Euler-Apéry Type Multiple Zeta Star Values and Multiple t-Star Values

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Abstract: By introducing an extra binomial factor in the summands of multiple zeta star series, we can define the Euler-Apéry type multiple zeta star series. Their convergent values at positive integers are called Euler-Apéry type multiple zeta star values. In this paper we establish several recurrence relations about these values and a parametric variant by the method of iterated integrals. We then find the explicit evaluations for some specific Euler-Apéry type multiple zeta star values and t-star values, together with a parametric variant of the star version.

Key-Words: (alternating) multiple zeta (star) values; multiple t-(star) values; colored multiple zeta values; iterated integrals; central binomial coefficients; multiple polylogarithm function.

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1 Introduction

In recent years, the study of multiple zeta values and their many variants has attracted numerous prominant mathematicians and physicists due to their deep connections to various branches in mathematics and theoretical physics. For example, F. Brown determined explicitly the mixed Tate motives oover $\mathbb{Z}$ and proved the Deligne-Ihara conjecture in [1], by applying the motivic theory of multiple zeta values while Broadhurst revealed the intimate relations between alternating multiple zeta values (also called Euler sums) and the knot invariants and Feynman integrals in [2], [3]. We also remark that the zeta function is a ubiquitous object in mathematics and it has appeared in many other types of research, e.g., in the study of cotangent sums, [4], prime numbers, [5], and even black holes, [6].

In his seminal work [7] Euler initiated the investigation of the special values of multiple zeta function in early 18th century. In fact, he considered the star version of these values in which the summation indices are allowed to be the same (see (1) below). As one of the mathematical giants, his contribution to math and science is still impacting our modern lives on a daily basis. For example, the Euler and Navier-Stokes equations are well-known in fluid mechanics, [8], and his method to numerically evaluate definite integrals is still being taught in standard calculus courses as well as used by professional mathematicians alike [9].

In this paper, we focus on one particular variant of multiple zeta values, namely, the Euler-Apéry type multiple zeta star series. The origin of these series goes back to Apéry who gave the first proof of irrationality of $\zeta(3)$ by expressing it as a variation of the Riemann zeta series with an extra binomial coefficient factor in the summands. Since then the multiple variable version of this type of series is often called a (multiple) Apéry series/sum which has appeared unexpectedly in the evaluation of Feynman integrals. Along this direction, a few important experimental work occurred at the beginning of this century, e.g., see, [10], [11], [12], for inverse binomial series of Apéry type (where the binomial coefficient appears in the denominator of the summands) and [13], for ordinary binomial series of Apéry type. Furthermore, odd variations (with some summation indices restricted to odd numbers) of both types already appeared implicitly in Eq. (1.1) of [12], and Eq. (A.25) of [13], respectively.

We now introduce some basic notations. Let $\mathbb{Z}$ and $\mathbb{N}$ be the set of integers and positive integers, respectively. A finite sequence $\boldsymbol{k} := (k_1, \ldots, k_r) \in \mathbb{N}^r$ is called a *composition*. Set $|\boldsymbol{k}| := k_1 + \cdots + k_r$. We call $|\boldsymbol{k}|$ and r the weight and the depth of $\boldsymbol{k}$, respectively. If $k_1 > 1$, $\boldsymbol{k}$ is called *admissible*.

For a composition $\boldsymbol{k} = (k_1, \dots, k_r)$ and $n \in \mathbb{N}$, the *multiple harmonic sums* and *multiple harmonic star sums* are defined by

$$\zeta_n(\boldsymbol{k}) := \sum_{n \geq n_1 > \cdots > n_r > 0} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \tag{1}$$

(2)

and

$$\zeta_n^\star(\boldsymbol{k}) := \sum_{n \geq n_1 \geq \cdots \geq n_r > 0} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}, \tag{3}$$

respectively. If $n < r$ then $\zeta_n(\boldsymbol{k}) := 0$ and $\zeta_n(\emptyset) = \zeta_n^\star(\emptyset) := 1$. As special cases,

$$H_n := \zeta_n(1) = \zeta_n^\star(1)$$

and

$$H_n^{(k)} := \zeta_n(k) = \zeta_n^\star(k)$$

are the *classical* and *generalized harmonic numbers*, respectively. When taking the limit $n \to \infty$ in (1) and (3) we obtain the so-called *multiple zeta values* (MZVs) and the *multiple zeta star values* (MZSVs), respectively:

$$\zeta(\boldsymbol{k}) := \lim_{n \to \infty} \zeta_n(\boldsymbol{k}), \quad \zeta^\star(\boldsymbol{k}) := \lim_{n \to \infty} \zeta_n^\star(\boldsymbol{k}),$$

defined for admissible compositions $\boldsymbol{k}$ to ensure convergence of the series. Although Euler studied the double zeta values almost three hundred years ago in [14], the systematic study of MZVs began in the early 1990s with the work in [15, 16, 17]. Due to their surprising and sometimes mysterious appearance in the study of many branches of mathematics and theoretical physics, these special values have attracted a lot of attention and interest in the past three decades (for example, see the survey article, [18], and the books, [19, 20]).

Recall that Hoffman introduced and studied odd variants of MZVs and MZSVs in [21]. They are defined for an admissible composition $\boldsymbol{k} = (k_1, k_2, \dots, k_r)$ by

$$t(\boldsymbol{k}) := \sum_{n_1 > n_2 > \cdots > n_r > 0} \prod_{j=1}^{r} \frac{1}{(2n_j - 1)^{k_j}}$$

and

$$t^\star(\boldsymbol{k}) := \sum_{n_1 \geq n_2 \geq \cdots \geq n_r > 0} \prod_{j=1}^{r} \frac{1}{(2n_j - 1)^{k_j}},$$

and are called *multiple t-value* and *multiple t-star value*, respectively. In fact, we can also restrict the summation indices to a fix parity pattern such as even and odd alternatively interlaced which leads to multiple T-values studied in [22] and the multiple S-values considered by the authors of this paper (see, [23], and preprints arXiv:2008.13157, 2009.10774 and 2208.09593).

Similar to multiple harmonic sums and multiple harmonic star sums, for a composition $\boldsymbol{k} = (k_1, \dots, k_r)$ and positive integer n, we can define the *multiple t-harmonic sums* and *multiple t-harmonic star sums* respectively by

$$t_n(\boldsymbol{k}) := \sum_{n \geq n_1 > n_2 > \cdots > n_r > 0} \prod_{j=1}^{r} \frac{1}{(2n_j - 1)^{k_j}}$$

and

$$t_n^\star(\boldsymbol{k}) := \sum_{n \geq n_1 \geq n_2 \geq \cdots \geq n_r > 0} \prod_{j=1}^{r} \frac{1}{(2n_j - 1)^{k_j}}.$$

In general, for any $\boldsymbol{k} = (k_1, \dots, k_r) \in \mathbb{N}^r$ and $\boldsymbol{z} = (z_1, \dots, z_r)$ where $z_1, \dots, z_r$ are Nth roots of unity, we can define the *colored MZVs* (CMZVs) of level N as

$$\mathrm{Li}_{\boldsymbol{k}}(\boldsymbol{z}) := \sum_{n_1 > \cdots > n_r > 0} \frac{z_1^{n_1} \dots z_r^{n_r}}{n_1^{k_1} \dots n_r^{k_r}} \tag{4}$$

which converges if $(k_1, z_1) \neq (1, 1)$ (see, Ch. 15 of [20]), in which case we call $(\boldsymbol{k}; \boldsymbol{z})$ *admissible*. In particular, if all $z_j \in \{\pm 1\}$ in (4), then the level two colored MZVs are called *alternating MZVs* (or *Euler sums*). In this case, namely, when $(z_1, \dots, z_r) \in \{\pm 1\}^r$ and $(k_1, z_1) \neq (1, 1)$, we set $\zeta(k_1, \dots, k_r; z_1, \dots, z_r) = \mathrm{Li}_{k_1, \dots, k_r}(z_1, \dots, z_r)$. Further, we put a bar on top of k_j if $z_j = -1$. For example,

$$\zeta(\bar{2}, 6, \bar{1}, 8) = \zeta(2, 6, 1, 8; -1, 1, -1, 1).$$

More generally, for any composition $\boldsymbol{k} = (k_1, \dots, k_r) \in \mathbb{N}^r$, the *classical multiple polylogarithm function* with r-variable is defined by

$$\mathrm{Li}_{\boldsymbol{k}}(x_1, \dots, x_r) := \sum_{n_1 > n_2 > \cdots > n_r > 0} \frac{x_1^{n_1} \cdots x_r^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}}$$

which converges if $|x_1 \cdots x_j| < 1$ for all $j = 1, \dots, r$. They can be analytically continued to a multi-valued meromorphic function on $\mathbb{C}^r$ [24].

In particular, if $x_1 = x, x_2 = \cdots = x_r = 1$, then $\mathrm{Li}_{k_1, \dots, k_r}(x, 1_{r-1})$ is the classical multiple polylogarithm function with single-variable. As a convention, we denote by 1_d the sequence of 1's with d repetitions. When $d = 1$ we recover the well-know polylogarithm function which, together with multiple polylogarithms, has been studied by many researchers since 1960s due to its deep connections to many branches of mathematics and theoretical physics[25], [26], [27], [28], [29], [30], [31], [32].

We now define an odd variation of the multiple polylogarithm. For any composition $\boldsymbol{k} = (k_1, \dots, k_r)$ the *multiple t-polylogarithm function* is defined by

$$\begin{aligned}
&\mathrm{ti}_{\boldsymbol{k}}(x)\\
&:= \sum_{n_1>n_2>\cdots>n_r>0} \frac{x^{2n_1-1}}{(2n_1-1)^{k_1}\dots(2n_r-1)^{k_r}}\\
&= \int_0^x \frac{dt}{1-t^2}\left(\frac{dt}{t}\right)^{k_r-1}\frac{tdt}{1-t^2}\left(\frac{dt}{t}\right)^{k_{r-1}-1}\\
&\qquad \cdots \frac{tdt}{1-t^2}\left(\frac{dt}{t}\right)^{k_1-1},
\end{aligned}$$

where $|x| \le 1$ with $(k_1, x) \neq (1, 1)$. Clearly, $t_{\boldsymbol{k}}(1) = t(\boldsymbol{k})$ with $k_1 \ge 2$.

Motivated by [33], [34], people also studied some Euler-Apéry type series of the form

$$\sum_{n=1}^{\infty} \frac{H_n^{(k_1)} H_n^{(k_2)} \cdots H_n^{(k_r)}}{n^p} a_n^{\pm 1}, \tag{5}$$

where $a_n = \binom{2n}{n}/4^n$. For the above and some other similar series they established the corresponding explicit formulas using the alternating MZVs. In particular, they discovered a few elegant explicit formulas for the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^p}, \quad \sum_{n=1}^{\infty} \frac{a_n H_n^{(m)}}{n^p}, \quad \sum_{n=1}^{\infty} \frac{a_n H_n H_n^{(m)}}{n^p},$$

and

$$\sum_{n=1}^{\infty} \frac{a_n H_n^3}{n^p}, \sum_{n=1}^{\infty} \frac{a_n \zeta_n^\star(1_m)}{n^p}, \sum_{n=1}^{\infty} \frac{a_n H_n \zeta_n^\star(1_m)}{n^p},$$

for $m, p \ge 1$, by using the method of iterated integrals. They also found some expressions of the Euler-Apéry type series

$$\sum_{n=1}^{\infty} \frac{a_n^{-1}}{n^p}, \sum_{n=1}^{\infty} \frac{H_n}{a_n n^p}, \sum_{n=1}^{\infty} \frac{H_{2n}}{a_n n^p}, \sum_{n=1}^{\infty} \frac{O_n}{a_n n^p},$$

for $p \ge 2$, and

$$\sum_{n=1}^{\infty} \frac{a_n}{n^p}, \quad \sum_{n=1}^{\infty} \frac{a_n H_{2n}}{n^p}, \quad \sum_{n=1}^{\infty} \frac{a_n O_n}{n^p},$$

for $p \ge 1$, by computing the contour integrals related to gamma functions, polygamma functions and trigonometric functions. Here $O_n = \sum_{k=1}^n \frac{1}{2k-1}$ are the classical *odd harmonic numbers*. Obviously, by applying the stuffle relations, also called quasi-shuffle relations, [35], we know that for any composition $\boldsymbol{k} = (k_1, \dots, k_r)$, the product $H_n^{(k_1)} \cdots H_n^{(k_r)}$ can be expressed in terms of a linear combination of multiple harmonic (star) sums (for the explicit formula, see, Eq. (2.4) of [36]). For example

$$\begin{aligned}
H_n H_n^{(2)} &= \zeta_n(1)\zeta_n(2)\\
&= \zeta_n(1,2) + \zeta_n(2,1) + \zeta_n(3).
\end{aligned}$$

Hence, we can study the Euler-Apéry type MZVs

$$\sum_{n=1}^{\infty} \frac{\zeta_n(k_2, \dots .k_r)}{n^{k_1}} a_n^{\pm 1}$$

to obtain some explicit evaluations of (5). Au proved in [33] that for $\boldsymbol{k} = (k_1, \dots, k_r) \in \mathbb{N}^r$, the Euler-Apéry type MZVs above can be expressed in terms of alternating MZVs (even though he did not give a general explicit formula), namely,

$$\sum_{n=1}^{\infty} \frac{\zeta_n(k_2, \dots .k_r)}{n^{k_1}} a_n^{\pm 1} \in \mathsf{CMZV}^2_{|\boldsymbol{k}|}.$$

Therefore, the Euler-Apéry type series (5) can be evaluated by alternating MZVs. Some related results may be found in [37], [38], [39], [40], [41], [42], [43], [44], and references therein. Further, Au showed that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{\zeta_n(k_2, \dots .k_r)}{n^{k_1}} a_n^{-2} \in \mathsf{CMZV}^4_{|\boldsymbol{k}|},\\
&\sum_{n=1}^{\infty} \frac{\zeta_n(k_2, \dots .k_r)}{n^{k_1}} a_n^{2} \in \frac{1}{\pi}\mathsf{CMZV}^4_{|\boldsymbol{k}|+1},
\end{aligned}$$

where CMZV^N_w is the $\mathbb{Q}$-span of CMZVs of weight w and level N.

In this paper, we will study the following Euler-Apéry type MZSVs and MtSVs

$$\begin{aligned}
Z_\zeta(k_1, \dots, k_r) &:= \sum_{n=1}^{\infty} \frac{\zeta_n^\star(k_2, \dots, k_r)}{n^{k_1}} a_n\\
&= \sum_{n=1}^{\infty} \frac{\zeta_n^\star(k_2, \dots, k_r)}{n^{k_1} 4^n} \binom{2n}{n}, \quad (6)
\end{aligned}$$

$$\begin{aligned}
Z_t(k_1, \dots, k_r) &:= \sum_{n=1}^{\infty} \frac{t_n^\star(k_2, \dots, k_r)}{n^{k_1}} a_n\\
&= \sum_{n=1}^{\infty} \frac{t_n^\star(k_2, \dots, k_r)}{n^{k_1} 4^n} \binom{2n}{n},
\end{aligned}$$

and a parametric variant of (6)

$$Z_\zeta(k_1,\dots,k_r;x):=\sum_{n=1}^{\infty}\frac{\zeta_n^\star(k_2,\dots,k_r;x)}{n^{k_1}}a_n =\sum_{n=1}^{\infty}\frac{\zeta_n^\star(k_2,\dots,k_r;x)}{n^{k_1}4^n}\binom{2n}{n}, \quad (7)$$

where the *parametric multiple harmonic star sum* $\zeta_n^\star(k_1,\cdots,k_r;x)$ is defined by

$$\zeta_n^\star(k_1,\cdots,k_r;x):=\sum_{n\ge n_1\ge\cdots\ge n_r\ge 1}\frac{x^{n_r}}{n_1^{k_1}\cdots n_r^{k_r}}$$

and $\zeta_n^\star(\emptyset;x):=x^n$.

The primary goal of this paper is to study the explicit relations of (6) and (7). We will use the method of iterated integrals to obtain some recurrence relations of (6) and (7), which in turn will lead to some explicit evaluations of (6) and (7).

2 Euler-Apéry type MZSVs and Its Parametric Variant

The theory of iterated integrals was developed first by K.T. Chen in his ground breaking works [45], [46] in the 1960's. It has played an indispensable role in the study of algebraic topology and algebraic geometry in past half century. We will use its following form:

$$\int_a^b f_p(t)dt\cdots f_1(t)dt := \int\limits_{a<t_p<\cdots<t_1<b} f_p(t_p)\cdots f_1(t_1)\,dt_1\cdots dt_p.$$

In this section, we use these integrals to establish two recurrence relations of (6) and (7). First, by Eqs. (3.1) and (3.2) of [47], we obtain the iterated integral expression

$$\mathrm{Li}_{k_1,\dots,k_r}(x_1,x_2/x_1\dots,x_r/x_{r-1}) =\int_0^1\frac{x_r\,dt}{1-x_rt}\left(\frac{dt}{t}\right)^{k_r-1}\cdots\frac{x_1\,dt}{1-x_1t}\left(\frac{dt}{t}\right)^{k_1-1}. \quad (8)$$

In particular, CMZVs can be expressed using iterated integrals

$$\mathrm{Li}_{\boldsymbol{k}}(\boldsymbol{z})=\int_0^1 \mathtt{x}_{\xi_r}\mathtt{x}_0^{k_r-1}\cdots\mathtt{x}_{\xi_1}\mathtt{x}_0^{k_1-1}, \quad (9)$$

where $\xi_j:=\prod_{i=1}^j z_i^{-1}$, and $\mathtt{x}_\xi=dt/(\xi-t)$ for any Nth roots of unity ξ, see, §2.1 of [20], for a brief summary of this theory.

To save space, for any composition $\boldsymbol{m}=(m_1,\dots,m_p)\in\mathbb{N}^p$ and $i,j\in\mathbb{N}$, we put

$$\overrightarrow{\boldsymbol{m}}_{i,j}:=\begin{cases}(m_i,\dots,m_j), & \text{if } i\le j\le p;\\ \emptyset, & \text{if } i>j,\end{cases}$$

$$\overleftarrow{\boldsymbol{m}}_{i,j}:=\begin{cases}(m_j,\dots,m_i), & \text{if } i\le j\le p;\\ \emptyset, & \text{if } i>j.\end{cases}$$

For all $1\le i\le p$, we set

$$\begin{aligned}\overrightarrow{\boldsymbol{m}}_i&=\overrightarrow{\boldsymbol{m}}_{1,i}=(m_1,\dots,m_i),\\ \overleftarrow{\boldsymbol{m}}_i&=\overleftarrow{\boldsymbol{m}}_{i,p}=(m_p,\dots,m_i),\\ \boldsymbol{m}_+&=(m_1,\dots,m_p+1),\\ \boldsymbol{m}_-&=(m_1,\dots,m_p-1)\quad\text{if } m_p>1.\end{aligned}$$

The Hoffman dual of a composition $\boldsymbol{m}=(m_1,\dots,m_p)$ is $\boldsymbol{m}^\vee=(m'_1,\dots,m'_{p'})$ determined by $|\boldsymbol{m}|:=m_1+\cdots+m_p=m'_1+\cdots+m'_{p'}$ and

$$\{1,2,\dots,|\boldsymbol{m}|-1\} =\Big\{\sum_{1\le i\le j}m_i\Big\}_{j=1}^{p-1}\coprod\Big\{\sum_{1\le i\le j}m'_i\Big\}_{j=1}^{p'-1}.$$

Equivalently, $\boldsymbol{m}^\vee$ can be obtained from $\boldsymbol{m}$ by swapping the commas "," and the plus signs "+" in the expression

$$\boldsymbol{m}=(\underbrace{1+\cdots+1}_{m_1\text{ times}},\dots,\underbrace{1+\cdots+1}_{m_p\text{ times}}).$$

For example, we have $(1,1,2,1)^\vee=(3,2)$ and $(1,2,1,1)^\vee=(2,3)$. More generally, we have

$$\boldsymbol{m}^\vee=(\underbrace{1,\dots,1}_{m_1}+\underbrace{1,\dots,1}_{m_2}+1,\dots,1+\underbrace{1,\dots,1}_{m_p}). \quad (10)$$

Put $\mathtt{x}_1=dt/(1-t)$ and $\mathtt{x}_0=dt/t$. Concerning this duality, from Eq. (2.8) of [48], we have the iterated integral expression

$$\zeta_n^\star(\boldsymbol{m}^\vee;x)-\sum_{j=1}^p(-1)^{p-j}\zeta_n^\star(\overrightarrow{\boldsymbol{m}}_j^\vee)\mathrm{Li}_{\overleftarrow{\boldsymbol{m}}_{j+1}}(1-x) =n(-1)^p\int_x^1\mathtt{x}_1^{m_p-1}\mathtt{x}_0\cdots\mathtt{x}_1^{m_1}\mathtt{x}_0t^{n-1}dt. \quad (11)$$

To state our result, we need the following lemmas. To save space, we set

$$\begin{aligned}\mathtt{x}_\xi&:=\frac{dt}{\xi-t}\quad(\xi\ne0),\\ \mathtt{y}&:=\mathtt{x}_{-i}+\mathtt{x}_i-\mathtt{x}_{-1}-\mathtt{x}_1,\\ \mathtt{z}&:=-\mathtt{x}_0-\mathtt{x}_{-i}-\mathtt{x}_i,\\ \omega_0&:=\frac{dt}{1-t^2},\quad \omega_1:=\frac{tdt}{1-t^2}.\end{aligned}$$

Lemma 2.1. (cf. Theorem 2.1 of [48]) *For any* $n, p \in \mathbb{N}$, $\boldsymbol{m} = (m_1, \dots, m_p) \in \mathbb{N}^p$ *and* $x \in [0, 1]$,

$$\begin{aligned} & n \int_0^x t^{n-1} dt \boldsymbol{x}_1 \boldsymbol{x}_0^{m_1-1} \cdots \boldsymbol{x}_1 \boldsymbol{x}_0^{m_p-1} \\ =& (-1)^p \zeta_n^\star(\boldsymbol{m}; x) \\ & - \sum_{j=1}^p (-1)^j \zeta_n^\star(\overrightarrow{\boldsymbol{m}}_{j-1}) \operatorname{Li}_{\overleftarrow{\boldsymbol{m}}_j}(x). \end{aligned} \tag{12}$$

Lemma 2.2. (cf. Theorem 3.6 of [48]) *For any* $n, p \in \mathbb{N}$, $\boldsymbol{m} = (m_1, \dots, m_p) \in \mathbb{N}^p$ *and* $x \in [0, 1]$,

$$\begin{aligned} & 2n \int_0^x t^{2n-1} dt \omega_0 \boldsymbol{x}_0^{m_1-1} \omega_1 \boldsymbol{x}_0^{m_2-1} \cdots \omega_1 \boldsymbol{x}_0^{m_p-1} \\ =& (-1)^p t_n^\star(\boldsymbol{m}; x) \\ & - \sum_{j=1}^p (-1)^j t_n^\star(\overrightarrow{\boldsymbol{m}}_{j-1}) \operatorname{ti}_{\overleftarrow{\boldsymbol{m}}_j}(x), \end{aligned} \tag{13}$$

where

$$t_n^\star(\boldsymbol{k}; x) := \sum_{n \geq n_1 \geq \cdots \geq n_r \geq 1} x^{2n_r - 1} \times \prod_{j=1}^r \frac{1}{(2n_j - 1)^{k_j}}.$$

Theorem 2.3. *For any* $k, p \in \mathbb{N}$, $\boldsymbol{m} = (m_1, \dots, m_p) \in \mathbb{N}^p$ *with* $m_p \geq 2$, *we have*

$$\begin{aligned} & \sum_{j=1}^p (-1)^j \zeta(\overleftarrow{\boldsymbol{m}}_j) \sum_{n=1}^\infty \frac{\zeta_n^\star(\overrightarrow{\boldsymbol{m}}_{j-1})}{n^k 4^n} \binom{2n}{n} \\ & \qquad - (-1)^p \sum_{n=1}^\infty \frac{\zeta_n^\star(\boldsymbol{m})}{n^k 4^n} \binom{2n}{n} \\ =& 2^{p+1} \sum_{\substack{\sigma_j \in \{\pm 1\}, \\ j = 1, 2, \dots |\widetilde{\boldsymbol{m}}|_p + k - 1}} \operatorname{Li}_{(k+1, \boldsymbol{m}_-)^\vee}(-1, \sigma_1, \sigma_1\sigma_2, \\ & \qquad \dots, \sigma_{|\widetilde{\boldsymbol{m}}|_p + k - 2} \sigma_{|\widetilde{\boldsymbol{m}}|_p + k - 1}) \in \mathsf{CMZV}^2_{|\boldsymbol{m}| + k}, \end{aligned}$$

where $|\widetilde{\boldsymbol{m}}|_j := m_1 + m_2 + \cdots + m_j - j$.

Proof. Multiplying (12) by $\frac{\binom{2n}{n}}{n^k 4^n}$, summing up, and using the well-known formula, [49], [50]

$$\begin{aligned} \sum_{n=1}^\infty \frac{\binom{2n}{n}}{n 4^n} x^n =& 2 \log \left(\frac{2}{1 + \sqrt{1-x}} \right) \\ =& \int_0^x \frac{dt}{h(t)} \quad \forall x \in [-1, 1), \end{aligned}$$

where $h(t) = 1 - t + \sqrt{1-t}$, we have

$$\begin{aligned} & (-1)^p \sum_{n=1}^\infty \frac{\zeta_n^\star(\boldsymbol{m}; x)}{n^k 4^n} \binom{2n}{n} \\ & - \sum_{j=1}^p (-1)^j \operatorname{Li}_{\overleftarrow{\boldsymbol{m}}_j}(x) \sum_{n=1}^\infty \frac{\zeta_n^\star(\overrightarrow{\boldsymbol{m}}_{j-1})}{n^k 4^n} \binom{2n}{n} \\ =& \int_0^x \frac{dt}{h(t)} \mathrm{x}_0^{k-1} \mathrm{x}_1 \mathrm{x}_0^{m_1-1} \cdots \mathrm{x}_1 \mathrm{x}_0^{m_p-1} \\ \overset{t \to 1-t}{=} & \int_{1-x}^1 \mathrm{x}_1^{m_p-1} \mathrm{x}_0 \cdots \mathrm{x}_1^{m_1-1} \mathrm{x}_0 \mathrm{x}_1^{k-1} \frac{dt}{t + \sqrt{t}} \\ \overset{t \to t^2}{=} & 2^{p+1} \int_{\sqrt{1-x}}^1 \left(\frac{2tdt}{1-t^2} \right)^{m_p - 1} \frac{dt}{t} \cdots \\ & \cdots \left(\frac{2tdt}{1-t^2} \right)^{m_1 - 1} \frac{dt}{t} \left(\frac{2tdt}{1-t^2} \right)^{k-1} \frac{dt}{1+t}. \end{aligned}$$

Letting $x = 1$ and noting the fact that

$$\frac{2tdt}{1-t^2} = \sum_{\sigma \in \{\pm 1\}} \frac{\sigma dt}{1 - \sigma t},$$

one can obtain the desired evaluation by applying (8). $\square$

Theorem 2.4. *For any* $k, p \in \mathbb{N}$, $\boldsymbol{m} = (m_1, \dots, m_p) \in \mathbb{N}^p$ *with* $m_p \geq 2$, *we have*

$$\sum_{j=1}^{p+1} (-1)^{j-1} t(\overleftarrow{\boldsymbol{m}}_j) \sum_{n=1}^\infty \frac{t_n^\star(\overrightarrow{\boldsymbol{m}}_{j-1})}{n^k 4^n} \binom{2n}{n} \in \mathsf{CMZV}^4_{|\boldsymbol{m}| + k}. \tag{14}$$

Proof. Multiplying (13) by $\frac{\binom{2n}{n}}{n^k 4^n}$, summing up, and noting the fact that

$$\sum_{n=1}^\infty \frac{\binom{2n}{n}}{n^k 4^n} x^{2n} = 2^k \int_0^x \frac{tdt}{g(t)} \left(\frac{dt}{t} \right)^{k-1} \tag{15}$$

where $g(t) = 1 - t^2 + \sqrt{1 - t^2}$, we have

$$\begin{aligned} & (-1)^p \sum_{n=1}^\infty \frac{t_n^\star(\boldsymbol{m}; x)}{n^k 4^n} \binom{2n}{n} \\ & - \sum_{j=1}^p (-1)^j \operatorname{ti}_{\overleftarrow{\boldsymbol{m}}_j}(x) \sum_{n=1}^\infty \frac{t_n^\star(\overrightarrow{\boldsymbol{m}}_{j-1})}{n^k 4^n} \binom{2n}{n} \\ =& 2^k \int_0^x \frac{tdt}{g(t)} \mathrm{x}_0^{k-1} \omega_0 \mathrm{x}_0^{m_1-1} \cdots \omega_1 \mathrm{x}_0^{m_p-1}. \end{aligned} \tag{16}$$

Applying $t \to \frac{1-t^2}{1+t^2}$, we get

$$\begin{aligned} \mathrm{x}_0 = \frac{dt}{t} & \to - \left(\frac{2tdt}{1+t^2} + \frac{2tdt}{1-t^2} \right) = \mathrm{y}, \\ \omega_0 = \frac{dt}{1-t^2} & \to - \frac{dt}{t} = -\mathrm{x}_0, \end{aligned}$$

$$\frac{tdt}{1-t^2+\sqrt{1-t^2}} \to -(\mathtt{x}_i+\mathtt{x}_{-i}-2\mathtt{x}_{-1}),$$
$$\omega_1=\frac{tdt}{1-t^2} \to -\left(\frac{dt}{t}-\frac{2tdt}{1+t^2}\right)=\mathtt{z}.$$

Hence, applying the above to (16) with $x=1$ yields

$$\sum_{j=1}^{p}(-1)^{j-1}t(\overleftarrow{\boldsymbol{m}}_j)\sum_{n=1}^{\infty}\frac{t_n^\star(\overrightarrow{\boldsymbol{m}}_{j-1})}{n^k4^n}\binom{2n}{n}$$
$$=2^k(-1)^{|\boldsymbol{m}|+k}\int_0^1 \mathtt{y}^{m_p-1}\mathtt{z}\cdots$$
$$\cdots\mathtt{y}^{m_2-1}\mathtt{z}\mathtt{y}^{m_1-1}\mathtt{x}_0\mathtt{y}^{k-1}(\mathtt{x}_i+\mathtt{x}_{-i}-2\mathtt{x}_{-1}).$$

Finally, the iterated integral expression (9) of CMZVs implies (14) immediately. □

Theorem 2.5. *For any $x\in[0,1]$, $k,p\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$ and $\boldsymbol{m}=(m_1,\dots,m_p)\in\mathbb{N}^p$, we have*

$$\sum_{n=1}^{\infty}\frac{\zeta_n^\star(\boldsymbol{m}^\vee;x)}{n^{k+2}4^n}\binom{2n}{n}$$
$$=\sum_{j=1}^{p}(-1)^{p-j}\operatorname{Li}_{\overleftarrow{\boldsymbol{m}}_{j+1}}(1-x)\sum_{n=1}^{\infty}\frac{\zeta_n^\star(\overrightarrow{\boldsymbol{m}}_j^\vee)}{n^{k+2}4^n}\binom{2n}{n}$$
$$+(-1)^{p+k}2\log(2)\operatorname{Li}_{(\overleftarrow{\boldsymbol{m}})_+,1_k}(1-x)$$
$$+\sum_{j=1}^{k}(-1)^{p+k-j}\operatorname{Li}_{\overleftarrow{\boldsymbol{m}}_+,1_{k-j}}(1-x)\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{n^{j+1}4^n}$$
$$+\frac{2^{|\boldsymbol{m}|+2-p}}{(-1)^{p+k}}\sum_{\substack{\sigma_j\in\{\pm1\},\\ j=1,2,\dots,p+k}}\operatorname{Li}_{(\overleftarrow{\boldsymbol{m}})_+,1_{k+1}}(\sigma_1\sqrt{1-x},$$
$$\sigma_1\sigma_2,\dots,\sigma_{p+k-1}\sigma_{p+k},-\sigma_{p+k}). \quad (17)$$

Proof. One can derive the result by applying (11) and using a similar argument as in the proof of Theorem 2.3. We leave the detail to the interested reader. □

Letting $k=0$ in (17) yields the following corollary.

Corollary 2.6. *For any $p\in\mathbb{N}_0$, $\boldsymbol{m}=(m_1,\dots,m_p)\in\mathbb{N}^p$ and $x\in[0,1]$, we have*

$$\sum_{n=1}^{\infty}\frac{\zeta_n^\star(\boldsymbol{m}^\vee;x)}{n^24^n}\binom{2n}{n}$$
$$=(-1)^p2\log(2)\operatorname{Li}_{m_p,\dots,m_2,m_1+1}(1-x)$$
$$+\sum_{j=1}^{p}(-1)^{p-j}\operatorname{Li}_{\overleftarrow{\boldsymbol{m}}_{j+1}}(1-x)\sum_{n=1}^{\infty}\frac{\zeta_n^\star(\overrightarrow{\boldsymbol{m}}_j^\vee)}{n^24^n}\binom{2n}{n}$$
$$+(-1)^p2^{|\boldsymbol{m}|+2-p}\sum_{\substack{\sigma_j\in\{\pm1\}\\ j=1,2,\dots,p}}\operatorname{Li}_{(\overleftarrow{\boldsymbol{m}})_+,1}(\sigma_1\sqrt{1-x},$$
$$\sigma_1\sigma_2,\dots,\sigma_{p-1}\sigma_p,-\sigma_p).$$

Example 2.7. Setting $p=1$ and $m_1=m$ gives

$$\sum_{n=1}^{\infty}\frac{\zeta_n^\star(m^\vee;x)}{n^24^n}\binom{2n}{n}$$
$$=\sum_{n=1}^{\infty}\frac{\zeta_n^\star(m^\vee)}{n^24^n}\binom{2n}{n}-2\log(2)\operatorname{Li}_{m+1}(1-x)$$
$$-2^{m+1}\operatorname{Li}_{m+1,1}(\sqrt{1-x},-1)$$
$$-2^{m+1}\operatorname{Li}_{m+1,1}(-\sqrt{1-x},1).$$

In particular, taking $x=0$ we get

$$\sum_{n=1}^{\infty}\frac{\zeta_n^\star(m^\vee)}{n^24^n}\binom{2n}{n}=2\log(2)\zeta(m+1)$$
$$+2^{m+1}\zeta(m+1,\bar{1})+2^{m+1}\zeta(\overline{m+1},1).$$

Remark 2.8. Having dealt with the case $k\geq 0$ in (17) we now provide a formula for $k=-1$. Replacing n by k in (11), summing both sides of it over $k=1,\dots,n$, then changing $m_1\to m_1-1$, we obtain

$$\zeta_n^\star(\boldsymbol{m}^\vee;x)-\sum_{j=1}^{p}(-1)^{p-j}\zeta_n^\star(\overrightarrow{\boldsymbol{m}}_j^\vee)\operatorname{Li}_{\overleftarrow{\boldsymbol{m}}_{j+1}}(1-x)$$
$$=(-1)^p\int_0^{1-x}\frac{1-(1-t)^n}{t}dt\,\mathtt{x}_0^{m_1-1}\mathtt{x}_1\mathtt{x}_0^{m_2-1}$$
$$\cdots\mathtt{x}_1\mathtt{x}_0^{m_p-1}, \quad (18)$$

where $m_j\geq 1$ with $m_1\geq 2$. Here we used the fact that $(1,\boldsymbol{m}^\vee)=(m_1+1,m_2,\dots,m_p)^\vee$. However, (18) still holds for $m_1=1$, the proof of which is left to the interested reader.

Corollary 2.9. *For any $p\in\mathbb{N}_0$, $\boldsymbol{m}=(m_1,\dots,m_p)\in\mathbb{N}^p$ and $x\in[0,1]$, we have*

$$\sum_{n=1}^{\infty}\frac{\zeta_n^\star(\boldsymbol{m}^\vee;x)}{n4^n}\binom{2n}{n}$$
$$=\sum_{j=1}^{p}(-1)^{p-j}\operatorname{Li}_{\overleftarrow{\boldsymbol{m}}_{j+1}}(1-x)\sum_{n=1}^{\infty}\frac{\zeta_n^\star(\overrightarrow{\boldsymbol{m}}_j^\vee)}{n4^n}\binom{2n}{n}$$
$$-(-1)^p2^{|\boldsymbol{m}|+2-p}\sum_{\substack{\sigma_j\in\{\pm1\}\\ j=1,2,\dots,p-1}}\operatorname{Li}_{\overleftarrow{\boldsymbol{m}}_+}(\sigma_1\sqrt{1-x},$$
$$\sigma_1\sigma_2,\dots,\sigma_{p-2}\sigma_{p-1},-\sigma_{p-1}).$$

Proof. Put $f(t)=\log(1+\sqrt{t})$ and $g(t)=\log(1+t)$ in this proof. Multiplying (18) by $\binom{2n}{n}/(4^nn)$, we have

$$\sum_{n=1}^{\infty}\frac{\zeta_n^\star(\boldsymbol{m}^\vee;x)}{n4^n}\binom{2n}{n}$$

$$
\begin{aligned}
&-\sum_{j=1}^{p}(-1)^{p-j}\,\mathrm{Li}_{\overleftarrow{\boldsymbol{m}}_{j+1}}(1-x)\sum_{n=1}^{\infty}\frac{\zeta_n^\star(\overrightarrow{\boldsymbol{m}}_j^\vee)}{n4^n}\binom{2n}{n}\\
&=(-1)^p 2\int_0^{1-x} f(t)\mathrm{x}_0^{m_1}\mathrm{x}_1\mathrm{x}_0^{m_2-1}\cdots\mathrm{x}_1\mathrm{x}_0^{m_p-1}\\
&\overset{t\to t^2}{=}(-1)^p 2^{|\boldsymbol{m}|+1}\int_0^{\sqrt{1-x}} g(t)\mathrm{x}_0^{m_1}\\
&\qquad\frac{tdt}{1-t^2}\mathrm{x}_0^{m_2-1}\cdots\frac{tdt}{1-t^2}\mathrm{x}_0^{m_p-1}\\
&=(-1)^p 2^{|\boldsymbol{m}|+2-p}\int_0^{\sqrt{1-x}} g(t)\mathrm{x}_0^{m_1}\\
&\qquad(\mathrm{x}_1+\mathrm{x}_{-1})\,\mathrm{x}_0^{m_2-1}\cdots(\mathrm{x}_1+\mathrm{x}_{-1})\,\mathrm{x}_0^{m_p-1}\\
&=(-1)^p 2^{|\boldsymbol{m}|+2-p}\sum_{\substack{\sigma_j\in\{\pm1\}\\ j=1,2,\dots,p-1}}\int_0^{\sqrt{1-x}} g(t)\mathrm{x}_0^{m_1}\\
&\qquad\frac{\sigma_{p-1}dt}{1-\sigma_{p-1}t}\mathrm{x}_0^{m_2-1}\cdots\frac{\sigma_1 dt}{1-\sigma_1 t}\mathrm{x}_0^{m_p-1}\\
&=(-1)^{p-1} 2^{|\boldsymbol{m}|+2-p}\sum_{\substack{\sigma_j\in\{\pm1\}\\ j=1,2,\dots,p-1}}\int_0^{\sqrt{1-x}} \mathrm{x}_{-1}\mathrm{x}_0^{m_1}\\
&\qquad\frac{\sigma_{p-1}dt}{1-\sigma_{p-1}t}\mathrm{x}_0^{m_2-1}\cdots\frac{\sigma_1 dt}{1-\sigma_1 t}\mathrm{x}_0^{m_p-1},
\end{aligned}
$$

where $\mathrm{x}_{-1}=dt/(-1-t)$. We have obtained the desired formula. $\square$

In particular, letting $p=1$ and $m_1=m$ in Corollary 2.9 yields

$$
\sum_{n=1}^{\infty}\frac{\zeta_n^\star(m^\vee;x)}{n4^n}\binom{2n}{n}=\sum_{n=1}^{\infty}\frac{\zeta_n^\star(m^\vee)}{n4^n}\binom{2n}{n}+2^{m+1}\,\mathrm{Li}_{m+1}(-\sqrt{1-x}).
$$

Noticing that $(m)^\vee=(1_m)$ by (10) and setting $x=0$ in the above equation we see that

$$
\sum_{n=1}^{\infty}\frac{\zeta_n^\star(1_m)}{n4^n}\binom{2n}{n}=-2^{m+1}\zeta(\overline{m+1}).
$$

Therefore we get

$$
\sum_{n=1}^{\infty}\frac{\zeta_n^\star(1_m;x)}{n4^n}\binom{2n}{n}=-2^{m+1}\zeta(\overline{m+1})+2^{m+1}\,\mathrm{Li}_{m+1}(-\sqrt{1-x}).
$$

3 Multiple Integrals Associated with 5-posets

Yamamoto first used a graphical representation to study the MZVs and MZSVs in [51]. In this section, we introduce the multiple integrals associated with 5-labeled posets, and use them to express some parametric Euler-Apéry type MZSVs.

Definition 3.1. A 5-*poset* is a pair (X,δ_X), where $X=(X,\leq)$ is a finite partially ordered set and the *label map* $\delta_X : X\to\{-2,-1,0,1,2\}$. We often omit δ_X and simply say "a 5-poset X".

Similar to 2-poset, a 5-poset (X,δ_X) is called *admissible* if $\delta_X(x)\neq 0$ for all maximal elements and $\delta_X(x)\neq 1,\pm2$ for all minimal elements $x\in X$.

Definition 3.2. For an admissible 5-poset X, we define the associated integral

$$
I_z(X)=\int_{\Delta_X}\prod_{x\in X}\mathrm{x}_{\delta_X(x)}(t_x), \tag{19}
$$

where for $z\in[0,1]$

$$
\Delta_X=\left\{(t_x)_x\in[0,z]^X \;\middle|\; t_x<t_y \text{ if } x>y\right\}
$$

and

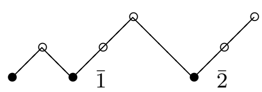
$$
\mathrm{x}_{-2}(t)=\frac{2tdt}{1-t^2},\quad \mathrm{x}_{-1}(t)=\frac{-dt}{1+t},
$$
$$
\mathrm{x}_0(t)=\frac{dt}{t},\quad \mathrm{x}_1(t)=\frac{dt}{1-t},\quad \mathrm{x}_2(t)=\frac{2dt}{1-t^2}.
$$

Clearly, $\mathrm{x}_{-2}=\mathrm{x}_1+\mathrm{x}_{-1}$ and $\mathrm{x}_2=\mathrm{x}_1-\mathrm{x}_{-1}$. Denote by $\emptyset$ the empty 5-poset and put $I_z(\emptyset):=1$.

Proposition 3.1. *For non-comparable elements a and b of a 5-poset X, X_a^b denotes the 5-poset that is obtained from X by adjoining the relation $a<b$. If X is an admissible 5-poset, then the 5-poset X_a^b and X_b^a are admissible and*

$$
I_z(X)=I_z(X_a^b)+I_z(X_b^a).
$$

Note that the admissibility of a 5-poset corresponds to the convergence of the associated integral. We use the Hasse diagrams to indicate 5-posets, with vertices $\circ$ and "$\bullet\ \sigma$" corresponding to $\delta(x)=0$ and $\delta(x)=\sigma$ ($\sigma\in\{\pm1\}$), respectively. For convenience, we replace "$\bullet$ 1" by $\bullet$ and replace "$\bullet\ -1$" (resp. "$\bullet\ -2$") by "$\bullet\ \bar{1}$" (resp. "$\bullet\ \bar{2}$"). For example, the diagram

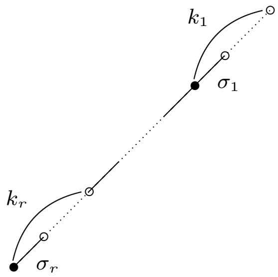


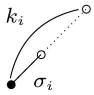
represents the 5-poset $X=\{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8\}$ with order $x_1<x_2>x_3<x_4<x_5>x_6<x_7<x_8$ and label $(\delta_X(x_1),\dots,\delta_X(x_8))=(1,0,-1,0,0,-2,0,0)$.

For composition $\boldsymbol{k} = (k_1, \ldots, k_r)$ and $\boldsymbol{\sigma} \in \{\pm 1, \pm 2\}^r$ (admissible or not), we write

 $(\boldsymbol{k},\boldsymbol{\sigma})$

for the "totally ordered" diagram:



If $k_i = 1$, we understand the notation 

as a single $\bullet\ \sigma_i$. We see from (8)

$I_z\Big($  $(\boldsymbol{k},\boldsymbol{\sigma}) \Big)$

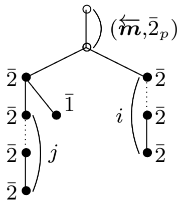
$$= \mathrm{Li}_{k_1,\ldots,k_r}(\sigma_1 z, \sigma_1\sigma_2, \ldots, \sigma_{r-1}\sigma_r), \quad (20)$$

where $(\sigma_1, \ldots, \sigma_r) \in \{\pm 1\}^r$.

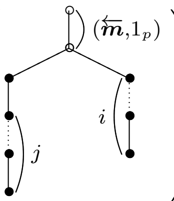
It is clear that all the multiple associated integral $I_z(\cdot)$ can be expressed in terms of the multiple polylogarithm function.

Theorem 3.2. *For nonnegative k and positive integers $m_1, m_2, \ldots, m_p$ and real $x \in [0, 1]$,*

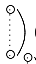
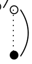
$$\sum_{n=1}^{\infty} \frac{\zeta_n^\star(\boldsymbol{m}^\vee; x)}{n^{k+2}4^n}\binom{2n}{n} - \sum_{j=1}^{p}(-1)^{p-j}\,\mathrm{Li}_{\overleftarrow{\boldsymbol{m}}_{j+1}}(1-x)\sum_{n=1}^{\infty}\frac{\zeta_n^\star(\overrightarrow{\boldsymbol{m}}_j^\vee)}{n^{k+2}4^n}\binom{2n}{n}$$
$$+ \sum_{\substack{i+j=k,\\ i\ge 1, j\ge 0}} (-1)^{p+i}\left\{\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{n^{j+2}4^n}\right\}\mathrm{Li}_{\overleftarrow{\boldsymbol{m}}_+,\{1\}_{i-1}}(1-x)$$
$$= c_1 \sum_{\substack{i+j=k,\\ i,j\ge 0}} (-1)^i I_{\sqrt{1-x}}$$



$$+ c_2 \sum_{\substack{i+j=k,\\ i,j\ge 0}} (-1)^i I_{1-x}$$



$$, \quad (21)$$

where $c_1 := (-1)^p 2^{|\boldsymbol{m}|+2-p}$, $c_2 := (-1)^p 2\log(2)$, and  $(\boldsymbol{m},\boldsymbol{\sigma})$ *represents the Hasse diagram obtained from*  $(\boldsymbol{m},\boldsymbol{\sigma})$ *by replacing its unique minimum $\bullet$ by $\circ$.*

Proof. From (15), by an elementary calculation, we deduce that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{n^{k+2}4^n}z^n \\
&=2\int_0^z \frac{\log\left(\frac{2}{1+\sqrt{1-t}}\right)}{t}dt\,\mathrm{x}_0 k \\
&=\frac{2}{k!}\int_0^z \frac{\log\left(\frac{2}{1+\sqrt{1-t}}\right)\log^k\left(\frac{z}{t}\right)}{t}dt \\
&=2\sum_{\substack{i+j=k,\\ i,j\ge 0}}\frac{\log^i(z)}{i!j!}\int_0^z\frac{b(t)\log^j\left(\frac{1}{t}\right)}{t}dt, \qquad (22)
\end{aligned}$$

where $b(t) = \log(2/(1+\sqrt{1-t}))$. Further, applying $t \to 1-t$ and using the special case $z = 1$ in (22) we get

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{n^{k+2}4^n}\Big(1-(1-z)^n\Big) &= 2\sum_{\substack{i+j=k,\\ i,j\ge 0}}(-1)^j \\
&\times \frac{\log^i(1-z)}{i!j!}\int_0^z\frac{\log\left(\frac{2}{1+\sqrt{t}}\right)\log^j(1-t)}{1-t}dt \\
&- \sum_{\substack{i+j=k,\\ i\ge 1, j\ge 0}}\left\{\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{n^{j+2}4^n}\right\}\frac{\log^i(1-z)}{i!}. \qquad (23)
\end{aligned}$$

Set $\mathrm{y}_1 = \frac{tdt}{1-t^2}$,

$$\mathbf{w}' = \frac{\log^i(1-t^2)}{t}dt\,\mathrm{x}_0^{m_1-1}\mathrm{y}_1\mathrm{x}_0^{m_2-1}\cdots\mathrm{y}_1\mathrm{x}_0^{m_p-1},$$
$$\mathbf{w} = \frac{\log^i(1-t)}{t}dt\mathrm{x}_0^{m_1-1}\,\mathrm{x}_1\mathrm{x}_0^{m_2-1}\cdots\mathrm{x}_1\mathrm{x}_0^{m_p-1}.$$

Multiplying (18) by $\frac{\binom{2n}{n}}{4^n n^{k+2}}$ and applying (23), we see that

$$\begin{aligned}
&(-1)^p\sum_{n=1}^{\infty}\frac{\zeta_n^\star(\boldsymbol{m}^\vee;x)}{n^{k+2}4^n}\binom{2n}{n} \\
&-\sum_{j=1}^{p}(-1)^j\,\mathrm{Li}_{\overleftarrow{\boldsymbol{m}}_{j+1}}(1-x)\sum_{n=1}^{\infty}\frac{\zeta_n^\star(\overrightarrow{\boldsymbol{m}}_j^\vee)}{n^{k+2}4^n}\binom{2n}{n} \\
&= -2^{|\boldsymbol{m}|+2}\sum_{\substack{i+j=k,\\ i,j\ge 0}}\frac{(-1)^j}{i!j!}
\end{aligned}$$

$$\times \int_0^{\sqrt{1-x}} \frac{\log(1+t)\log^j(1-t^2)}{1-t^2} t dt \sqcup\!\sqcup \mathbf{w}'$$
$$+ 2\log(2) \sum_{\substack{i+j=k,\\ i,j\geq 0}} \frac{(-1)^j}{i!j!} \int_0^{1-x} \frac{\log^j(1-t)}{1-t} dt \sqcup\!\sqcup \mathbf{w}$$
$$- \sum_{\substack{i+j=k,\\ i\geq 1, j\geq 0}} \left\{ \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{n^{j+2}4^n} \right\} \frac{(-1)^i}{i!} \int_0^{1-x} \mathbf{w}.$$

Here, $\sqcup\!\sqcup$ is the shuffle product of 1-forms as defined in [45]. Finally, according to the definition of multiple associated integral $I_z(\cdot)$ we obtain the desired evaluation using (20). □

We remark that letting $k = 0$ in (21) we can recover Corollary 2.6 again.

4 Concluding Remarks

In this paper, we have studied one particular variant of multiple zeta values. By introducing an extra binomial factor in the summands of multiple zeta star series, we can define the Euler-Apéry type multiple zeta star series. We have established several recurrence relations about these values and a parametric variant by the method of iterated integrals. In particular, the application of Yamamotos grahpic representation of the iterated integrals is proved to be a powerful tool. Using this mechinery, we can then find the explicit evaluations for some of these values.

As another application, we are able to define and study a parametric variant of the Euler-Apéry type multiple zeta star series. It is our intention and hope that these newly discovered series will play some roles in the computation of more complicated Feynman diagrams in quantum field physics. For related works, [2], [3], [10], [11], [12], [13].

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Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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