{n+2}) - m_{r_{a_n, a_{n+2}}} \\
& + \left[\sum_{i=n+2}^{n+p-1} m_r(a_i, a_{i+1}) - m_{r_{a_i, a_{i+1}}} \right] \\
& \leq m_r(a_n, a_{n+2}) - m_{r_{a_n, a_{n+2}}} \\
& + \left[\sum_{i=n+2}^{\infty} m_r(a_i, a_{i+1}) - m_{r_{a_i, a_{i+1}}} \right] \\
& \leq m_r(a_n, a_{n+2}) - m_{r_{a_n, a_{n+2}}} + \sum_{i=n+2}^{\infty} \frac{1}{i^{\frac{1}{\alpha}}} \\
& \leq \frac{1}{i^{\frac{1}{\alpha}}} + \sum_{i=n+2}^{\infty} \frac{1}{i^{\frac{1}{\alpha}}}.
\end{aligned}
$$

From the convergence of the series we have

$$
\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{\alpha}}} < \infty \ \Rightarrow
$$
$$
\lim_{n \to \infty} m_r(a_n, a_{n+p}) - m_{r_{a_n, a_{n+p}}} = 0.
$$

By Lemma 2.9 we obtain that, for any $n, m \in \mathbb{N}$,

$$
m_r^s(a_n, a_m) = m_r(a_n, a_m) - m_{r_{a_n, a_m}} \to 0 \quad \text{as } n \to \infty.
$$

This implies that $\{a_n\}$ is a m_r-Cauchy sequence with respect to m_r^s and converges by Lemma 2.10. Thus,

$$\lim_{n,m\to\infty} m_r^s\left(a_n, a_{n+2m+1}\right) = 0$$

and

$$\lim_{n,m\to\infty} m_r^s\left(a_n, a_{n+2m}\right) = 0.$$

We received by Lemma 2.8 that $\{a_n\}$ is a m_r-Cauchy sequence. From the completeness of $\mathcal{X}$, there exists $a \in \mathcal{X}$ such that

$$\lim_{n\to\infty} a_n = a.$$

Thus, by Lemma 2.9,

$$\lim_{n\to\infty} m_r(a_n, a) - m_{r_{a_n,a}} = 0.$$

Finally, the continuity of $\mathcal{T}$ yields

$$\begin{aligned} & m_r\left(a, \mathcal{T}a\right) - m_{r_{a,\mathcal{T}a}} \\ &= \lim_{n\to\infty} m_r\left(a_n, \mathcal{T}a_n\right) - m_{r_{a_n,\mathcal{T}a_n}} \\ &= \lim_{n\to\infty} m_r\left(a_n, a_{n+1}\right) - m_{r_{a_n,a_{n+1}}} = 0. \end{aligned}$$

So $a = \mathcal{T}a$.
Now, we show the uniqueness of the fixed point of $\mathcal{T}$. Assume that $\mathcal{T}$ has two distinct fixed points $a, b \in X$, such that $a = \mathcal{T}a$ and $b = \mathcal{T}b$.
From the condition (5), we have

$$\begin{aligned} \theta(m_r\left(a, b\right)) &= \theta(m_r\left(\mathcal{T}a, \mathcal{T}b\right)) \\ &\leq \left[\theta(m_r\left(\mathcal{T}a, \mathcal{T}b\right))\right]^k. \end{aligned}$$

So, since $0 < k < 1$, we conclude that

$$\theta(m_r\left(a, b\right)) < \theta(m_r\left(\mathcal{T}a, \mathcal{T}b\right)) = \theta(m_r\left(a, b\right)),$$

which is a contradiction. Hence $\mathcal{T}$ has a unique fixed point. □

Definition 3.3. Let $(\mathcal{X}, m_r)$ be a rectangular M-metric space and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a mapping. $\mathcal{T}$ is said to be a $\theta - C$-contraction on $\mathcal{X}$, if there exist $\theta \in \Theta_C$ and $0 < \gamma < 1$ such that, for any $u, v \in \mathcal{X}$,

$$\begin{aligned} & m_r\left(\mathcal{T}u, \mathcal{T}v\right) - m_{r\mathcal{T}u,\mathcal{T}v} > 0 \;\; \Rightarrow \\ & \theta\left(m_r\left(\mathcal{T}u, \mathcal{T}v\right) - m_{r\mathcal{T}u,\mathcal{T}v}\right) \\ & \leq \left[\theta\left(m_r(u, v) - m_{r_{u,v}}\right)\right]^\gamma. \end{aligned} \tag{15}$$

Theorem 3.4. *Let $(\mathcal{X}, m_r)$ be a complete rectangular M-metric space and let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a continuous $\theta - C$-contraction. Consider the sequence $\{a_n\}_{n\in\mathbb{N}}$ defined by $a_{n+1} = \mathcal{T}a_n$, $n = 0, 1, \ldots$. Then, $\mathcal{T}$ has a unique fixed point $a \in \mathcal{X}$ and for every $a_0 \in \mathcal{X}$ the sequence $\{\mathcal{T}^n(a_0)\}_{n\in\mathbb{N}}$ is convergent to a.*

Proof. Similarly to the proof of Theorem 3.2, we can conclude that

$$\lim_{n\to\infty} m_r\left(a_n, a_{n+1}\right) - m_{r_{a_n,a_{n+1}}} = 0. \tag{16}$$

and

$$\lim_{n\to\infty} m_r\left(a_n, a_{n+2}\right) - m_{r_{a_n,a_{n+2}}} = 0. \tag{17}$$

We shall prove that $\{a_n\}$ is a Cauchy sequence in $(\mathcal{X}, m_r)$, that is

$$\lim_{n\to\infty} m_r\left(a_n, a_m\right) - m_{r_{a_n,a_m}} = 0 \;\text{ for all }\; n, m \in \mathbb{N}. \tag{18}$$

If otherwise there exists $\varepsilon > 0$ for which we can find a sequence of positive integers $\{a_{n_k}\}_k$ and $\{a_{m_k}\}_k$ of $\{a_n\}$ such that, for all positive integers k, with $n_k > m_k > k$,

$$\begin{aligned} m_r(a_{m_k}, a_{n_k}) - m_{r_{a_{m_k},a_{n_k}}} &\geq \varepsilon, \\ m_r\left(a_{m_k}, a_{n_{k-1}}\right) - m_{r_{a_{m_k},a_{n_{k-1}}}} &< \varepsilon. \end{aligned} \tag{19}$$

Now, using (16), (17), (19) and the rectangular M-inequality, we find

$$\begin{aligned} \varepsilon &\leq m_r\left(a_{m_k}, a_{n_k}\right) - m_{r_{a_{m_k}},a_{n_k}} \\ &\leq m_r\left(a_{m_k}, a_{m_{k+1}}\right) - m_{r_{a_{m_k}},a_{m_{k+1}}} \\ &+ m_r\left(a_{m_{k+1}}, a_{m_{k-1}}\right) - m_{r_{a_{m_{k+1}}},a_{m_{k-1}}} \\ &+ m_r\left(a_{m_{k-1}}, a_{n_k}\right) - m_{r_{a_{m_{k-1}}},a_{n_k}} \\ &< m_r\left(a_{m_k}, a_{m_{k+1}}\right) - m_{r_{a_{m_k}},a_{m_{k+1}}} \\ &+ m_r\left(a_{m_{k+1}}, a_{m_{k-1}}\right) - m_{r_{a_{m_{k+1}}},a_{m_{k-1}}} + \varepsilon. \end{aligned}$$

Since

$$\lim_{k\to\infty} m_r\left(a_{m_k}, a_{m_{k+1}}\right) - m_{r_{a_{m_k}},a_{m_{k+1}}} = 0.$$

and

$$\lim_{k\to\infty} m_r\left(a_{m_{k+1}}, a_{m_{k-1}}\right) - m_{r_{a_{m_{k+1}}},a_{m_{k-1}}} = 0.$$

Then

$$\lim_{k\to\infty} m_r\left(a_{m_k}, a_{n_k}\right) - m_{r_{a_{m_k}},a_{n_k}} = \varepsilon. \tag{20}$$

Now, by M-rectangular inequality, we have

$$\begin{aligned} & m_r\left(a_{m_{k+1}}, a_{n_{k+1}}\right) - m_{r_{a_{m_{k+1}}},a_{n_{k+1}}} \\ &\leq m_r\left(a_{m_{k+1}}, a_{m_k}\right) - m_{r_{a_{m_{k+1}}},a_{m_k}} \\ &+ m_r\left(a_{m_k}, a_{n_k}\right) - m_{r_{a_{m_k}},a_{n_k}} \\ &+ m_r\left(a_{n_k}, a_{n_{k+1}}\right) - m_{r_{a_{n_k}},a_{n_{k+1}}}. \end{aligned}$$

$$
\begin{aligned}
\varepsilon &\le m_r\left(a_{m_k}, a_{n_k}\right) - m_{r_{a_{m_k}, a_{n_k}}} \\
&\le m_r\left(a_{m_k}, a_{n_{k-1}}\right) - m_{r_{a_{m_k}, a_{n_{k-1}}}} \\
&\quad + m_r\left(a_{n_{k-1}}, a_{n_{k+1}}\right) - m_{r_{a_{n_{k-1}}, a_{n_{k+1}}}} \\
&\quad + m_r\left(a_{n_{k+1}}, a_{n_k}\right) - m_{r_{a_{n_{k+1}}, a_{n_k}}}.
\end{aligned}
$$

Letting $k \to \infty$ in the above inequalities, we obtain

$$
\lim_{k\to\infty} m_r\left(a_{m_{k+1}}, a_{n_{k+1}}\right) - m_{r_{a_{m_{k+1}}, a_{n_{k+1}}}} = \varepsilon \quad (21)
$$

and

$$
\lim_{k\to\infty} m_r\left(a_{m_k}, a_{n_{k-1}}\right) - m_{r_{a_{m_k}, a_{n_{k-1}}}} = \varepsilon. \quad (22)
$$

By (21), let $\mathcal{B} = \frac{\varepsilon}{2} > 0$, from the definition of limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \ge n_0$,

$$
\left|m_r\left(a_{m_{k+1}}, a_{n_{k+1}}\right) - m_{r_{a_{m_{k+1}}, a_{n_{k+1}}}} - \varepsilon\right| \le \mathcal{B}, \ .
$$

This implies that, for all $n \ge n_0$,

$$
m_r\left(a_{m_{k+1}}, a_{n_{k+1}}\right) - m_{r_{a_{m_{k+1}}, a_{n_{k+1}}}} \ge \mathcal{B} > 0,
$$

Applying (15) with $u = a_{m_k}$ and $v = a_{n_k}$, we obtain

$$
\begin{aligned}
&\theta\left(m_r\left(\mathcal{T}a_{m_k}, \mathcal{T}a_{n_k}\right) - m_{r_{\mathcal{T}a_{m_k}, \mathcal{T}a_{n_k}}}\right) \\
&\quad \le \left[\theta\left(m_r\left(a_{m_k}, a_{n_k}\right) - m_{r_{a_{m_k}, a_{n_k}}}\right)\right]^{\gamma}.
\end{aligned}
$$

Letting $k \to \infty$ in the above inequality and using property (θ_3) in Definition 2.14, we obtain

$$
\begin{aligned}
&\theta\left(\lim_{k\to\infty}\left(m_r\left(\mathcal{T}a_{m_k}, \mathcal{T}a_{n_k}\right) - m_{r_{\mathcal{T}a_{m_k}, \mathcal{T}a_{n_k}}}\right)\right) \\
&\le \left[\theta\left(\lim_{k\to\infty}\left(m_r\left(a_{m_k}, a_{n_k}\right) - m_{r_{a_{m_k}, a_{n_k}}}\right)\right)\right]^{\gamma}.
\end{aligned}
$$

Therefore,

$$
\theta(\varepsilon) \le [\theta(\varepsilon)]^{\gamma} < \theta(\varepsilon).
$$

It is a contradiction. So

$$
\lim_{n,m\to\infty} m_r\left(a_m, a_n\right) - m_{r_{a_m, a_n}} = 0.
$$

From the completeness of $\mathcal{X}$, there exists $a \in \mathcal{X}$ such that

$$
\lim_{n\to\infty} a_n = a.
$$

We show that $m_r\left(\mathcal{T}a, a\right) - m_{r_{\mathcal{T}a, a}} = 0$. Arguing by contradiction, we assume that

$$
m_r\left(\mathcal{T}a, a\right) - m_{r_{\mathcal{T}a, a}} > 0.
$$

By the rectangular inequality we get,

$$
\begin{aligned}
&m_r\left(\mathcal{T}a_n, \mathcal{T}a\right) - m_{r_{\mathcal{T}a_n, \mathcal{T}a}} \\
&\le m_r\left(\mathcal{T}a_n, a_n\right) - m_{r_{\mathcal{T}a_n, a_n}} \\
&+ m_r\left(a_n, a\right) - m_{r_{a_n, a}} \\
&+ m_r\left(a, \mathcal{T}a\right) - m_{r_{a, \mathcal{T}a}}
\end{aligned} \quad (23)
$$

and

$$
\begin{aligned}
&m_r\left(a, \mathcal{T}a\right) - m_{r_{a, \mathcal{T}a}} \\
&\le m_r\left(a, a_n\right) - m_{r_{a, a_n}} \\
&+ m_r\left(a_n, \mathcal{T}a_n\right) - m_{r_{a_n, \mathcal{T}a_n}} \\
&+ m_r\left(\mathcal{T}a_n, \mathcal{T}a\right) - m_{r_{\mathcal{T}a_n, \mathcal{T}a}}.
\end{aligned} \quad (24)
$$

By letting $n \to \infty$ in inequality (23) and (24), we obtain

$$
\begin{aligned}
&m_r\left(a, \mathcal{T}a\right) - m_{r_{a, \mathcal{T}a}} \\
&\le \lim_{n\to\infty} m_r\left(\mathcal{T}a_n, Ta\right) - m_{r_{\mathcal{T}a_n, \mathcal{T}a}} \\
&\le m_r\left(a, Ta\right) - m_{r_{a, \mathcal{T}a}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\lim_{n\to\infty} m_r\left(\mathcal{T}a_n, \mathcal{T}a\right) - m_{r_{\mathcal{T}a_n, \mathcal{T}a}} \\
&= m_r\left(a, \mathcal{T}a\right) - m_{r_{a, \mathcal{T}a}}.
\end{aligned} \quad (25)
$$

Applying (15) with $u = a$ and $v = a_n$, we obtain

$$
\begin{aligned}
&\theta\left(m_r\left(\mathcal{T}a, \mathcal{T}a_n\right) - m_{r_{\mathcal{T}a, \mathcal{T}a_n}}\right) \\
&\le \left[\theta\left(m_r\left(a, a_n\right) - m_{r_{a, a_n}}\right)\right]^{\gamma},
\end{aligned} \quad (26)
$$

with $0 < \gamma < 1$.
Letting $n \to \infty$ in the above inequality and using property (θ_3) in Definition 2.14, we obtain

$$
\begin{aligned}
&\theta\left(\lim_{n\to\infty}\left(m_r\left(\mathcal{T}a, \mathcal{T}a_n\right) - m_{r_{\mathcal{T}a, \mathcal{T}a_n}}\right)\right) \\
&\le \left[\theta\left(\lim_{n\to\infty}\left(m_r\left(a, a_n\right) - m_{r_{a, a_n}}\right)\right)\right]^{\gamma}.
\end{aligned}
$$

Therefore, by (25),

$$
1 \le \theta\left(m_r\left(a, \mathcal{T}a\right) - m_{r_{a, \mathcal{T}a}}\right) \le [\theta(0)]^{\gamma} = 1.
$$

Thus $m_r\left(a, \mathcal{T}a\right) - m_{r_{a, \mathcal{T}a}} = 0$. Hence $\mathcal{T}a = a$. Similarly to the proof of Theorem 3.2, we can conclude that a is the unique fixed point. □

Example. Let $\mathcal{X} = [1, +\infty)$. Define $m_r : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ by

$$
m_r\left(a, b\right) = \frac{|a-b|}{2}
$$

and consider

$$
\theta(t) = e^{\sqrt{t}}, \ k = \frac{1}{\sqrt{2}}.
$$

Then $(\mathcal{X}, m_r)$ is a complete rectangular M-metric space, $k \in (0, 1)$.
Since

$$
\lim_{t\to 0} \frac{e^{\sqrt{t}} - 1}{\sqrt{t}} = 1.
$$

then $\theta \in \Theta_C \bigcap \Theta_G$.

Define $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ by

$$T(t) = \sqrt{t} \text{ for all } t \in [1, +\infty).$$

case 1: $1 \leq a \leq b$.

$$m_r(Ta, Tb) = \frac{\sqrt{b} - \sqrt{a}}{2},$$

$$\theta\left(m_r(\mathcal{T}a, \mathcal{T}b)\right) = e^{\sqrt{\frac{\sqrt{b}-\sqrt{a}}{2}}}.$$

Since $a, b \in [1, +\infty)$, then

$$\sqrt{b} - \sqrt{a} \leq \frac{b-a}{2}.$$

Thus

$$e^{\sqrt{\frac{\sqrt{b}-\sqrt{a}}{2}}} \leq \left[e^{\sqrt{\frac{b-a}{2}}}\right]^{\frac{1}{\sqrt{2}}},$$

hence

$$\theta\left(m_r(\mathcal{T}a, \mathcal{T}b)\right) \leq \left[\theta\left(m_r(a, b)\right)\right]^k.$$

case 2: $a > b \geq 1$.

$$m_r(Ta, Tb) = \frac{\sqrt{a} - \sqrt{b}}{2},$$

$$\theta\left(m_r(\mathcal{T}a, \mathcal{T}b)\right) = e^{\sqrt{\frac{\sqrt{a}-\sqrt{b}}{2}}}.$$

Since $a, b \in [1, +\infty)$, then

$$\sqrt{a} - \sqrt{b} \leq \frac{a-b}{2}.$$

Thus

$$e^{\sqrt{\frac{\sqrt{a}-\sqrt{b}}{2}}} \leq \left[e^{\sqrt{\frac{a-b}{2}}}\right]^{\frac{1}{\sqrt{2}}},$$

hence

$$\theta\left(m_r(\mathcal{T}a, \mathcal{T}b)\right) \leq \left[\theta\left(m_r(a, b)\right)\right]^k.$$

Hence, the conditions (5) and (15) are satisfied, since $m_{r\mathcal{T}a,\mathcal{T}b} = 0$ and $m_{ra,b} = 0$ and, consequently, we can apply Theorems 3.2 and 3.4, respectively. Therefore, $\mathcal{T}$ has a unique fixed point $z = 1$.

4 Application to Nonlinear Integral Equations

In this section, we apply Theorems 3.2 and 3.4 to prove the existence and uniqueness of the solution of the integral equation of Fredholm type:

$$u(t) = \nu \int_m^n \mathcal{H}(t, s, u(s))ds, \tag{27}$$

where $m, n \in \mathbb{R}^+$, $u \in C([m, n], \mathbb{R})$ and $\mathcal{H} : [m, n]^2 \times \mathbb{R} \to \mathbb{R}$ is a continuous function and ν is a constant depending on the parameters m and n.

Theorem 4.1. *Let $m, n \in \mathbb{R}^+$ and let $\mathcal{H},\ u$ be functions as above, such that*

$$|\mathcal{H}(t, s, u(s)) - \mathcal{H}(t, s, v(s))| \leq |u(s) - v(s)|,$$
$$\forall\, t, s \in [m, n],\ \ \forall\, u, v \in C([m, n], \mathbb{R}).$$

Then the equation (27) ~~h~~as a unique solution $u \in C([m, n], \mathbb{R})$ and $|\nu| \leq \frac{m}{n} \cdot (n-m)^{-1}$.

Proof. Let $\mathcal{X} = C([m, n], \mathbb{R})$ and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ defined by

$$\mathcal{T}(u)(t) = \nu \int_m^n \mathcal{H}(t, s, u(s))ds,$$
$$\forall\, u \in \mathcal{X},\ \ t \in [m, n].$$

Let $m_r : \mathcal{X} \times \mathcal{X} \to [0, +\infty[$ given by

$$m_r(u, v) = \sup_{t \in [m,n]} \frac{\overline{|u(t) - v(t)|}}{2}.$$

Clearly, $\mathcal{X}$ is a complete rectangular M-metric space. Assume that, $u, v \in \mathcal{X}$ and $t, s \in [m, n]$. Then we get, for any $t \in [m, n]$,

$$\begin{aligned}
&|\mathcal{T}u(t) - \mathcal{T}v(t)| \\
&= |\nu|\left(\left|\int_m^n \mathcal{H}(t, s, u(s))ds - \int_m^n \mathcal{H}(t, s, v(s))ds\right|\right) \\
&= |\nu| \left|\int_m^n \left(\mathcal{H}(t, s, u(s)) - \mathcal{H}(t, s, v(s))\right) ds\right| \\
&\leq |\nu| \int_m^n |\mathcal{H}(t, s, u(s)) - \mathcal{H}(t, s, v(s))|\, ds \\
&\leq |\nu| \int_m^n |u(s) - v(s)|\, ds \\
&\leq |\nu| \int_m^n \left(\sup_{s \in [m,n]} |u(s) - v(s)|\right) ds \\
&= |\nu| \cdot (n-m) \sup_{s \in [m,n]} |u(s) - v(s)|.
\end{aligned}$$

Thus

$$\begin{aligned}
m_r(\mathcal{T}u, \mathcal{T}v) &= \sup_{t \in [m,n]} \frac{\overline{|\mathcal{T}u(t) - \mathcal{T}v(t)|}}{2} \\
&\leq |\nu| \cdot (n-m) \sup_{s \in [m,n]} \frac{\overline{|u(s) - v(s)|}}{2} \\
&= |\nu| \cdot (n-m) \cdot m_r(u, v).
\end{aligned}$$

As $m_r(\mathcal{T}u, \mathcal{T}v) > 0$ and $m_r(u, v) > 0$ for any $u \neq v$, then we can take natural exponential sides and, taking $\theta(t) = e^t$, we get

$$\exp\left(m_r(\mathcal{T}u, \mathcal{T}v)\right) \leq \left[\exp(m_r(u, v))\right]^{|\nu| \cdot (n-m)},$$

hence, since $|\nu| \leq \frac{m}{n} \cdot (n-m)^{-1}$, we have

$$\theta\left(m_r(\mathcal{T}u, \mathcal{T}v)\right) \leq \left[\theta\left(m_r(u,v)\right)\right]^k, \qquad (28)$$

for all $u, v \in \mathcal{X}$, with $k = \frac{m}{n} < 1$. Then $\mathcal{T}$ satisfies the conditions (5) and (15), since $m_{r\mathcal{T}u,\mathcal{T}v} = 0$ and $m_{ru,v} = 0$ and, consequently, we can apply Theorems 3.2 and 3.4, respectively. This completes the proof. □

Remark 4.2. Particular cases of rectangular M-metric spaces are the Grand Lebesgue spaces considered in [19], [20], were an application to a linear convolution integral equation was given, using the contraction property and the fixed point theorem, [19].

5 Conclusion

We introduced a new version of the θ-contraction for rectangular M-metric spaces and we proved fixed point theorems, as an extension of previous existing results in literature. Moreover we illustrated an example and we gave an application to a nonlinear integral equation of Fredholm type. As future project, we aim to investigate other practical applications of the obtained result to some areas mentioned in the Introduction.

Acknowledgments M. R. Formica is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). This work has been partially funded by the European Union - NextGenerationEU within the framework of PNRR Mission 4 - Component 2 - Investment 1.1 under the Italian Ministry of University and Research (MUR) program PRIN 2022 - grant number 2022BCFHN2 - *Advanced theoretical aspects in PDEs and their applications* - CUP: I53D23002300006.

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Competing interest The authors declare that they have no competing interests.

Authors contributions The authors equally contributed to the manuscript and approved the final manuscript.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself
No funding was received for conducting this study.

Conflict of Interest
The authors have no conflicts of interest to declare that are relevant to the content of this article.