On the Iterated Method for the Solution of Functional Equations with Shift Whose Fixed Points are Located at the Ends of a Contour

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Abstract: - In this paper, we offer an approach for solving functional equations containing a shift operator and its iterations. With the help of an algorithm, the initial equation is reduced to the first iterated equation, then, applying the same algorithm, we obtain the second iterated equation. Continuing this process, we obtain the n-th iterated equation and the limit iterated equation. We prove a theorem on the equivalence of the initial equation and the iterated equations. Based on the analysis of the solvability of the limit equation, we find a solution to the initial equation. Equations of this type appear when modeling renewable systems with elements in different states, such as being sick, healthy with immunity, and without immunity. The obtained results represent appropriate mathematical tools for the study of such systems.

Key-Words: - Functional operator with shift, Inverse operator, Iterated method, Hölder functions, Hölder functions with weight, Renewable systems.

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1 Introduction

When modeling systems with renewable resources [1], functional equations with shift arise $B_\alpha\varphi(x) = \varphi[\alpha(x)]$. To solve equations with the operator B_α, $I\varphi(x) - bB_\alpha\varphi(x) = g(x)$, the results on the invertibility of the operator $I - bB_\alpha$ in Hölder space with weight were used.

More complex models have been developed to study renewable systems with elements in three states. The balance relations of the model included not only the shift operator B_α, but also its second-order compositions $B_\alpha{}^2 = B_\alpha B_\alpha$. To solve the equation:

$$\varphi(x) - b(x)B_\alpha\varphi(x) - c(x)B_\alpha{}^2\varphi(x) = g(x),$$

iterative methods of solution were developed and applied.

In this work, we generalized the iterative method proposed in [1] and obtained formulas for solving the equation with compositions of the operator B_α of the l-th order:

$$\varphi(x) - b_1B_\alpha\varphi(x) - \dots - b_lB_\alpha{}^l\varphi(x) = g(x).$$

Here, operators are considered in the class of Hölder functions and in the spaces of Hölder functions with weight, [2], [3].

The statements and theorems proved in this work will serve as an effective mathematical apparatus in the study of renewable systems with elements in different states.

2 Iteration Equations. Connection between the Initial Values of the Sought Function and the Free Term

From [3], we recall the definitions of the space of Hölder class functions with weight $H_\mu^0(J,\rho)$, where $\rho(x) = x^{\mu_0}(1-x)^{\mu_1},\ 0 < \mu < 1,\ J = [0,1]$,
$\mu < \mu_1 < 1 + \mu,\ \mu < \mu_2 < 1 + \mu$.

Functions $\varphi(x)$ that satisfy the following condition: $\varphi(x_1) - \varphi(x_2)| \le C|x_1 - x_2|^\mu,\ \ x_1 \in J,\ x_2 \in J$
are called Hölder functions and form the class $H_\mu(J)$. When $\mu = 1$, then such functions are called Lipschitz functions.

Functions that become Hölder functions and turn into zero at points $x = 0,\ x = 1$, after being multiplied by $\rho(x)$, form a Banach space. Functions of this space are called Hölder functions with weight

$\rho(x)$ and have the notation $H_\mu^0(J,\rho)$. The norm in the space $H_\mu^0(J,\rho)$ is defined by:

$$\|f(x)\|_{H_\mu^0(J,\rho)} = \|\rho(x)f(x)\|_{H_\mu(J)},$$

where

$$\|\rho(x)f(x)\|_{H_\mu(J)} = \|\rho(x)f(x)\|_C + \|\rho(x)f(x)\|_\mu$$

and

$$||\rho(x)f(x)||_C = \max_{x\in J}|\rho(x)f(x)|,$$

$$||\rho(x)f(x)||_\mu \sup_{x_{1,2}\in\, J,\, x_1\neq x_2} \frac{|\rho(x_1)f(x_1)-\rho(x_2)f(x_2)|}{|x_1-x_2|^\mu}.$$

Let $\alpha(x)$ be a bijective orientation-preserving shift on J: if $x_1 < x_2$, then $\alpha(x_1) < \alpha(x_2)$ and $\alpha(x)$ has only two fixed points $\alpha(0) = 0$, $\alpha(1) = 1$ and $\alpha(x) \neq x$, when $x \neq 0$, $x \neq 1$. In addition, let $\alpha(x)$ be a differentiable function with $\frac{d}{dx}\alpha(x) \neq 0$ and $\frac{d}{dx}\alpha(x) \in H_\mu(J)$.

From the properties of non-Carleman shift $\alpha(x)$, an important property of the functional operator $B_\alpha\varphi(x) = \varphi[\alpha(x)]$ follows:

$$\lim_{n\to\infty} {B_\alpha}^n\varphi(x) = \varphi(1). \qquad (1)$$

We consider the equation:

$$\varphi(x) - b_1(x)B_\alpha\varphi(x) - b_2(x){B_\alpha}^2\varphi(x) - \dots - b_l(x){B_\alpha}^l\varphi(x) = g(x) \qquad (2)$$

or, in operator form,

$$L = \sum_{i=1}^{l} b_i(\mathrm{x})\, {B_\alpha}^{l-1},$$

we write

$$(I - LB_\alpha)\varphi(x) = g(x). \qquad (3)$$

We will consider equation (2) in two cases:
I. The first case of the problem in space $H_\mu(J)$:

$$\xi(x) - b_1(x)B_\alpha\xi(x) - b_2(x){B_\alpha}^2\xi(x) - \dots - b_l(x){B_\alpha}^l\xi(x) = q(x) \qquad (4)$$

Coefficients $b_i(\mathrm{x}), b_i(\mathrm{x}), \dots, b_i(\mathrm{x})$ belong to $H_\mu(J)$, the known free term on the right side $q(x)$ belongs to $H_\mu(J)$, the unknown function ξ is sought in space $H_\mu(J)$.

II. The second case of the problem in space $H_\mu^0(J,\rho)$:

$$\psi(x) - b_1(x)B_\alpha\psi(x) - b_2(x){B_\alpha}^2\psi(x) - \dots - b_l(x){B_\alpha}^l\psi(x) = h(x) \qquad (5)$$

Coefficients $b_i(\mathrm{x}), b_i(\mathrm{x}), \dots, b_i(\mathrm{x})$ belong to $H_\mu(J)$, the known free term on the right side $h(x)$ belongs to $H_\mu^0(J,\rho)$, the unknown function $\psi(x)$ is sought in space $H_\mu^0(J,\rho)$.

We will now proceed to describe the process of constructing iterated equations. We write the original equation (3) in recurrent form:

$$\varphi(x) = LB_\alpha\, \varphi(x) + g(x). \qquad (6)$$

Substituting the expression for the unknown function into the right side of the same equation (4), we get an equation after the first iteration:

$$\varphi(x) = (LB_\alpha)[LB_\alpha\alpha\, \varphi(x) + g(x)] + g(x),$$
$$\varphi(x) = (LB_\alpha)(LB_\alpha)\varphi(x) + (LB_\alpha)g(x) + g(x).$$

We denote the obtained equation as the first iterated equation. Using the same algorithm, we construct the second iterated equation and move operator B_α to the end of it, in front of $\varphi(x)$.

Here, we have indicated the results of the first and the second iteration. Continuing the iterative process, at the step n, we obtain n-th iterated equation:

$$\varphi(x) = (LB_\alpha)^{n+1}\varphi(x) + G_n(x), \qquad (7)$$

where

$$G_n(x) = (LB_\alpha)^n g(x) \dots + (LB_\alpha)g(x) + g(x) \qquad (8)$$

Now, in the first case of the problem, space $H_\mu(J)$, we establish connections between the initial values of the free term and the initial values of the unknown function of the original equation (2). These values will be useful in the construction of the solution of the considered equation. The following simple statement holds:

Lemma 1. There are relationships

$$\xi(1) = \frac{q(1)}{1-b_1(1)-b_2(1)-\dots-b_l(1)},$$

where $1 - b_1(1) - b_2(1) - \dots - b_l(1) \neq 0$ and $\xi(0)\frac{q(0)}{1-b_1(0)-b_2(0)-\dots-b_l(0)}$, where $1 - b_1(0) - b_2(0) - \dots - b_l(0) \neq 0$.

Proof. It is easy to see that from original equation (2) we obtain:

$$\xi(1) - b_1(1)B_\alpha\xi(1) - b_2(1){B_\alpha}^2\xi(1) - \dots - b_l(1){B_\alpha}^l\xi(1) = q(1)$$
$$\xi(0) - b_1(0)B_\alpha\xi(0) - b_2(0){B_\alpha}^2\xi(0) - \dots - b_l(0){B_\alpha}^l\xi(0) = q(0)$$

and

$$\xi(1) - b_1(1)\xi(1) - b_2(1)\xi(1) - \dots - b_l(1)\xi(1) = q(1),$$

$\xi(0) - b_1(0)\xi(0) - b_2(0)\xi(0) - \cdots - b_l(0)\xi(0) = q(0)$.

This follows from the fact that constants do not change under the action of the shift and $x = 1$, $x = 0$ is a fixed point for non-Carleman shift $\alpha(x)$.

3 Equivalence of the Original Equation and the Iterated Equations

That the solution to the original equation is a solution to the iterated equations directly follows from the description of the algorithm for constructing the iterated equations. We formulate this affirmation as a theorem due to its importance to what follows in this article and we will provide its proof.

Theorem 1.
The initial equation and iterated equations are equivalent to each other.

Proof. At first, we will prove that if φ is a solution to the original equation, then this function φ is a solution to all iterated equations. This follows directly from the algorithm of construction of iterated equations. So, it is obvious that all solutions of original equation (6): $\varphi(x) = LB_\alpha\,\varphi(x) + g(x)$ are included in the set of solutions of the first iterated equation $\varphi(x) = (LB_\alpha)[LB_\alpha\alpha\,\varphi(x) + g] + g(x)$. Continuing our reasoning in this way, we come to the conclusion of the first part of the proof. Now we will show that the iterated equations do not have any other solutions except for the solutions of the original equation.

Let $f(x)$ be the solution of the first iterated equation: $\varphi(x) = (LB_\alpha)^2\,\varphi(x) + G_1(x)$, $G_1(x) = (LB_\alpha)g + g$, then the function f will be the solution of the second iterated equation: $f(x) = (LB_\alpha)[(LB_\alpha)^2\,\varphi(x) + G_1] + G_2$, $G_2 = LB_\alpha G_1 + g$. Now, we add and subtract the term $LB_\alpha\, f(x) + g(x)$ and perform some transformations,

$$f = LB_\alpha\, f + g + LB_\alpha[(LB_\alpha)^2 f + G_1] + G_2 - LB_\alpha\, f - g$$

$$f = LB_\alpha\, f + LB_\alpha[(LB_\alpha)^2 f + G_1 - f - G_1] + LB_\alpha G_1 + G_2 - g,$$

$$f = LB_\alpha\, f + g - LB_\alpha\, G_1 + G_2 - g.$$

Here the fact that f is a solution of the first iterated equation $f(x) = [(LB_\alpha)^2\, f(x) + G_1]$ was used. Counting $-LB_\alpha\, G_1 + G_2 - g = 0$ we come to initial equation $\varphi(x) = LB_\alpha\,\varphi(x) + g(x)$. The theorem has been proved. Note that we did not separate cases 1 and 2 since differences in statements do not affect the proof of the theorem.

We will continue the analysis of the solvability of original equation (2) using its representation in the form of n-th iterated equation (7), (8). We represent the operator $(LB_\alpha)^{n+1}$, obtained after n iterations, in another way:

$(LB_\alpha)^{n+1} = L\left(B_\alpha L B_\alpha{}^{-1}\right)\left(B_\alpha{}^2\, L B_\alpha{}^{-2}\right) \dots (B_\alpha{}^n\, L B_\alpha{}^{-n}) B_\alpha{}^{n+1}$.

We introduce an operator:

$\Omega_n^L = L\left(B_\alpha L B_\alpha{}^{-1}\right)\left(B_\alpha{}^2\, L B_\alpha{}^{-2}\right) \dots (B_\alpha{}^n\, L B_\alpha{}^{-n})$,

that will play an important role and

$\Gamma_n^L = (LB_\alpha)^n + (LB_\alpha)^{n-1} \dots + (LB_\alpha) + I$.

Note that $\Gamma_0^L g(x) = g(0)$.

Operator Ω_n^L can be written out in detail:

$\left(\left(B_\alpha{}^1 b\right)I + \left(B_\alpha{}^1 c\right)I\right) \dots \left((B_\alpha{}^n b)I + (B_\alpha{}^n c)\right)$:

The relation $(LB_\alpha)^{n+1} = \Omega_n B_\alpha{}^{n+1}$ holds.

The n-th iterated equation is represented as:

$$\varphi(x) = \Omega_n^L B_\alpha{}^{n+1}\varphi(x) + \Gamma_n^L g(x).$$

4 Solution to the Initial Equation

Passing to the limit, directing n to infinity, we have:

$$\varphi(x) = \lim_{n\to\infty}[\Omega_n^L B_\alpha{}^{n+1}\varphi(x) + \Gamma_n^L g(x)].$$

We start with the first statement in space $H_\mu(J)$:

$[I - b_1(x)B_\alpha - \dots - b_l(x)B_\alpha{}^l)]\xi(x) = q(x)$.

To calculate the value of the unknown function at the end points by Lemma 1, we assume $1 - b_1(1) - b_2(1) - \dots - b_l(1) \neq 0$.

We can introduce operators Ω^L and $\lim\limits_{n\to\infty}[\Gamma_n^L q(x)] = \Gamma^L q(x)$ when their limits exist in operator norm and functional norm of the Hölder space, respectively. Taking into account the linearity of the operators, we have a unique solution to the original equation:

$\varphi(x)[\lim\limits_{n\to\infty}\Omega_n^L][\lim\limits_{n\to\infty}B_\alpha{}^{n+1}\varphi(x)] + \lim\limits_{n\to\infty}\Gamma_n^L g(x)$

and $\varphi(x) = [\Omega^L] \times [\,\varphi(1)] + \Gamma^L g(x)$.

Here,

$\Omega^L = L(B_\alpha L\, B_\alpha^{-1})\left(B_\alpha{}^2\, L\, B_\alpha^{-2}\right) \dots (B_\alpha{}^n\, L\, B_\alpha^{-n}) \dots$ (9)

and

$\Gamma^L = [I + (LB_\alpha) + (LB_\alpha)^2 \dots + (LB_\alpha)^n + \cdots]$. (10)

Paying attention to formula for the initial values from Lemma 1 and to the limit operators (9, 10), the solution of the initial equation can be written as follows:

$$\varphi(x) = \Omega \frac{g(1)}{1-b_1(1)-b_2(1)-\ldots-b_l(1)} + \Gamma g(x). \qquad (11)$$

Theorem 2.
Let the sequence of operators $\{\Omega_n\}$ converge in operator norm to the operator Ω acting in $H_\mu(J)$, the functional sequence $\{G_n(x)\}$ converge to some function $G(x)$ from space $H_\mu(J)$ and $1 - b_1(1) - b_2(1) \ldots - b_l(1) \neq 0$.
Then the equation in space $H_\mu(J)$:
$\varphi(x) - b_1(x)B_\alpha\varphi(x) \ldots - b_l(x)\, B_\alpha^l\varphi(x) = g(x)$,
where coefficients and free term belong to $H_\mu(J)$ and unknown function is searched from $H_\mu(J)$, has a unique solution that is determined by the formula:
$\varphi(x)\ \left(\Omega \frac{g(1)}{1-b_1(1)-b_2(1)-\ldots-b_l(1)}\right)(x)\Gamma_n^L g(x)$.

Now, consider the second statement in space $H_\mu^0(J,\rho)$.

Let us remember that we study the equation (5)
$\psi(x) - b_1(x)B_\alpha\psi(x) \ldots - b_l(x)\, B_\alpha^l\psi(x) = h(x)$.
with the free term $h(x)$ belonging to $H_\mu^0(J,\rho)$, the unknown function $\psi(x)$ belonging to $H_\mu^0(J,\rho)$ and the coefficients taken from $H_\mu(J)$.

We translate the equation from $H_\mu^0(J,\rho)$ into $H_\mu(J)$. We multiply the left and the right sides of (5) by the weight function $\rho(x)$ and obtain
$\rho\psi(x) - \rho b_1 B_\alpha\psi(x) \ldots - \rho b_l\, B_\alpha^l\psi(x) = \rho h(x)$.

Here, according to the definition of the Hölder space with weight $H_\mu^0(J,\rho)$, functions $\eta(x)=\rho(x)\,\psi(x)$ and $f(x) = \rho(x)h(x)$ belong to the space of Hölder $H_\mu(J)$ and vanish at the end points, $\eta(1)=\eta(0)=0$, $f(1) = f(0) = 0$.

We write down equivalent equations for the unknown function $\eta(x) = \rho(x)\psi(x)$,
$\eta(x) - \rho b_1 B_\alpha\rho^{-1}\eta(x) \ldots - \rho b_l\, B_\alpha^l\rho^{-1}\eta(x) = f(x)$.
and, finally, we get the equation:

$$\eta(x) - w_1(x)B_\alpha\eta(x) \ldots - w_l(x)\, B_\alpha^l\eta(x) = f(x), \qquad (12)$$

where
$w_1(x) = b_1(\rho B_\alpha\rho^{-1}), \ldots (w_l(x) = b_l(\rho {B_\alpha}^l\rho^{-1})$.

Here, the following was used: $B_\alpha(\rho^{-i}\eta) = (B_\alpha\rho^{-i})B_\alpha\eta, \qquad i = 1 \ldots l$.

Let's introduce the operator:

$$S = \sum_{i=1}^{l} w_i(\mathrm{x})\, {B_\alpha}^{l-1}$$

and write equation (12) in the form:
$\eta(x)=\mathrm{S}B_\alpha\eta(x) + f(x), \quad \eta(x) \in H_\mu(J)$.

To simply the proof of the following statements, we will add to the requirements imposed on the shift one more: $\alpha(x)$ has a second derivative and $\alpha''(1) \neq 0$, $\alpha''(1) \neq 0$. In what follows we will assume that $\alpha(x)$ has this property.

We take the weight function from the definition of space $H_\mu^0(J,\rho)$: $\rho(x) = (x-0)^{\mu_0}(1-x)^{\mu_1}$, $\mu < \mu_0 < 1 + \mu$, $\quad \mu < \mu_1 < 1 + \mu$.

Lemma 2.
Functions $w(x) = \frac{x}{\alpha(x)}$, $\frac{\rho(x)}{B_\alpha\rho(x)}$, … , $\frac{\rho(x)}{{B_\alpha}^l\rho(x)}$ belong to $H_\mu(J)$.

Proof. First, we prove that $w(x)$ belongs to the Lipschitz class $H_1(J)$:
$|w(x_1) - w(x_2)| \leq C|x_1 - x_2|$.

We use Lagrange´s Theorem. If function $w(x)$ is continuous on a closed segment $[x_1, x_2]$ and differentiable on the open interval (x_1, x_2), then the following relation holds:

$$w(x_1) - w(x_2) = w'(c)(x_1 - x_2), x_1 < c < x_2; \qquad (13)$$

in more detail: $\frac{x_1}{\alpha(x_1)} - \frac{x_2}{\alpha(x_2)} = [\frac{x}{\alpha(x)}]'_{x=c}(x_1 - x_2)$.

The function $w'(c) = [\frac{x}{\alpha(x)}]'_{x=c}$ is bounded by some constant M. We calculate this constant.

The derivative of the function $w(x)$ is equal to $[\frac{\alpha(x)-.x\alpha'(x).}{(\alpha(x))^2}]$ and $\left(\frac{x}{\alpha(x)}\right)' = [\frac{x}{\alpha(x)}]^2\,[\frac{\alpha(x)-.x\alpha'(x).}{(x)^2}]$.

The derivative has no singularities at $x \neq 0$. In order to assert the continuity of the derivative we consider the limit when x tends to zero:

$$\lim_{x\to 0}([w(x)]^2)\left(\frac{\alpha(x)-.x\alpha'(x)}{(x)^2}\right) = [\frac{1}{\alpha'(0)}]^2 \lim_{x\to 0}\frac{\alpha'(x)-\alpha'(x)-x\alpha''(x)}{2x} = \frac{1}{(\alpha'(0))^2}\frac{-\alpha''(0)}{2}.$$

The continuous function $w'(x)$ is defined on segment [0,1] and therefore reaches its extreme values on this segment, which we will denote by w'_{min} and w'_{max} . We came to an estimate:
$|\left(\frac{x}{\alpha(x)}w(x)\right)'| \leq \mathrm{M} = \{\max|w'_{min}|, |w'_{max}|\}$.

Coming back to (13), we conclude that function $w(x)$ is a Lipschitz function.

Note that $1 - \alpha(x) = \alpha(x)(1-x)$ and that the product of a Hölder function from $H_\mu(J)$ and a Lipschitz function from $H_1(J)$ gives us a Hölder function from $H_\mu(J)$.

The following equality holds:

$$\frac{\rho(x)}{B_\alpha\rho(x)}\ \frac{(\alpha(x)-0)^{\mu_0}(1-\alpha(x))^{\mu_1}}{(\alpha(x)-0)^{\mu_0}(1-\alpha(x))^{\mu_1}} = \left(\frac{x}{\alpha(x)}\right)^{\mu_0}\left(\frac{1-x}{1-\alpha(x)}\right)^{\mu_1}$$

and $\left(\frac{x}{\alpha(x)}\right)^{\mu_0}\left(\frac{1-x}{1-\alpha(x)}\right)^{\mu_1}$ belongs to $H_\mu(J)$.

We move on to:

$$\frac{\rho(x)}{{B_\alpha}^2\rho(x)} = \frac{\rho(x)}{B_\alpha\rho(x)}\frac{B_\alpha\rho(x)}{{B_\alpha}^2\rho(x)} = \frac{\rho(x)}{B_\alpha\rho(x)}B_\alpha\left(\frac{\rho(x)}{B_\alpha\rho(x)}\right).$$

The requirements imposed on the shift and the belonging of the shift to Hölder class with the exponent μ imply that $B_\alpha(w)$ belongs $H_\mu(J)$. Presenting $\frac{\rho(x)}{{B_\alpha}^i\rho(x)}$ as:

$$\frac{\rho(x)}{B_\alpha\rho(x)}B_\alpha\left(\frac{\rho(x)}{B_\alpha\rho(x)}\right)\ B_\alpha^2\left(\frac{\rho(x)}{B_\alpha\rho(x)}\right)\dots B_\alpha^{i-1}\left(\frac{\rho(x)}{B_\alpha\rho(x)}\right),$$

we, similarly, have that $\frac{\rho(x)}{B_\alpha^i\rho(x)}$ belongs to $H_\mu(J)$.

From Lemma 2 it follows that, the coefficients $w_1(\mathrm{x}), w_2(\mathrm{x}), \dots, w_l(\mathrm{x})$ belong to the space $H_\mu(J)$, since according to the statement of the problem, the coefficients $b_1(\mathrm{x}), b_2(\mathrm{x}), \dots, b_l(\mathrm{x})$ are taken from the space $H_\mu(J)$.

To facilitate understanding of the ideas and the algorithms for constructing operators used in the article, the authors propose a list of sources that provide classical definitions of operator theory and functional analysis [4], [5] [6], [7], as well as specific features of Hölder spaces of weighted functions and operators acting in them, and indicate some applications [8], [9].

We introduce the operators:

$\Omega_n^S = S\left(B_\alpha S {B_\alpha}^{-1}\right)\left({B_\alpha}^2 S{B_\alpha}^{-2}\right)\dots\left({B_\alpha}^n S{B_\alpha}^{-n}\right)$;

the limit operator $\Omega^S = \lim_{n\to\infty}\Omega_n^S$, if it exists in operator norm and acts on $H_\mu(J)$; operator:

$\Gamma_n^S = I + (SB_\alpha) + (SB_\alpha)^2 + \dots + (SB_\alpha)^n$ and operator as the sum of functional series $\Gamma^R r = r + (RB_\alpha)r + (RB_\alpha)^2 r \dots + (RB_\alpha)^n r + \cdots$ when it converges to some function from $H_\mu(J)$.

Equation (12) is a special case of the original equation (4), with the right-hand side vanishing at the point $x = 1$, $\varphi(1) = 0$. We write down its solution using the formula (11): $\varphi(x) = \Gamma^S(f(x)$, where the term $\Omega^S\frac{f(1)}{1-b(1)-c(1)}$ disappears because $f(1) = 0$. We get a theorem:

Theorem 3.
Let the sequence of operators $\{\Omega_n^S\}$ converge in operator norm to operator Ω^S acting in $H_\mu(J)$ and the functional series $\{\Gamma_n^S\rho(x)h(x)\}$ converge in norm of space $H_\mu(J)$ to some function $\Gamma^S\rho(x)h(x)$ from the space $H_\mu(J)$ and, moreover, let $1 - b_1(1) - b_2(1) \dots - b_l(1) \neq 0$.

Then, the equation:

$\psi(x) - b_1(x)B_\alpha\psi(x) \dots - b_l(x){B_\alpha}^l\psi(x) = h(x)$,

where the coefficients belong to $H_\mu(J)$, free term belongs to $H_\mu^0(J,\rho)$ and unknown function $\psi(x)$ is sought from $H_\mu^0(J,\rho)$, has a unique solution:

$\psi(x) = \rho^{-1}(x)\Gamma^S[\rho(x)h(x)$,

where $\rho(x) = x^{\mu_0}(1-x)^{\mu_1}$,

$$\mu < \mu_0 < 1+\mu, \qquad \mu < \mu_1 < 1+\mu.$$

4 Conclusion

The equation considered in this paper is a linear one. The authors plan to generalize the proposed method and apply it to the study of some nonlinear equations that arise when modeling systems with renewable resources. In order to achieve this, there is a need to develop the theory of continued fractions and infinite products [10], which complicates the construction of solutions to nonlinear equations. The first advances in this direction have already been made and the results obtained will serve as new tools for studying renewable systems with elements in different states; for example, infected, not infected, with immunity, and without immunity. Another direction of research is the introduction of several shifts and their iterations into the equation under consideration, which will allow taking into account more complex relationships and interactions between the elements of the system.

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Conflict of Interest

The authors have no conflicts of interest to declare.