Matrix Transforms into a Set of α-absolutely A^{λ}-summable Sequences

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Abstract: - Suppose *A* is a matrix having real or complex entries and λ - a monotonically increasing strictly positive sequence, i.e., the speed. In this paper, the notions of λ-reversibility of *A*, A^{λ}-boundedness, and A^{λ}-summability of sequences are recalled, and the notion of α-absolute A^{λ}-summability of sequences is introduced. Also, there are characterized matrix transforms from the set of all A^{λ}-bounded, or the set of all A^{λ}-summable, or the set of all 1- absolute A^{λ}-summable sequences into the set of all α-absolutely ($\alpha > 1$) B^{μ}-summable sequences for a normal or λ-reversible matrix *A* and a matrix $B = (b_{nk})$ with $b_{nk} = 0$, $k > n$, and for another speed μ.

Key-Words: - Matrix transforms, λ-reversibility of matrices, boundedness with speed, convergence with speed, zero-convergence with speed, summability with speed, α-absolute summability with speed.

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1 Introduction

First, we note that the present paper is the continuation of [1]. Therefore, we strictly follow the notations and concepts of [1].

The speed of convergence of sequences can be treated in different ways, see, for example, [2], [3], [4], [5], [6], [7], [8]. We use, as in [1], the tools, introduced in [2] and [7].

However, in general, we do not expect the boundedness of λ, the most important case is $\lambda_k \neq O(1)$, because in this case relation

$$\lim_k x_k := s \text{ and } v_k := \lambda_k (x_k - s) \tag{1}$$

for a convergent sequence $x = (x_k)$ allows to evaluate the quality of convergence of converging sequences. Indeed, let x^1 and x^2 be two convergent sequences with the finite limit *s*. If $(v_k) \in c$ (or $v_k = O(1)$) for $x = x^1$, and $(v_k) \notin c$ (or $v_k \neq O(1)$)` for $x = x^2$, then the sequence x^1 converges "better" (more precisely, faster) than sequence x^2. Thus λ, in the case $\lambda_k \neq O(1)$, measures the speed of approaching to the limit *s* for the observed sequences.

Let $A = (a_{nk})$ be an arbitrary matrix with real or complex entries. Following [6] (see also [2]), a sequence $x = (x_k)$ is called A^{λ}-bounded (A^{λ}-summable), if $Ax \in l_{\infty}^{\lambda}$ (correspondingly $Ax \in c^{\lambda}$).

Definition 1. If $Ax \in l_{\alpha}^{\lambda}$, then we tell that a convergent sequence $x = (x_k)$ is α-absolutely A^{λ}-summable,

Let $(l_{\infty})_A^{\lambda}$ - the set of A^{λ}-bounded sequences, c_A^{λ} - the set of all A^{λ}-summable sequences, and $(l_{\alpha})_A^{\lambda}$ – the set of all α-absolutely A^{λ}-summable sequences. Let

$$(c_0)_A^{\lambda} := \left\{ x \in c_A^{\lambda} : Ax \in c_0^{\lambda} \right\},$$

$$c_A := \left\{ x \in \omega : Ax \in c \right\}.$$

It is easy to see that

$$(l_{\alpha})_A^{\lambda} \subset (c_0)_A^{\lambda} \subset c_A^{\lambda} \subset (l_{\infty})_A^{\lambda} \subset c_A,$$

and

$$(l_{\infty})_A^{\lambda} = c_A^{\lambda} = (c_0)_A^{\lambda} = c_A$$

a bounded λ.

Matrix transformations, boundedness and convergence with speed are widely used in approximation theory to transform non-convergent sequences into convergent ones or to transform convergent sequences into “better” convergent sequences, [6], [9], [10], [11].

In general, the problems of improvement of the quality of convergence of sequences by matrix transformations are considered in many works, [1], [12], [13], [14], [15], [16]. Moreover, the applications in theoretical physics can be found, for example, in [17] and [18].

Let further $\mu:=(\mu_k)$ be another speed, $B=(b_{nk})$ a lower triangular infinite matrix, and a matrix $A=(a_{nk})$ is normal or λ-reversible. We recall that A is normal, if $a_{nk}=0$ for $k>n$, and $a_{nn}\neq 0$ for each n, and λ-reversible, if the system of equations $z_n=A_nx$ has a unique solution for each sequence $(z_n)\in c^{\lambda}$. Matrix transforms from $X\to Y$, where X is one of the sets c^{λ}, c_0^{λ}, l_{∞}^{λ}, l_1^{λ}, c_A^{λ}, $(c_0)_A^{\lambda}$, $(l_{\infty})_A^{\lambda}$ or $(l_1)_A^{\lambda}$, and Y one of the sets c^{μ}, c_0^{μ}, l_{∞}^{μ}, l_{α}^{μ} ($\alpha>1$), c_B^{μ}, $(c_0)_B^{\mu}$, $(l_{\infty})_B^{\mu}$ or $(l_1)_B^{\mu}$ have been studied in [6], and in several works of the authors of the current paper, we mention only [1] and [2].

In this paper, we characterize the matrix transforms from c_A^{λ}, $(c_0)_A^{\lambda}$, $(l_{\infty})_A^{\lambda}$ and $(l_1)_A^{\lambda}$ into $(l_{\alpha})_B^{\mu}$ ($\alpha>1$).

2 Auxiliary Results

In this section, we present some results, which we use in next sections in the proofs of the main theorems of the present paper.

The following Lemmas 1 – 3 have been proved in [19] and [20].

Lemma 1. *For $A=(a_{nk})\in(c_0,c)$ it is necessary and sufficient that*

$$\exists\lim_m a_{nk}:=a_k \text{ (finite)} \tag{2}$$

$$\sup_n\sum_k|a_{nk}|<\infty. \tag{3}$$

Moreover,

$$\lim_n A_nx=\sum_k a_kx_k \tag{4}$$

for every $x=(x_k)\in c_0$.

Lemma 2. *For $A=(a_{nk})\in(c,c)$ it is necessary and sufficient that conditions* (2), (3) *hold, and*

$$\exists\,\tau \text{ such that } \lim_n\sum_k a_{nk}:=\tau. \tag{5}$$

Moreover, if $\lim_k x_k=s$ for $x=(x_k)\in c$, then

$$\lim_n A_nx=s\tau+\sum_k(x_k-s)a_k.$$

Lemma 3. *Next assertions are equivalent:*

(a) $A=(a_{nk})\in(l_{\infty},c)$.

(b) *Relations* (2), (3) *hold and*

$$\lim_n\sum_k|a_{nk}-a_k|=0. \tag{6}$$

(c) *Relation* (2) *is satisfied and*

$$\sum_k|a_{nk}|<\infty \text{ uniformly with respect to } n. \tag{7}$$

In addition, if one of the assertions (a)-(c) *holds, then* (4) *is satisfied for each $x=(x_k)\in l_{\infty}$.*

The following Lemmas 4 – 6 have been proved in [20].

Lemma 4. *For $A=(a_{nk})\in(l_1,c)$ it is necessary and sufficient that* (2) *holds, and*

$$\sup_{n,k}a_{nk}<\infty. \tag{8}$$

In addition, relation (4) *is valid for each $x=(x_k)\in l_1$.*

Lemma 5. *For $A=(a_{nk})\in(l_1,l_{\alpha})$ ($\alpha>1$) it is necessary and sufficient that*

$$\sup_k\sum_n|a_{nk}|^{\alpha}<\infty.$$

Lemma 6. *For $A=(a_{nk})\in(l_{\infty},l_{\alpha})=(c_0,l_{\alpha})$ ($\alpha>1$) it is necessary and sufficient that*

$$\sup_K\sum_n\left|\sum_{k\in K}a_{nk}\right|^{\alpha}<\infty$$

for all finite subsets K from N: = $\{0,1,2,...\}$, *or*

$$\sum_n\left|\sum_{k\in K^*}a_{nk}\right|^{\alpha}<\infty$$

for every $K^*\subset$N..

Let $A=(a_{nk})$ be a λ-reversible matrix. For $x\in c_A^{\lambda}$, let

$$\phi:=\lim_n A_nx,\ d_n:=\lambda_n(A_nx-\phi),\ d:=\lim_n d_n,$$

and $\eta:=(\eta_k)$, $\varphi:=(\varphi_k)$, (η_{kj}) (for every j) satisfy the system $y=Ax$ respectively to $y=(\delta_{nn})$, $y=(\delta_{nn}/\lambda_n)$ $y=y^j:=(\delta_{nj})$ (where $\delta_{nj}=1$ if $n=j$ and $\delta_{nj}=0$ if $n\neq j$).

Lemma 7 ([2], Corollary 9.1). *For a λ-reversible $A=(a_{nk})$, each member x_k of a sequence $x=(x_k)\in c_A^{\lambda}$ can be represented in the form*

$$x_k = \phi\eta_k + d\varphi_k + \sum_n \frac{\eta_{kn}}{\lambda_n}(d_n - d), \sum_n \left|\frac{\eta_{kn}}{\lambda_n}\right| < \infty. \quad (9)$$

Remark 1. If $x=(x_k)\in(c_0)_A^\lambda$, then $d=0$. Hence for every member x_k of $x=(x_k)\in(c_0)_A^\lambda$ we may write

$$x_k = \phi\eta_k + \sum_n \frac{\eta_{kn}}{\lambda_n}(d_n - d), \sum_n \left|\frac{\eta_{kn}}{\lambda_n}\right| < \infty. \quad (10)$$

3 The Set $\left((l_\infty)_A^\lambda, (l_\alpha)_B^\mu\right), \alpha > 1$

Let throughout in this section $A=(a_{nk})$ be a normal matrix with its inverse matrix $A^{-1}:=(\eta_{nk})$, $B=(b_{nk})$ a triangular matrix, and $M=(m_{nk})$ an arbitrary matrix

In the beginning, we find necessary and sufficient conditions for the existence of Mx on $(l_\infty)_A^\lambda$. In this case for all $x:=(x_k)\in(l_\infty)_A^\lambda$ we can write

$$\sum_{k=0}^j m_{nk}x_k = \sum_{k=0}^j m_{nk}\sum_{l=0}^k \eta_{kl}y_l = \sum_{l=0}^j h_{jl}^n y_l \quad y_l := A_l x,$$

where $H^n := \left(h_{jl}^n\right)$ is the lower triangular matrix for each *n*, defined by

$$h_{jl}^n := \sum_{k=l}^j m_{nk}\eta_{kl}, \quad l \le j.$$

Consequently, for the existence of Mx for each $x\in(l_\infty)_A^\lambda$ it is necessary and sufficient that $H^n\in(l_\infty^\lambda, c)$ for all *n*. Thus, we obtain ([2], Proposition 8.1)

Proposition 1. *For the existence of Mx on $(l_\infty)_A^\lambda$ it is necessary and sufficient that for every n,*

$$\exists \lim_j h_{jl}^n := h_{nl} \ (\textit{finite}), \quad (11)$$

$$\exists \lim_j \sum_{l=0}^j h_{jl}^n \ (\textit{finite}) \quad (12)$$

$$\sup_j \sum_l \frac{\left|h_{jl}^n\right|}{\lambda_l} < \infty, \quad (13)$$

$$\lim_j \sum_l \frac{\left|h_{jl}^n - h_{nl}\right|}{\lambda_l} = 0. \quad (14)$$

Moreover, instead of (13) *it is possible to set*

$$\sum_l \frac{\left|h_{nl}\right|}{\lambda_l} < \infty \quad (15)$$

for all n

Remark 2. With the help of Lemma 3 c) it is possible to prove that instead of (13) and (14) we can put

$$\sum_l \frac{\left|h_{jl}^n\right|}{\lambda_l} < \infty \text{ uniformly in } j \text{ for all } n. \quad (16)$$

Next, we need the matrix $G=(g_{nk})=BM$; i.e.,

$$g_{nk} := \sum_{l=0}^n b_{nl}m_{lk},$$

and lower triangular matrices $\Gamma^n := \left(\gamma_{nl}^j\right)$, where

$$\gamma_{nl}^j := \sum_{k=l}^j g_{nk}\eta_{kl}, \quad l \le j.$$

Theorem 1. *Let* $\lambda_n \ne O(1)$ *and* $\alpha > 1$. *For* $M\in\left((l_\infty)_A^\lambda, (l_\alpha)_B^\mu\right)$ *it is necessary and sufficient that* (11) – (14) *hold,*

$$\exists \lim_j \gamma_{nl} := \gamma_l \ (\textit{finite}), \quad (17)$$

$$\lim_n \sum_l \frac{\left|\gamma_{nl} - \gamma_l\right|}{\lambda_l} = 0, \quad (18)$$

$$\sup_n \sum_l \frac{\left|\gamma_{nl}\right|}{\lambda_l} < \infty, \quad (19)$$

$$\sup_K \sum_n \mu_n \left|\sum_{l\in K} \frac{\gamma_{nl} - \gamma_l}{\lambda_l}\right|^\alpha < \infty \quad (20)$$

for every finite $K \subset \mathbb{N}$, *where*

$$\gamma_{nl} := \lim_j \gamma_{nl}^j, \quad (21)$$

and

$$(\rho_n) \in l_\alpha^\mu, \quad \rho_n := \lim_j \sum_{l=0}^j \gamma_{nl}^j. \quad (22)$$

Moreover, instead of (19) *it is possible to set*

$$\sum_l \frac{\left|\gamma_l\right|}{\lambda_l} < \infty. \quad (23)$$

Proof. Necessity. Suppose that $M\in\left((l_\infty)_A^\lambda, (l_\alpha)_B^\mu\right)$. In this case Mx exists on $(l_\infty)_A^\lambda$. This implies that (11) - (14) hold by Proposition 1, and

$$B_n y = G_n x \quad (24)$$

on $(l_\infty)_A^\lambda$, since *B* is lower triangular.

Hence $G\in\left((l_\infty)_A^\lambda, l_\alpha^\mu\right)$ by (24) .

Moreover,

$$\sum_{k=0}^j g_{nk}x_k = \sum_{l=0}^j \gamma_{nl}^j A_l x \quad (25)$$

for all $x \in (l_\infty)_A^\lambda$. As A is normal, then we can find $x \in (l_\infty)_A^\lambda$ satisfying the equality $A_l x = e$. Consequently (22) holds by (25).

As A is normal, then for every bounded sequence (β_n) we can find $x \in (l_\infty)_A^\lambda$ satisfying the relations

$$\lim_n A_n x := \delta \text{ and } \beta_n = \lambda_n (A_n x - \delta). \tag{26}$$

As from (26) we have

$$A_n x = \delta + \frac{\beta_n}{\lambda_n},$$

then, using (25) and (26), we obtain for all $x \in (l_\infty)_A^\lambda$ that

$$\sum_{k=0}^{j} g_{nk} x_k = \delta \sum_{l=0}^{j} \gamma_{nl}^j + \sum_{l=0}^{j} \frac{\gamma_{nl}^j}{\lambda_l} \beta_l. \tag{27}$$

Since $G_n x$ converges on $x \in (l_\infty)_A^\lambda$, and there exist the finite limits ρ_n by (22), then $\Gamma_\lambda^n := (\gamma_{nl}^j / \lambda_l) \in (l_\infty, c)$ for all n. Consequently, using Lemma 3, we conclude from (27) that

$$G_n x = \delta \rho_n + \sum_l \frac{\gamma_{nl}}{\lambda_l} \beta_l \tag{28}$$

for every $x \in (l_\infty)_A^\lambda$. We note that the existence of the finite limits γ_{nl} (see (21)) follows from the existence of the transform Mx on $(l_\infty)_A^\lambda$. Moreover, the finite limit $\lim_n \rho_n := \rho$ exists by (22). Hence from (28), we can conclude that the matrix $\Gamma_\lambda := (\gamma_{nl} / \lambda_l) \in (l_\infty, c)$. This implies that conditions (17) – (19) hold and

$$\lim_n G_n x = \delta \rho + \sum_l \frac{\gamma_l}{\lambda_l} \beta_l \tag{29}$$

for every $x \in (l_\infty)_A^\lambda$ by Lemma 3. Writing

$$\mu_n (G_n x - \lim_n G_n x) = \delta \mu_n (\rho_n - \rho) + \mu_n \sum_l \frac{\gamma_{nl} - \gamma_l}{\lambda_l} \beta_l \tag{30}$$

for every $x \in (l_\infty)_A^\lambda$, we obtain with the help of (22) that $\Gamma_{\lambda,\mu} := (\mu_n (\gamma_{nl} - \gamma_l) / \lambda_l) \in (l_\infty, l_\alpha)$. Therefore condition (20) is satisfied by Lemma 6.

Finally, (23) holds by (18) and (19).

Sufficiency. Let relations (11) – (14) and (17) – (22) be fulfilled. In this case Mx exists on $(l_\infty)_A^\lambda$ by Proposition 1, and (24) – (27) are satisfied on $(l_\infty)_A^\lambda$. Similarly to the necessity part, with the help of (22) and Lemma 3, it is possible to show that (27) implies (28) for all $x \in (l_\infty)_A^\lambda$. As (22) holds and $\Gamma_\lambda \in (l_\infty, c)$ by Lemma 3 (due to the validity of (17) – (19)), then from (28) we can conclude (again by Lemma 3) that equation (29) is valid for every $x \in (l_\infty)_A^\lambda$. Hence relation (30) also is satisfied on $(l_\infty)_A^\lambda$, and $\Gamma_{\lambda,\mu} \in (l_\infty, l_\alpha)$ by Lemma 6. Therefore $M \in ((l_\infty)_A^\lambda, (l_\alpha)_B^\mu)$ due to (22).

Finally, instead of (19) we can put (23) since (19) holds by (18) and (23).

If $\sup_n \lambda_n < \infty$, then $\lim_n \beta_n = 0$ in (26). Therefore, in this case $\Gamma_\lambda^n \in (c_0, c)$ for every n. Thus, applying Lemma 1, we can immediately to formulate

Theorem 2. *Let* $\lambda_n = O(1)$ *and* $\alpha > 1$. *For* $M \in ((l_\infty)_A^\lambda, (l_\alpha)_B^\mu)$ *it is necessary and sufficient that* (11) – (14), (17) *and* (19), (20), (22) *hold.*

Remark 3. With the help of Lemma 3 c) we obtain that in Theorem 1, instead of (13), (14) we may set (16), and instead of (18) and (19) – the relation:

$$\sum_l \frac{|\gamma_{nl}|}{\lambda_l} < \infty \text{ uniformly with respect to } n.$$

Remark 4. It is possible to show with the help of Lemma 6 that instead of (20) in Theorems 1 and 2 we can put:

$$\sum_n \mu_n \left| \sum_{l \in K^*} \frac{\gamma_{nl} - \gamma_l}{\lambda_l} \right|^\alpha < \infty \ (\forall K^* \subset \mathrm{N}). \tag{31}$$

4 The Sets $(c_A^\lambda, (l_\alpha)_B^\mu), ((c_0)_A^\lambda, (l_\alpha)_B^\mu)$ and $((l_1)_A^\lambda, (l_\alpha)_B^\mu)$, $\alpha > 1$

Let throughout in this section $A = (a_{nk})$ be a λ-reversible matrix, $B = (b_{nk})$ a triangular matrix, and $M = (m_{nk})$ an arbitrary matrix

Proposition 2 ([2], Proposition 9.5). *For the existence of Mx on c_A^λ it is necessary and sufficient that* (11), (13) *hold, and for every n,*

$$\sum_k m_{nk}\eta_k < \infty, \tag{32}$$

$$\sum_k m_{nk}\varphi_k < \infty. \tag{33}$$

Proposition 3. *For the existence of Mx on $(c_0)_A^\lambda$ it is necessary and sufficient that* (11), (13) *and* (32) *hold.*

Proposition 4. *For the existence of Mx on $(l_1)_A^\lambda$ it is necessary and sufficient that* (11), (32) *hold, and*

$$\frac{h_{jl}^n}{\lambda_l} = O_n(1). \tag{34}$$

Propositions 3 and 4 can be proved in a similar fashion to Proposition 2 (presented in [2]); therefore, we omit the proofs of these propositions. Further, we apply matrices $\Gamma^n := \left(\gamma_{nl}^j\right)$, where

$$\gamma_{nl}^j := \sum_{k=l}^{j} g_{nk}\eta_{kl}.$$

If Mx exists on c_A^λ (on $(c_0)_A^\lambda$, or on $(l_1)_A^\lambda$), then finite limits (21) exist.

Theorem 3. *Let $\alpha > 1$. For $M \in \left(c_A^\lambda, (l_\alpha)_B^\mu\right)$ it is necessary and sufficient that* (11), (13), (17), (19), (20), (32), (33) *are satisfied, and*

$$\eta, \varphi \in \left(l_\alpha\right)_G^\mu. \tag{35}$$

Proof. Necessity. Suppose that $M \in \left(c_A^\lambda, (l_\alpha)_B^\mu\right)$. In this case Mx exists on c_A^λ. This implies that (11), (13), (32), and (33) are satisfied by Proposition 2, and relation (24) is valid on c_A^λ since *B* is lower triangular. Consequently $c_A^\lambda \subset \left(l_\alpha\right)_G^\mu$, and $G_n x$ converges on c_A^λ. This implies that condition (35) holds due to $\eta, \varphi \in c_A^\lambda$. As each member x_k of $x := (x_k) \in c_A^\lambda$ may be presented in the form (9) by Lemma 7, then, we have:

$$\sum_{k=0}^{j} g_{nk}x_k = \phi\sum_{k=0}^{j} g_{nk}\eta_k + d\sum_{k=0}^{j} g_{nk}\varphi_k + \sum_l \frac{\gamma_{nl}^j}{\lambda_l}(d_l - d) \tag{36}$$

for all $x \in c_A^\lambda$. As $G_n x$ converges on c_A^λ, then from (36) it follows, by (35), that the matrix Γ_λ^n for each *n* transforms this sequence $(d_l - d) \in c_0$ into *c*. We show that Γ_λ^n transforms every sequence $(\bar{d}_l) \in c_0$ into *c*. Indeed, for every sequence $(\bar{d}_l) \in c_0$ there exists a sequence $(d_l) \in c$ with $\bar{d}_l = d_l - d$, where $\lim_l d_l = d$. Then $\left((d_l - d)/\lambda_l\right) \in c_0$. Moreover, for this sequence, we can find $z = (z_l) \in c$ so that

$$\left(d_l - d\right)/\lambda_l = z_l - \phi, \;\; \phi := \lim_l z_l,$$

As *A* is λ-reversible, then for each $z = (z_l) \in c$ with $\phi := \lim_l z_l$, there exists a convergent sequence *x* with $z_l = A_l x$. So we showed that, for each $(\bar{d}_l) \in c_0$ we can find $x \in c_A^\lambda$ so that $\bar{d}_l = \lambda_l(A_l x - \phi)$. Hence $\Gamma_\lambda^n \in (c_0, c)$. Therefore (36) implies, by Lemma 1, that:

$$G_n x = \phi G_n \eta + d G_n \varphi + \sum_l \frac{\gamma_{nl}}{\lambda_l}(d_l - d) \tag{37}$$

for all $x \in c_A^\lambda$. From (37) we conclude by (35) that $\Gamma_\lambda \in (c_0, c)$. This implies by Lemma 1 that conditions (17), (19) hold and:

$$\lim_n G_n x = \phi \lim_n G_n \eta + d \lim_n G_n \varphi + \sum_l \frac{\gamma_l}{\lambda_l}(d_l - d) \tag{38}$$

for every $x \in c_A^\lambda$. Consequently we can write:

$$\mu_n\left(G_n x - \lim_n G_n x\right) = \phi\mu_n\left(G_n \eta - \lim_n G_n \eta\right) + d\mu_n\left(G_n \varphi - \lim_n G_n \varphi\right) + \mu_n \sum_l \frac{\gamma_{nl} - \gamma_l}{\lambda_l}(d_l - d) \tag{39}$$

for each $x \in c_A^\lambda$. From (39) it follows, by (35), that $\Gamma_{\lambda,\mu} \in (c_0, l_\alpha)$. Thus (20) holds by Lemma 6.

Sufficiency. Let relations (11), (13), (17), (19), (20), (32), (33) and (35) be fulfilled. In this case Mx exists on c_A^λ by Proposition 2. Hence (24), (36) are satisfied for every $x \in c_A^\lambda$ and $\Gamma_\lambda^n \in (c_0, c)$ for all *n*. Then it is possible to take the limit under the summation sign in the third summand of (36) by Lemma 1. Thus, using (35), from (36) we obtain that (37) holds on c_A^λ by Lemma 1. Moreover, $\Gamma_\lambda \in (c_0, c)$ by (17) and (19). Hence from (37) we have, by (35) and Lemma 1, that (38) holds for every $x \in c_A^\lambda$. Then relation (39) also is satisfied for every $x \in c_A^\lambda$, and $\Gamma_{\lambda,\mu} \in (c_0, l_\alpha)$ by Lemma 6 because condition (20) holds. Therefore $M \in \left(c_A^\lambda, (l_\alpha)_B^\mu\right)$ by (35).

If instead of c_A^λ to take $(c_0)_A^\lambda$ in Theorem 3, then $d=0$ in the proof of this theorem, and instead of Proposition 2 it is necessary to use Proposition 3. Therefore, we can immediately to formulate the following result.

Theorem 4. *Let* $\alpha>1$. *For* $M\in\left((c_0)_A^\lambda,(l_\alpha)_B^\mu\right)$ *it is necessary and sufficient that* (11), (13), (17), (19), (20), (32) *are satisfied, and* $\eta\in(l_\alpha)_G^\mu$.

Remark 5. As in Section 3. it is possible to show by Lemma 6 that condition (20) in Theorems 3 and 4 can be replaced by condition (31).

Theorem 5. *Let* $\alpha>1$. *For* $M\in\left((l_1)_A^\lambda,(l_\alpha)_B^\mu\right)$ *it is necessary and sufficient that* (11), (17), (32), (34) *are satisfied,* $\eta\in(l_1)_G^\mu$, and

$$\sup_{n,l}\left|\frac{\gamma_{nl}}{\lambda_l}\right|<\infty,$$

$$\sup_n\frac{1}{\lambda_l}\sum_n\mu_n\left|\gamma_{nl}-\gamma_l\right|^\alpha<\infty.$$

5 Conclusion

This paper is the continuation [1]. Let $\alpha>1$, and λ,μ be speeds of convergence; i.e., monotonically increasing strictly positive sequences In the current paper, we characterized the matrix transforms from the λ-boundedness domain of a normal matrix A (with real or complex entries), and from the λ-convergence or from the absolute λ-convergence domain of a λ-reversible matrix A into the α-absolute μ-convergence domain of a triangular matrix B (with real or complex entries), .

Further, we intend to generalize the results of this paper to abstract structures. For example, we will study matrix transforms over ultrametric spaces. Also, we try to apply our results in approximation theory, for example, for the estimation of approximation orders of Fourier expansions.

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