Confidence Intervals for the Difference between Two Coefficients of Variation of the delta-Birnbaum-Saunders Distributions

USANEE JANTHASUWAN, SA-AAT NIWITPONG, SUPARAT NIWITPONG*
Department of Applied Statistics, Faculty of Applied Science,
King Mongkut's University of Technology North Bangkok,
Bangkok 10800,
THAILAND

*Corresponding Author

Abstract: - The delta-Birnbaum-Saunders distribution is a relatively novel concept that combines the Birnbaum-Saunders with the binomial distributions. As a result, datasets containing both positive and zero values conform well with this distribution, making it particularly intriguing. Additionally, coefficients of variation are among the important statistics for comparing the dispersion of data. Therefore, there is an interest in proposing methods for constructing confidence intervals, which play a crucial role in statistical inference for the difference between two coefficients of variation in delta-Birnbaum-Saunders distributions. There are four methods proposed: the method of variance estimates recovery, the bootstrap confidence interval, the generalized confidence interval based on the variance stabilized transformation, and the generalized confidence interval based on the Wilson score method. All the methods are compared in terms of performance using coverage probability and average width through Monte Carlo simulations. The simulation results show that the bootstrap confidence interval performs similarly to the method of variance estimate recovery, except in cases where the shape parameter is large. In addition, it is shown that the generalized confidence interval based on variance stabilized transformation and the generalized confidence interval based on the Wilson score method yield similar results and demonstrate the highest efficiency compared to other methods. Finally, two datasets are used to illustrate the application of the proposed confidence intervals.

Key-Words: - Bootstrap confidence interval, Coefficients of variation, Confidence intervals, Delta-Birnbaum-Saunders distribution, generalized confidence interval, Method of variance estimates recovery.

Received: June 14, 2024. Revised: October 25, 2024. Accepted: November 16, 2024. Published: December 27, 2024.

1 Introduction

The original Birnbaum-Saunders distribution, a positively skewed distribution, has been widely studied for describing random variables. This distribution, also known as the fatigue-life distribution, was first proposed, [1]. It is a distribution that specifies the entire period until the accumulated damage from the expansion of a primary crack exceeds a set threshold, leading to material failure. Additionally, the Birnbaum-Saunders distribution has been applied in various fields such as engineering [2], air pollution [3], agriculture [4], finance [5], medicine [6], and environmental science [7], [8]. Naturally, most real-world data are skewed and may also include zero values. In data that include both positive and zero values, the Birnbaum-Saunders distribution may not be suitable. Thus, a new distribution called the delta-Birnbaum-Saunders distribution has been developed. It combines positive values derived from the Birnbaum-Saunders distribution with zero values, which is a binomial proportion. The concept of this combination distribution originated from research by [9]. Subsequently, the concept of incorporating zeros has been applied to other distributions, such as the log-normal distribution [10], and the gamma distribution [11].

The coefficient of variation can be calculated from the ratio of the standard deviation to the mean. The coefficient of variation is a statistical measure of the dispersion of data when different datasets have different units or significantly different means. Making it a useful tool in statistics and applications in various fields such as finance, quality control, science, meteorology, and risk assessment [12], [13], [14], [15]. When there are two independent populations, it is possible to extend the analysis to compare the dispersion between both populations. By estimating the confidence interval for the difference in coefficients of variation, if the confidence interval contains zero, there is no significant difference between them. In addition, there are a considerable number of researchers who are interested in studying the construction of confidence intervals for the difference between two coefficients of variation using various distributions, such as [16] applied the concept of the fiducial generalized confidence interval, the Bayesian methods, and the standard bootstrap to generate confidence intervals for the difference between two coefficients of variation for delta-lognormal distributions. Subsequently, [17] proposed four methods to create confidence intervals based on the Birnbaum-Saunders distribution's coefficients of variation and the difference between them. The methods they presented include the highest posterior density interval, the Bayesian credible interval, the generalized confidence interval, and the bootstrapped confidence interval. Recently, [18] examined confidence intervals for the difference between coefficients of variation of two delta gamma distributions, and they presented four different methods: the method of variance of estimates recovery, the generalized confidence intervals based on fiducial inference, the parametric bootstrap, and the Box-Cox transformation.

A thorough review of the literature revealed that no researchers have studied or published any work on the statistical comparison of the difference between two delta-Birnbaum-Saunders coefficients of variation. Therefore, the objective of this study is to construct confidence intervals for the difference between two coefficients of variation in the Delta-Birnbaum-Saunders distributions, utilizing the method of variance estimates recovery and the bootstrap confidence interval. Additionally, the study utilizes the generalized confidence interval based on the variance-stabilized transformation and compares it with the generalized confidence interval based on the Wilson score method. The criteria for comparing the performance of all methods are based on the coverage probability and the average width. Furthermore, to illustrate the performance of these methods, we applied them to wind speed data from the Chachoengsao Agrometeorological Station and Lamphun Weather Observing Station in Thailand.

The remaining sections of the article are organized as follows: The second section presents the definition of the difference between two delta-Birnbaum-Saunders coefficients of variation. The third section examines the method that is employed for constructing confidence intervals concerning the difference between two delta-Birnbaum-Saunders coefficients of variation. The fourth section presents simulation studies to compare the coverage probabilities and average widths of the proposed methods. The fifth section applies the proposed methods to analyze real data on wind speed. In the final section, a summary of the study's conclusions is presented.

2 Preliminary

Consider a non-negative random sample represented as $D_{ij} = \left(D_{i1}, D_{i2}, \ldots, D_{im_i}\right)$; where $i = 1, 2$ and $j = 1, 2, \ldots, m_i$ from two independent delta-Birnbaum-Saunders distributions with the proportion of zero (γ_i), shape parameter (α_i), and scale parameter (β_i), denoted as $D_{ij} \sim DBS(\gamma_i, \alpha_i, \beta_i)$. This random sample includes both zero and positive observed values. The occurrences with zero observations adhere to the binomial distribution, whereas the occurrences with positive observations conform to the Birnbaum-Saunders distribution. The numbers of zero and positive observations are denoted as $m_{i(0)}$ and $m_{i(1)}$, respectively, where $m_i = m_{i(0)} + m_{i(1)}$. Then, the probability density function of D_{ij} is as follows:

$$f\left(d_{ij};\gamma_i,\alpha_i,\beta_i\right) = \gamma_i \mathrm{I}_0\left[d_{ij}\right] + \left(1-\gamma_i\right)\frac{1}{2\alpha_i\beta_i\sqrt{2\pi}}\left[\left(\frac{\beta_i}{d_{ij}}\right)^{1/2} + \left(\frac{\beta_i}{d_{ij}}\right)^{3/2}\right] \times \exp\left[-\frac{1}{2}\left(\frac{d_{ij}}{\beta_i} + \frac{\beta_i}{d_{ij}} - 2\right)\right]\mathrm{I}_{(0,\infty)}\left[d_{ij}\right],$$

if $d_{ij} = 0$, then $\mathrm{I}_0\left[d_{ij}\right] = 1$ and $\mathrm{I}_{(0,\infty)}\left[d_{ij}\right] = 0$, and if $d_{ij} > 0$, then $\mathrm{I}_{(0,\infty)}\left[d_{ij}\right] = 1$. The distribution function of D_{ij} can be written as:

$$G\left(d_{ij};\gamma_i,\alpha_i,\beta_i\right) = \begin{cases} \gamma_i & ; d_{ij} = 0 \\ \gamma_i + \left(1-\gamma_i\right)F\left(d_{ij};\alpha_i,\beta_i\right) & ; d_{ij} > 0 \end{cases}, \quad (1)$$

where $F\left(d_{ij};\alpha_i,\beta_i\right)$ is the Birnbaum-Saunders distribution function. The mean (ω_i), variance (ν_i), and coefficient of variation (ζ_i) for D_{ij} can be calculated as follows, using the idea provided by [9]:

$$\omega_i = (1-\gamma_i)\beta_i\left(1+\frac{\alpha_i^2}{2}\right),$$

$$\nu_i = (1-\gamma_i)(\alpha_i\beta_i)^2\left(1+\frac{5\alpha_i^2}{4}\right)+\gamma_i(1-\gamma_i)\beta_i^2\left(1+\frac{\alpha_i^2}{2}\right)^2,$$

and

$$\zeta_i = \frac{\sqrt{\omega_i}}{\nu_i} = \frac{1}{2+\alpha_i^2}\sqrt{\frac{\alpha_i^2\left(4+5\alpha_i^2\right)+\gamma_i\left(2+\alpha_i^2\right)^2}{1-\gamma_i}}. \quad (2)$$

Consequently, the difference between two coefficients of variation of the delta-Birnbaum-Saunders distributions can be expressed as:

$$\begin{aligned}\Psi &= \zeta_1-\zeta_2\\ &= \frac{1}{2+\alpha_1^2}\sqrt{\frac{\alpha_1^2\left(4+5\alpha_1^2\right)+\gamma_1\left(2+\alpha_1^2\right)^2}{1-\gamma_1}} \quad (3)\\ &\quad -\frac{1}{2+\alpha_2^2}\sqrt{\frac{\alpha_2^2\left(4+5\alpha_2^2\right)+\gamma_2\left(2+\alpha_2^2\right)^2}{1-\gamma_2}}\end{aligned}$$

For the method of constructing confidence intervals for Ψ, we will present it in the following section.

3 Proposed Methods

3.1 Method of Variance Estimates Recovery (MOVER)

The MOVER method was implemented by [19] to present a closed-form mathematical approximation for the confidence interval of parameter differences $\zeta_1-\zeta_2$. In accordance with the research of [20], the asymptotic joint distribution of $\hat{\alpha}_i$ and $\hat{\beta}_i$ follows a bivariate normal, which is shown by

$$\begin{pmatrix}\hat{\alpha}_i\\ \hat{\beta}_i\end{pmatrix} \sim N\left(\begin{pmatrix}\alpha_i\\ \beta_i\end{pmatrix}, \begin{pmatrix}\frac{\alpha_i^2}{2m_{i(1)}} & 0\\ 0 & \frac{(\alpha_i\beta_i)^2}{m_{i(1)}}\left(\frac{1+\frac{3}{4}\alpha_i^2}{\left(1+\frac{1}{2}\alpha_i^2\right)^2}\right)\end{pmatrix}\right).$$

Note that the modified moment estimators of α_i and β_i are asymptotically independent, which is denoted as:

$$\hat{\alpha}_i = \left\{2\left[\left(\bar{d}_i\sum_{j=1}^{m_{i(1)}}\frac{d_{ij}^{-1}}{m_{i(1)}}\right)^{1/2}-1\right]\right\}^{1/2}$$

and

$$\hat{\beta}_i = \left\{\bar{d}_i\left(\sum_{j=1}^{m_{i(1)}}\frac{d_{ij}^{-1}}{m_{i(1)}}\right)^{-1}\right\}^{1/2},$$

where $\bar{d}_i = \sum_{j=1}^{m_{i(1)}}\frac{d_{ij}}{m_{i(1)}}$. The asymptotic distribution of $\hat{\gamma}_i$ is calculated by applying the delta method, which is given by

$$\sqrt{m_i}\left(\hat{\gamma}_i-\gamma_i\right)\xrightarrow{D} N\left(0,\gamma_i\left(1-\gamma_i\right)\right),$$

where $\hat{\gamma}_i = m_{i(0)}/m_i$ represents the maximum likelihood estimate of γ_i.

Now, the estimated value of ζ_i can be expressed as:

$$\hat{\zeta}_i = \frac{1}{2+\hat{\alpha}_i^2}\sqrt{\frac{\hat{\alpha}_i^2\left(4+5\hat{\alpha}_i^2\right)+\hat{\gamma}_i\left(2+\hat{\alpha}_i^2\right)^2}{1-\hat{\gamma}_i}}.$$

Subsequently, we applied the delta method to calculate the asymptotic variance of the estimator ζ_i, expressed by the Taylor series as:

$$\begin{aligned} g(\hat{\alpha}_i,\hat{\gamma}_i) &\approx g(\alpha_i,\gamma_i)+\frac{\partial g(\alpha_i,\gamma_i)}{\partial\alpha_i}(\hat{\alpha}_i-\alpha_i)\\ &\quad +\frac{\partial g(\alpha_i,\gamma_i)}{\partial\gamma_i}(\hat{\gamma}_i-\gamma_i)\\ &\approx \frac{1}{2+\alpha_i^2}\sqrt{\frac{\alpha_i^2\left(4+5\alpha_i^2\right)+\gamma_i\left(2+\alpha_i^2\right)^2}{1-\gamma_i}}\\ &\quad +\frac{8\alpha_i\left(1+2\alpha_i^2\right)}{\left(2+\alpha_i^2\right)^2\sqrt{(1-\gamma_i)\left[\alpha_i^2\left(4+5\alpha_i^2\right)+\gamma_i\left(2+\alpha_i^2\right)^2\right]}}(\hat{\alpha}_i-\alpha_i)\\ &\quad +\frac{2+\alpha_i^2\left(4+3\alpha_i^2\right)}{\left(2+\alpha_i^2\right)\sqrt{(1-\gamma_i)^3\left[\alpha_i^2\left(4+5\alpha_i^2\right)+\gamma_i\left(2+\alpha_i^2\right)^2\right]}}(\hat{\gamma}_i-\gamma_i).\end{aligned}$$

Remind that

$$\zeta_i = g(\alpha_i,\gamma_i) = \frac{1}{2+\alpha_i^2}\sqrt{\frac{\alpha_i^2\left(4+5\alpha_i^2\right)+\gamma_i\left(2+\alpha_i^2\right)^2}{1-\gamma_i}}.$$

Hence, the asymptotic variance of estimator ζ_i, is obtained as

$$V\left(\hat{\zeta}_i\right)=V\left[g\left(\hat{\alpha}_i,\hat{\gamma}_i\right)\right]$$
$$\approx\frac{1}{\Lambda_i}\left[\frac{32\alpha_i^4\left(1+2\alpha_i^2\right)^2}{m_{i(1)}\left(2+\alpha_i^2\right)^2}+\frac{\gamma_i\left[2+\alpha_i^2\left(4+3\alpha_i^2\right)\right]^2}{m_i\left(1-\gamma_i\right)}\right].$$

where
$$\Lambda_i=\left(2+\alpha_i^2\right)^2\left(1-\gamma_i\right)\left[\alpha_i^2\left(4+5\alpha_i^2\right)+\gamma_i\left(2+\alpha_i^2\right)^2\right].$$

At this point, the values of α_i and γ_i are unknown parameters. As a result, the plug-in estimators of $V\left(\hat{\zeta}_i\right)$ are implemented, which are represented as:

$$\hat{V}\left(\hat{\zeta}_i\right)\approx\frac{1}{\Lambda_i^{\#}}\left\{\begin{array}{l}\dfrac{32\hat{\alpha}_i^4\left(1+2\hat{\alpha}_i^2\right)^2}{m_{i(1)}\left(2+\hat{\alpha}_i^2\right)^2}\\+\dfrac{\hat{\gamma}_i\left[2+\hat{\alpha}_i^2\left(4+3\hat{\alpha}_i^2\right)\right]^2}{m_i\left(1-\hat{\gamma}_i\right)}\end{array}\right\},\quad(4)$$

where
$$\Lambda_i^{\#}=\left(2+\hat{\alpha}_i^2\right)^2\left(1-\hat{\gamma}_i\right)\left[\hat{\alpha}_i^2\left(4+5\hat{\alpha}_i^2\right)+\hat{\gamma}_i\left(2+\hat{\alpha}_i^2\right)^2\right].$$

Now, the $(1-\eta)100\%$ asymptotic confidence interval for $\hat{\zeta}_i$ can be written as:

$$\left[l_i,u_i\right]=\left[\hat{\zeta}_i-z_{1-\eta/2}\sqrt{\hat{V}\left(\hat{\zeta}_i\right)},\hat{\zeta}_i+z_{1-\eta/2}\sqrt{\hat{V}\left(\hat{\zeta}_i\right)}\right],\quad(5)$$

where l_i and u_i are the lower and upper limits of the interval for $\hat{\zeta}_i$; $i=1,2$, respectively. Therefore, the $(1-\eta)100\%$ confidence interval for Ψ employing the MOVER method is given by

$$CI_{MOVER}^{\Psi}=\left[L_{MOVER}^{\Psi},U_{MOVER}^{\Psi}\right],\quad(6)$$

where $L_{MOVER}^{\Psi}=\left(\hat{\zeta}_1-\hat{\zeta}_2\right)-\sqrt{\left(\hat{\zeta}_1-l_1\right)^2+\left(u_2-\hat{\zeta}_2\right)^2}$

and $U_{MOVER}^{\Psi}=\left(\hat{\zeta}_1-\hat{\zeta}_2\right)+\sqrt{\left(u_1-\hat{\zeta}_1\right)^2+\left(\hat{\zeta}_2-l_2\right)^2}$.

Algorithm 1: For the MOVER method

Step 1: Calculate $\hat{\alpha}_i$ and $\hat{\gamma}_i$.

Step 2: Calculate $\hat{\zeta}_i$.

Step 3: Calculate $\hat{V}\left(\hat{\zeta}_i\right)$ using equation (4).

Step 4: Calculate l_i and u_i using equation (5).

Step 5: Calculate $(1-\eta)100\%$ CI for Ψ using equation (6).

3.2 Bootstrap Confidence Interval (BCI)

As proposed by [21], the bootstrap method is a resampling approach used to estimate the sampling distribution of a statistic by repeatedly resampling data. Later, [22] showed that the constant-bias-correcting parametric bootstrap method for the Birnbaum-Saunders distribution was the most effective in reducing bias. Thus, to construct a confidence interval for the difference between two coefficients of variation, the constant-bias-correcting parametric bootstrap is used. A bootstrap sample, represented by d_{ij}^*, where $i=1,2$ and $j=1,2,\ldots m_i$ is a sample of size m_i drawn with a replacement from the original sample. The maximum likelihood estimates of α_i can be calculated by maximizing the log-likelihood function using the Broyden-Fletcher-Goldfarb-Shanno quasi-Newton nonlinear optimization algorithm. Suppose that R bootstrap samples are available. In the bootstrap sample, the corresponding bootstraps $\hat{\alpha}_i$ and $\hat{\gamma}_i$ are represented by $\tilde{\alpha}_i$ and $\tilde{\gamma}_i$, respectively. The computation of $\tilde{\alpha}_i$ has utilized the Broyden-Fletcher-Goldfarb-Shanno quasi-Newton nonlinear optimization algorithm. Assume that $r\left(\hat{\alpha}_i,\alpha_i\right)$ is the bias of estimator $\hat{\alpha}_i$ such that $r\left(\hat{\alpha}_i,\alpha_i\right)=E\left(\hat{\alpha}_i\right)-\alpha_i$. Thus, the estimator for the bias can be expressed as

$$\hat{r}\left(\hat{\alpha}_i,\alpha_i\right)=\tilde{\alpha}_i^{\#}-\alpha_i,$$

where $\tilde{\alpha}_i^{\#}=1/R\sum_{k=1}^{R}\tilde{\alpha}_{ik}$. After that, applying the constant-bias-correcting estimates as defined by [23], the bias-corrected estimator is calculated as

$$\tilde{\alpha}_{ik}^*=\tilde{\alpha}_{ik}-2\hat{r}\left(\hat{\alpha}_i,\alpha_i\right),\quad k=1,2,\ldots,R.\quad(7)$$

According to the research by [24], they introduced a Jeffreys interval for the binomial proportion, applying Jeffreys prior, which can be expressed as

$$\hat{\gamma}_i^*\sim Beta\left(m_i\tilde{\gamma}_i+\frac{1}{2},\ m_i\left(1-\tilde{\gamma}_i\right)+\frac{1}{2}\right).\quad(8)$$

Now, the bootstrap estimator of ζ_i, can be written as

$$\hat{\zeta}_i^{Boot}=\lambda_i^*\sqrt{\frac{\left(\tilde{\alpha}_{ik}^*\right)^2\left(4+5\left(\tilde{\alpha}_{ik}^*\right)^2\right)+\hat{\gamma}_i^*\left(2+\left(\tilde{\alpha}_{ik}^*\right)^2\right)^2}{1-\hat{\gamma}_i^*}},\quad(9)$$

where $\lambda_i^*=\dfrac{1}{2+\left(\tilde{\alpha}_{ik}^*\right)^2}$. From equation (9), the bootstrap estimator of the coefficient of variation difference can be calculated as

$$\hat{\Psi}^{Boot}=\hat{\zeta}_1^{Boot}-\hat{\zeta}_2^{Boot}.\quad(10)$$

Consequently, the $(1-\eta)100\%$ confidence interval for Ψ employing the BCI method is given by

$$\left[L_{BCI}^{\Psi}, U_{BCI}^{\Psi}\right] = \left[\hat{\Psi}^{Boot}(\eta/2), \hat{\Psi}^{(Boot)}(1-\eta/2)\right], \quad (11)$$

where $\hat{\Psi}^{Boot}(\eta)$ as the $100\eta th$ percentile of $\hat{\Psi}^{Boot}$.

Algorithm 2: For the BCI method

Step 1: At the bth step
- i) Calculate $d_{i1}^*, d_{i2}^*, \ldots, d_{im_i}^*$ with replacement from $d_{i1}, d_{i2}, \ldots, d_{im_i}$.
- ii) Calculate $\tilde{\alpha}_i$ and $\hat{r}(\hat{\alpha}_i, \alpha_i)$.
- iii) Calculate $\tilde{\alpha}_{ik}^*$ and $\hat{\gamma}_i^*$ using equations (7) and (8), respectively.
- iv) Calculate $\hat{\zeta}_i^{Boot}$ and $\hat{\Psi}^{Boot}$ using equations (9) and (10), respectively.

Step 2: Repeat step 1, a number of times, B=500.

Step 3: Calculate $(1-\eta)100\%$ CI for Ψ using equation (11).

3.3 Generalized Confidence Interval (GCI)

The method known as the generalized confidence interval was first proposed by [25]. It operates on the principle of a generalized pivotal quantity (GPQ) and is utilized to construct confidence intervals. Here, we aim to construct a confidence interval for Ψ using GCI, considering the GPQs of both β_i and α_i in the analysis, which was introduced by [26] and [27], respectively. Thus, the GPQ of β_i is computed by applying:

$$Q_{\beta_i}(d_{ij};\Omega_i) = \begin{cases} \max(\beta_{i1}, \beta_{i2}) & ;\Omega_i \le 0 \\ \min(\beta_{i1}, \beta_{i2}) & ;\Omega_i > 0 \end{cases}, \quad (12)$$

where Ω_i has the t-distribution, which has $m_{i(1)}-1$ degrees of freedom. From equation (12), we have obtained a new algebraic rearrangement, and both solutions for β_i are expressed as β_{i1} and β_{i2}, which can be obtained by solving the following equation:

$$X_i\beta_i^2 - 2\Upsilon_i\beta_i + (m_{i(1)}-1)C_i^2 - \frac{1}{m_{i(1)}}C_i^{\#}\Omega_i^2 = 0, \quad (13)$$

where $X_i = (m_{i(1)}-1)A_i^2 - \frac{1}{m_{i(1)}}B_i\Omega_i^2$,

$\Upsilon_i = (m_{i(1)}-1)A_iC_i - (1-A_iC_i)\Omega_i^2$,

$A_i = \frac{1}{m_{i(1)}}\sum_{j=1}^{m_{i(1)}}\frac{1}{\sqrt{D_{ij}}}$, $B_i = \sum_{j=1}^{m_{i(1)}}\left(\frac{1}{\sqrt{D_{ij}}} - A_i\right)^2$,

$C_i = \frac{1}{m_{i(1)}}\sum_{j=1}^{m_{i(1)}}\sqrt{D_{ij}}$, and $C_i^{\#} = \sum_{j=1}^{m_{i(1)}}\left(\sqrt{D_{ij}} - C_i\right)^2$, while the GPQ of α_i can be computed using

$$Q_{\alpha_i}(d_{ij};\mathrm{H}_i,\Omega_i) = \sqrt{\frac{E_{i1} + E_{i2}Q_{\beta_i}^2 - 2m_{i(1)}Q_{\beta_i}}{Q_{\beta_i}\mathrm{H}_i}}, \quad (14)$$

where $E_{i1} = \sum_{j=1}^{m_{i(1)}}D_{ij}$, $E_{i2} = \sum_{j=1}^{m_{i(1)}}\frac{1}{D_{ij}}$, and H_i follows the Chi-squared distribution with $m_{i(1)}$ degrees of freedom.

In constructing the confidence interval for Ψ, in addition to considering β_i and α_i, another parameter that needs to be taken into account is γ_i. Two concepts are utilized for this purpose: the variance stabilized transformation (VST) and the Wilson score method (WS). Details on their usage will be explained in the following subsections.

3.3.1 GCI based on VST (G.VST)

[28] used the delta method to construct the VST. Later, [29] proposed an application of GPQ based on the VST to construct confidence intervals. Thus, the GPQ of γ_i is determined as

$$Q_{\gamma_i}^{VST} = \sin^2\left[\arcsin\sqrt{\hat{\gamma}_i} - \frac{V_i}{2\sqrt{m_i}}\right], \quad (15)$$

where $V_i = 2\sqrt{m_i}\left(\arcsin\sqrt{\hat{\gamma}_i} - \arcsin\sqrt{\gamma_i}\right) \sim N(0,1)$.

Since GPQs for Q_{α_i} and $Q_{\gamma_i}^{VST}$ do not depend on the unknown parameters, and the observed value of Q_i does not depend on the nuisance parameter, the pivotal quantity for ζ_i is given by

$$Q_{\zeta_i}^{VST} = \frac{1}{2+Q_{\alpha_i}^2}\sqrt{\frac{Q_{\alpha_i}^2\left(4+5Q_{\alpha_i}^2\right) + Q_{\gamma_i}^{VST}\left(2+Q_{\alpha_i}^2\right)^2}{1-Q_{\gamma_i}^{VST}}}. \quad (16)$$

Then, the GPQ based on the variance stabilized transformation for the coefficient of variation difference can be calculated as:

$$Q_{\Psi}^{VST} = Q_{\zeta_1}^{VST} - Q_{\zeta_2}^{VST}. \quad (17)$$

Consequently, the $(1-\eta)100\%$ confidence interval for Ψ employing the G.VST method can be created as follows:

$$\left[L_{G.VST}^{\Psi}, U_{G.VST}^{\Psi}\right] = \left[Q_{\Psi}^{VST}(\eta/2), Q_{\Psi}^{VST}(1-\eta/2)\right], \quad (18)$$

where $Q_{\Psi}^{VST}(\eta)$ as the $100\eta th$ percentile of Q_{Ψ}^{VST}.

3.3.2 GCI based on WS (G.WS)

According to the research by [30], the GPQ of γ_i is defined as:

$$Q_{\gamma_i}^{WS} = Q_i^* - \left[\frac{W_i}{m_i + W_i^2} \sqrt{m_{i(0)}\left(1 - \frac{m_{i(0)}}{m_i}\right) + \frac{W_i^2}{4}} \right], (19)$$

where $Q_i^* = \dfrac{m_{i(0)} + \left(W_i^2/2\right)}{m_i + W_i^2}$ and $W_i = \dfrac{m_{i(0)} - m_i\gamma_i}{\sqrt{m_i\gamma_i\left(1-\gamma_i\right)}}$.

From equations (14) and (19), the GPQ for ζ_i can be calculated as:

$$Q_{\zeta_i}^{WS} = \frac{1}{2 + Q_{\alpha_i}^2} \sqrt{\frac{Q_{\alpha_i}^2\left(4 + 5Q_{\alpha_i}^2\right) + Q_{\gamma_i}^{WS}\left(2 + Q_{\alpha_i}^2\right)^2}{1 - Q_{\gamma_i}^{WS}}}. (20)$$

Now, from equation (20), the GPQ based on Wilson score method for the coefficient of variation difference can be expressed as:

$$Q_{\Psi}^{WS} = Q_{\zeta_1}^{WS} - Q_{\zeta_2}^{WS}. \quad (21)$$

Finally, the $(1-\eta)100\%$ confidence interval for Ψ employing the G.WS method can be written as

$$\left[L_{G.WS}^{\Psi}, U_{G.WS}^{\Psi}\right] = \left[Q_{\Psi}^{WS}(\eta/2), Q_{\Psi}^{WS}(1-\eta/2)\right], (22)$$

where $Q_{\Psi}^{WS}(\eta)$ as the $100\eta th$ percentile of Q_{Ψ}^{WS}.

Algorithm 3: For the GCI method

Step 1: Calculate $A_i, B_i, C_i, C_i^{\#}, \mathrm{X}_i, \Upsilon_i, E_{i1}$ and E_{i2}, respectively.

Step 2: At the *g*th step

i) Generate $\Omega_i \sim t\left(m_{i(1)} - 1\right)$, and then calculate $Q_{\beta_i}\left(d_{ij}; \Omega_i\right)$ using equation (12).

ii) If $Q_{\beta_i}\left(d_{ij}; \Omega_i\right) < 0$, regenerate $\Omega_i \sim t\left(m_{i(1)} - 1\right)$.

iii) Generate $H_i \sim \chi^2_{m_{i(1)}}$, and then calculate $Q_{\alpha_i}\left(d_{ij}; \mathrm{H}_i, \Omega_i\right)$ using equation (14).

iv) F or the *G.VST* method, calculate $Q_{\gamma_i}^{VST}$, $Q_{\zeta_i}^{VST}$, and Q_{Ψ}^{VST} using equations (15), (16), and (17), respectively.

v) F or the *G.WS* method, calculate $Q_{\gamma_i}^{WS}$, $Q_{\zeta_i}^{WS}$, and Q_{Ψ}^{WS} using equations (19), (20), and (21), respectively.

Step 3: Repeat step 2, a number of times, *G*=1,000.

Algorithm 3: *Continued.*

Step 3: Repeat step 2, a number of times, *G*=1,000.

Step 4: Calculate $(1-\eta)100\%$ CI for Ψ employing the G.VST method using equation (18).

Step 5: Calculate $(1-\eta)100\%$ CI for Ψ employing the G.WS method using equation (22).

4 Simulation Studies

The results of the simulation study on the methods we proposed, MOVER, BCI, G.VST, and G.WS, are presented in this section. The performance of these methods is evaluated in terms of coverage probability and average width through Monte Carlo simulations using the R statistical software. A nominal confidence level of 0.95 was chosen for generating 3,000 replications, including 500 for the BCI and 1,000 for the GCI. Since β_i is the scale parameter, its value remains constant at β_i=1.0 without losing any general characteristics. The settings for shape parameters are (0.25, 0.25), (0.5, 0.5), (1.0, 1.0), and (2.0, 2.0). For the proportion of zero values, the configurations are (0.1, 0.1), (0.1,0.3), (0.1,0.5), (0.3,0.3), (0.3, 0.5), and (0.5,0.5). Additionally, for sample sizes, the settings are (30,30), (30,50), (30,100), (50,50), (50,100), and (100,100). The most efficient method for the specified scenario is the one with a coverage probability greater than or equal to 0.95, along with the narrowest average width. Moreover, we have prepared a flowchart detailing the process of studying the simulated scenario, as depicted in Figure 1. The results from the simulation study are presented in Table 1 (Appendix) and Figure 2, Figure 3, Figure 4.

From the results in Table 1 (Appendix) and Figure 2, it is generally observed that the MOVER method provides a coverage probability greater than the specified value of 0.95, except in scenarios where the shape parameters are (0.25, 0.25) with a proportion of zero equal to (0.1, 0.1) and when the shape parameters are (2.0, 2.0). In contrast, the BCI method tends to result in coverage probabilities lower than the specified level across almost all the scenarios considered. However, when considering the average width, the MOVER and BCI methods show similar results, except in scenarios where the shape parameters are (2.0, 2.0). In most cases, the MOVER method produces a slightly narrower average width compared to the BCI method. This suggests that the MOVER method demonstrates stronger performance in most scenarios, particularly when the shape parameters are moderate or small.

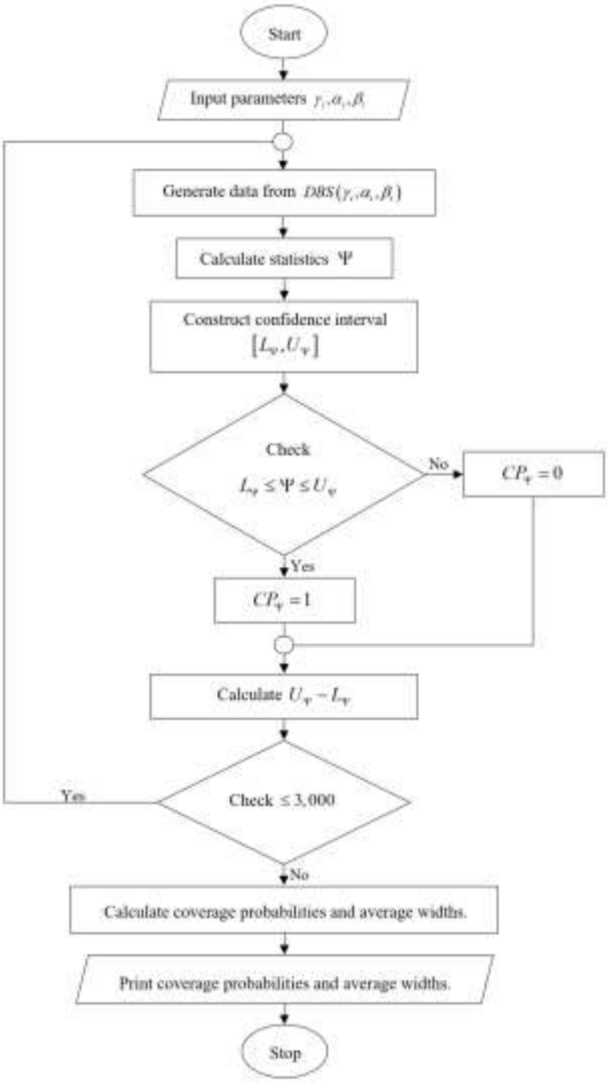

Fig. 1: A flowchart of the simulation study

For both the G.VST and G.WS methods, the coverage probabilities are close to 0.95 in almost all scenarios, and the average widths are narrower compared to the MOVER and BCI methods. This indicates that G.VST and G.WS not only maintain the desired coverage probability but also provide more efficient intervals with narrower average widths. Additionally, it is found that an increase in the shape parameter leads to a continuous increase in the average width of all methods. Subsequently, based on the results in Figure 3(C), it is evident that the coverage probability values in the context of the proportion of zero are relatively stable in each scenario. The G.VST and G.WS methods have values that are closer to 0.95 compared to the other methods. Examining Figure 3(D), it can be observed that an increase in the proportion of zero leads to an increase in the average width, indicating that a higher proportion of zero reduces the overall performance. When considering the results in Figure 4(E), it can be observed that the coverage probability values in the context of sample size for each method are quite similar. Furthermore, when examining Figure 4(F), it becomes clear that increasing the sample size results in a narrower average width for all methods, indicating that larger sample sizes lead to improved performance across all methods.

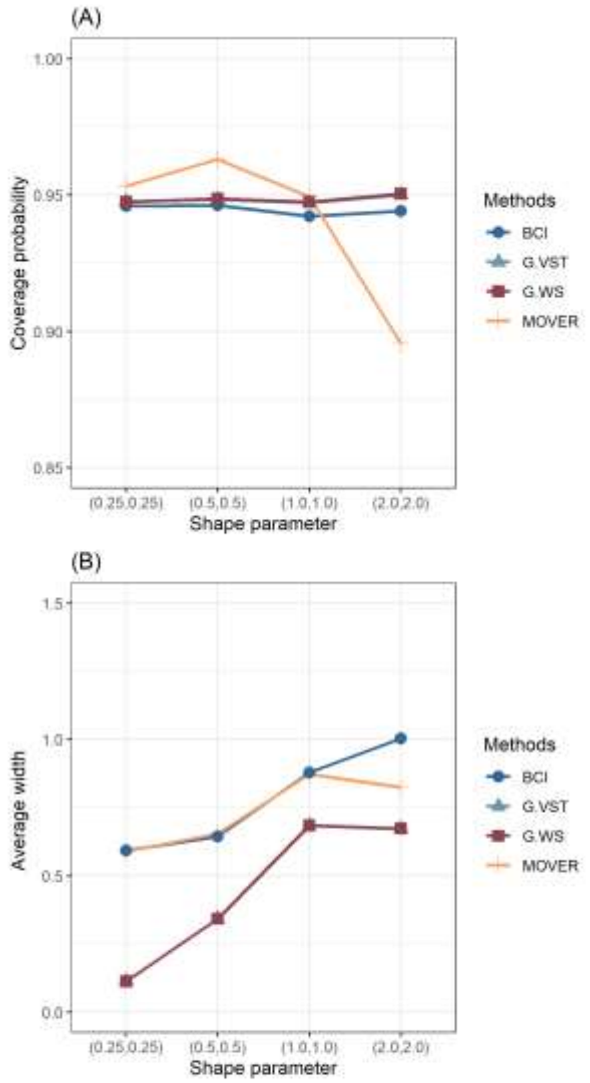

Fig. 2: Graphs compare the efficiency using (A) coverage probability and (B) average width relative to each shape parameter

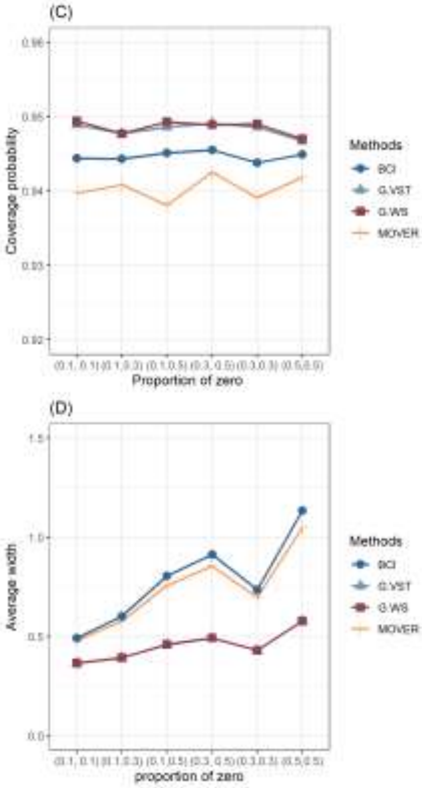

Fig. 3: Graphs compare the efficiency using (C) coverage probability and (D) average width relative to each proportion of zero

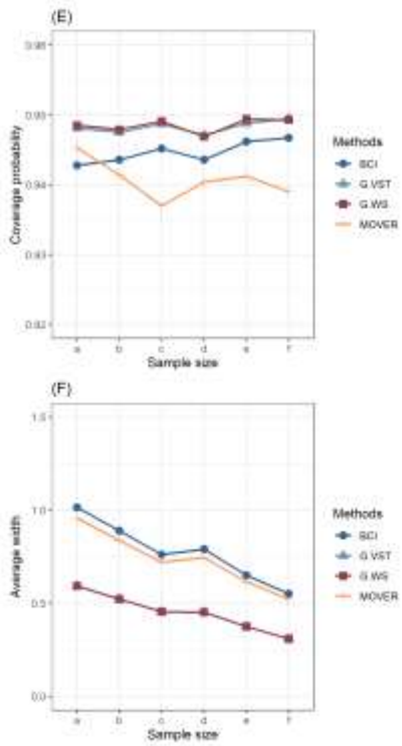

Fig. 4: Graphs compare the efficiency using (E) coverage probability and (F) average width relative to each sample size (a=(30,30), b=(30,50), c=(30,100), d=(50,50), e=(50,100), and f=(100,100))

5 An Empirical Application

In Thailand, wind speed exhibits variability throughout the year, with some areas experiencing low wind speeds while others have higher speeds, influenced by various factors such as geography, seasons, and topography. Additionally, wind speed plays a significant role in several aspects, including the generation of renewable energy, increasing agricultural productivity, and heat dissipation. Moreover, it has implications for weather conditions, with wind speed being crucial in monitoring and predicting weather patterns, contributing to disaster preparedness and prevention. In summary, wind speed in Thailand is of importance to farmers and serves as a valuable resource for renewable energy. Understanding and leveraging the characteristics of wind speed is crucial for the efficient development and utilization of this resource. Consequentially, we have chosen to use wind speed samples in Thailand for this research. We will utilize daily wind speed data from Chachoengsao Agrometeorological Station and Lamphun Weather Observing Station for the months of October, November, and December 2023, which represent the last quarter of the year. All the data used is presented in Table 2 (Appendix), [31]. From the presented data, it is observed that the wind speed data contains both zero values (indicating no wind) and positive values. To assess the appropriateness of the data distribution for positive values, the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) are utilized and computed using formulas:

$$\mathrm{AIC} = -2\ln(L) + 2p$$

and

$$\mathrm{BIC} = -2\ln(L) + 2p\ln(o),$$

respectively, where p is the number of parameters estimated, o is the number of observations, and L is the likelihood function. The distribution models considered for comparison include Normal, Logistic, Exponential, Cauchy, and Birnbaum-Saunders, as shown in Table 3 (Appendix). The results from Table 3 (Appendix) reveal that the Birnbaum-Saunders distribution has the lowest AIC and BIC values when compared to other distributions. This indicates that the Birnbaum-Saunders distribution is the most suitable for the positive wind speed data. Therefore, the wind speed data, comprising both zero and positive values, is appropriate for the delta-Birnbaum-Saunders distribution. Additionally, we have plotted histograms of wind speed data from both stations to visualize the distribution of the utilized data, as shown in Figure 5. As for Table 4 (Appendix), it presents statistical information regarding the wind speed data from both stations. The delta-Birnbaum-Saunders distributions are employed to calculate confidence intervals for the difference between two coefficients of variation for the wind speed data. From the presented data in Table 5 (Appendix), it can be observed that the G.VST method has the narrowest confidence interval width compared to other methods. Because of this, the G.VST method is the most efficient method for calculating confidence intervals for the difference between two coefficients of variation of the delta-Birnbaum-Saunders distributions in this wind speed data.

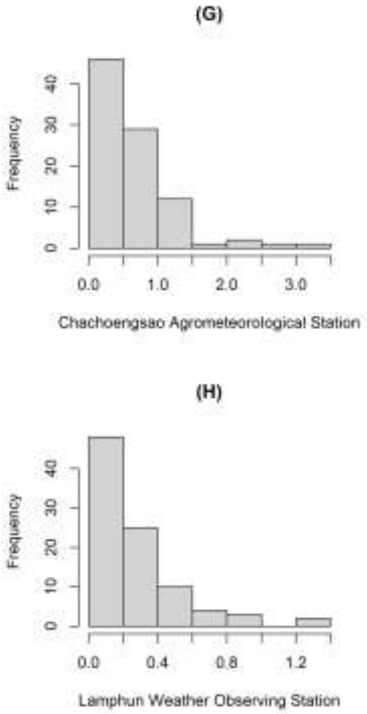

Fig. 5: Histograms of wind speed data for (G) Chachoengsao Agrometeorological Station and (H) Lamphun Weather Observing Station

6 Conclusions

In this paper, our focus is to present confidence intervals for the difference between two coefficients of variation within the context of the delta-Birnbaum-Saunders distribution. We have introduced methods for constructing confidence intervals, namely MOVER, BCI, G.VST, and G.WS. Additionally, performance comparison considers both coverage probability and average width obtained from Monte Carlo simulations. Importantly, we have applied the proposed methods to wind speed data. The results of the simulation study indicate that the BCI method produces findings similar to the MOVER method, except in cases where the shape parameter is (2.0, 2.0). However, the BCI method provides a lower coverage probability than the specified target and exhibits a wider average width compared to other methods. Meanwhile, the MOVER method achieves the target coverage probability and performs well when the shape parameters are small. The G.VST and G.WS methods provide similar results in terms of both coverage probability and average width, with coverage probability values close to the specified target for almost all methods and the narrowest interval. It was also found that as the sample size increases, all methods tend to show improved performance. Conversely, when the shape parameter and the proportion of zeros increase, the performance tends to decrease. Moreover, the simulation results are consistent with the results obtained from applying the methods to real data. Therefore, the methods recommended for constructing confidence intervals are the G.VST and G.WS. Additionally, the MOVER method is recommended for very small shape parameters.

In future research, we will investigate estimation for more than two parameters and alternative methods for constructing confidence intervals, such as Bayesian estimation or Highest posterior density, to broaden the scope of the analysis. Additionally, we aim to apply our findings to diverse real-world scenarios, such as medical, economic, and environmental data, to highlight the practical applications of our research.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work, the authors used ChatGPT and QuillBot in order to improve the clarity and accuracy of the language. After using this tool, the authors reviewed and edited the content as needed and took full responsibility for the content of the publication.

References:

[1] Birnbaum, Z.W. and Saunders, S.C., A new family of life distribution, *Journal of applied probability*, Vol. 6, 1969a, pp. 319-327. DOI: 10.2307/3212003.

[2] Birnbaum, Z. W., and Saunders, S. C., Estimation for a family of life distributions with applications to fatigue, *Journal of applied probability*, Vol. 6, No. 2, 1969, pp. 328-347. DOI: 10.2307/3212004.

[3] Leiva, V., Barros, M., Paula, G. A., and Sanhueza, A., Generalized Birnbaum-

Saunders distributions applied to air pollutant concentration. *Environmetrics: The official journal of the International Environmetrics Society*, Vol. 19, No. 3, 2008, pp. 235-249. DOI: 10.1002/env.861.

[4] Lio, Y. L., Tsai, T. R., and Wu, S. J., Acceptance sampling plans from truncated life tests based on the Birnbaum–Saunders distribution for percentiles. *Communications in Statistics-Simulation and Computation*, Vol. 39, No. 1, 2009, pp. 119-136. DOI: 10.1080/03610910903350508.

[5] Martínez-Flórez, G., Tovar-Falón, R., and Barrera-Causil, C., Inflated Unit-Birnbaum-Saunders Distribution, *Mathematics*, Vol. 10, No. 4, 2022, pp. 667. DOI: 10.3390/math 10040667.

[6] Shakil, M., Munir M., Kausar, N., Ahsanullah M., Khadim, A., Sirajo, M., and Kibria, BMG., Some Inferences on Three Parameters Birnbaum-Saunders Distribution: Statistical Properties, Characterizations and Applications, *Computational Journal of Mathematical and Statistical Sciences*, Vol. 2, No. 2, 2023, pp. 197-222. DOI: 10.21608/cjmss.2023.224583.1011.

[7] Leiva, V., Marchant, C., Ruggeri, F., and Saulo, H., A criterion for environmental assessment using Birnbaum–Saunders attribute control charts, *Environmetrics*, Vol. 26, No. 7, 2015, pp. 463-476. DOI: 10.1002/env.2349.

[8] Ratasukharom, N., Niwitpong, S.A., and Niwitpong, S., Confidence Intervals for the Mean and Difference of Means of Birnbaum-Saunders Distributions with Application to Wind Speed Data, *WSEAS Transactions on Mathematics*, Vol. 23, 2024, pp. 515-528. https://doi.org/10.37394/23206.2024.23.54.

[9] Aitchison, J., On the distribution of a positive random variable having a discrete probability mass at the origin, *Journal of the American Statistical Association*, Vol. 50, No. 271, 1955, pp. 901-908. DOI: 10.1080/01621459.1955.10501976.

[10] Hasan, M. S., and Krishnamoorthy, K., Confidence intervals for the mean and a percentile based on zero-inflated lognormal data, *Journal of Statistical Computation and Simulation*, Vol. 88, No. 8, 2018, pp. 1499-1514. DOI: 10.1080/00949655.2018.1439033.

[11] Wang, X., Li, M., Sun, W., Gao, Z., and Li, X., Confidence intervals for zero-inflated gamma distribution, *Communications in Statistics-Simulation and Computation*, Vol. 53, No. 7, 2022, pp. 3418-3435. DOI: 10.1080/03610918.2022.2104315.

[12] Bell, K. J., Hekker, S., and Kuszlewicz, J. S., Coefficients of variation for detecting solar-like oscillations, *Monthly Notices of the Royal Astronomical Society*, Vol. 482, No. 1, 2019, pp. 616-625. DOI: 10.1093/mnras/sty2731.

[13] Tran, K. P., Heuchenne, C., and Balakrishnan, N., On the performance of coefficient of variation charts in the presence of measurement errors, *Quality and Reliability Engineering International*, Vol. 35, No. 1, 2019, pp. 329-350. DOI: 10.1002/qre.2402.

[14] Pélabon, C., Hilde, C. H., Einum, S., and Gamelon, M., On the use of the coefficient of variation to quantify and compare trait variation, *Evolution letters*, Vol. 4, No. 3, 2020, pp. 180-188. DOI: 10.1002/evl3.171.

[15] Jalilibal, Z., Amiri, A., Castagliola, P., and Khoo, M. B., Monitoring the coefficient of variation: A literature review, *Computers & Industrial Engineering*, Vol. 161, 2021, pp. 107600. DOI: 10.1016/j.cie.2021.107600.

[16] Yosboonruang, N., Niwitpong, S.A., and Niwitpong, S., The Bayesian confidence intervals for measuring the difference between dispersions of rainfall in Thailand, *PeerJ*, Vol. 8, 2020, e9662. DOI: 10.7717/peerj.9662.

[17] Puggard, W., Niwitpong, S.A., and Niwitpong, S., Bayesian estimation for the coefficients of variation of Birnbaum–Saunders distributions, *Symmetry*, Vol. 13, No. 11, 2021, pp. 2130. DOI: 10.3390/sym13112130.

[18] Guo, H., Qian, Y., Zhu, Y., Dai, X., and Wang, X., Confidence intervals for the difference between coefficients of variation of zero-inflated gamma distributions, *AIMS Mathematics*, Vol. 8, No. 12, 2023, pp. 29713-29733. DOI:10.3934/math.20231521.

[19] Zou, G. Y. and Donner, A., Construction of confidence limits about effect measures: a general approach, *Statistics in medicine*, Vol. 27, No. 10, 2008, pp. 1693-1702. DOI: 10.1002/sim.3095.

[20] Ng, H. K. T., Kundu, D., and Balakrishnan, N., Modified moment estimation for the two-parameter Birnbaum–Saunders distribution, *Computational Statistics & Data Analysis*, Vol. 43, No. 3, 2003, pp. 283-298. DOI: 10.1016/S0167-9473(02)00254-2.

[21] Efron, B., Bootstrap methods: another look at the jackknife. *The Annals of Statistics*, Vol. 7, No. 1, 1979, pp. 1-26. DOI: 10.1214/aos/ 1176344552.

[22] Lemonte, A.J., Simas, A.B., and Cribari-Neto, F., Bootstrap-based improved estimators for the two-parameter Birnbaum-Saunders distribution, *Journal of Statistical Computation and Simulation*, Vol. 78, No. 1, 2008, pp. 37-49. DOI: 10.1080/10629360600903882.

[23] MacKinnon, J. G., and Smith Jr, A. A., Approximate bias correction in econometrics, *Journal of Econometrics*, Vol. 85, No. 2, 1998, pp. 205-230. DOI: 10.1016/S0304-4076(97)00099-7.

[24] Brown, L. D., Cai, T. T., and DasGupta, A., Interval estimation for a binomial proportion, *Statistical science*, Vol. 16, No. 2, 2001, pp. 101-133. DOI: 10.1214/ss/1009213286.

[25] Weerahandi, S., Generalized confidence intervals, *Journal of the American Statistical Association*, Vol. 88, No. 423, 1993, pp. 899–905. DOI: 10.2307/2290779.

[26] Sun, Z.L., The confidence intervals for the scale parameter of the Birnbaum-Saunders fatigue life distribution, *Acta Armament*, Vol. 30, No. 11, 2009, pp. 1558-1561.

[27] Wang, B. X., Generalized interval estimation for the Birnbaum–Saunders distribution, *Computational Statistics & Data Analysis*, Vol. 56, No. 12, 2012, pp. 4320-4326. DOI: 10.1016/j.csda.2012.03.023.

[28] DasGupta, A., *Asymptotic theory of statistics and probability*, New York, Springer, 2008.

[29] Wu, W. H. and Hsieh, H. N., Generalized confidence interval estimation for the mean of delta-lognormal distribution: an application to New Zealand trawl survey data, *Journal of Applied Statistics*, Vol. 41, No. 7, 2014, pp. 1471-1485. DOI: 10.1080/02664763.2014.881780.

[30] Li, X., Zhou, X., and Tian, L., Interval estimation for the mean of lognormal data with excess zeros, *Statistics & Probability Letters*, Vol. 83, No. 11, 2013, pp. 2447-2453. DOI: 10.1016/j.spl.2013.07.004.

[31] Thai Meteorological Department Automatic Weather System. 2024, [Online]. https://www.tmd.go.th/service/tmdData (Accessed Date: September 1, 2024).

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed to the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of funding for research presented in a scientific article or scientific article itself

This research has received funding support from the National Science, Research and Innovation Fund (NSRF), and King Mongkut's University of Technology North Bangkok: KMUTNB-FF-67-B-14.

Conflict of Interest

The authors have no conflicts of interest to declare.

APPENDIX

Table 1. The coverage probabilities and average widths for the 95% confidence intervals for Ψ

α_1,α_2	γ_1,γ_2	m_1,m_2	Coverage probabilities				Average widths			
			MOVER	BCI	G.VST	G.WS	MOVER	BCI	G.VST	G.WS
0.25,0.25	0.1,0.1	30,30	0.9353	0.9463	**0.9537**	**0.9557**	0.4814	0.4866	0.1483	**0.1482**
		30,50	0.9357	0.9423	**0.9500**	0.9487	0.4323	0.4336	**0.1301**	0.1302
		30,100	0.9250	0.9407	**0.9530**	**0.9530**	0.3896	0.3902	**0.1149**	0.1150
		50,50	0.9460	0.9453	0.9450	0.9457	0.3780	0.3769	0.1085	0.1085
		50,100	0.9427	0.9473	**0.9517**	**0.9517**	0.3274	0.3249	**0.0916**	**0.0916**
		100,100	0.9477	0.9447	**0.9520**	**0.9510**	0.2683	0.2655	**0.0727**	**0.0727**
	0.1,0.3	30,30	**0.9523**	0.9433	0.9493	0.9467	**0.6186**	0.6234	0.1463	0.1461
		30,50	0.9467	0.9417	0.9417	0.9410	0.5229	0.5250	0.1252	0.1253
		30,100	0.9393	0.9427	0.9437	0.9447	0.4420	0.4420	0.1118	0.1117
		50,50	**0.9597**	**0.9527**	**0.9500**	**0.9503**	0.4780	0.4768	**0.1031**	**0.1031**
		50,100	0.9443	0.9373	0.9390	0.9410	**0.3864**	0.3840	0.0881	0.0881
		100,100	**0.9500**	0.9480	**0.9503**	0.9490	0.3383	0.3355	**0.0684**	0.0684
	0.1,0.5	30,30	**0.9667**	**0.9503**	**0.9533**	**0.9513**	0.8342	0.8504	**0.1671**	0.1672
		30,50	**0.9567**	0.9463	0.9440	0.9417	**0.6759**	0.6808	0.1353	0.1352
		30,100	0.9443	0.9463	0.9453	0.9483	0.5312	0.5321	0.1162	0.1162
		50,50	**0.9557**	0.9450	0.9467	0.9480	**0.6378**	0.6406	0.1143	0.1140
		50,100	**0.9507**	0.9460	0.9470	0.9480	**0.4857**	0.4838	0.0927	0.0928
		100,100	**0.9553**	0.9493	0.9463	0.9477	**0.4489**	0.4460	0.0740	0.0739
	0.3,0.3	30,30	**0.9533**	0.9377	0.9460	0.9437	**0.7303**	0.7366	0.1417	0.1419
		30,50	**0.9550**	0.9437	0.9427	0.9393	**0.6520**	0.6543	0.1208	0.1209
		30,100	**0.9573**	0.9447	0.9483	0.9490	**0.5861**	0.5860	0.1064	0.1066
		50,50	**0.9533**	0.9423	**0.9503**	**0.9507**	0.5611	0.5602	**0.0985**	**0.0985**
		50,100	**0.9573**	0.9497	**0.9523**	**0.9540**	0.4851	0.4825	**0.0825**	**0.0825**
		100,100	**0.9563**	**0.9510**	**0.9547**	**0.9537**	0.3945	0.3906	0.0644	**0.0643**
	0.3,0.5	30,30	**0.9670**	**0.9517**	**0.9510**	0.9483	0.9243	0.9422	**0.1657**	0.1654
		30,50	**0.9643**	0.9493	0.9423	0.9443	**0.7778**	0.7845	0.1308	0.1308
		30,100	**0.9613**	**0.9567**	0.9477	0.9467	**0.6575**	0.6592	0.1111	0.1111
		50,50	**0.9560**	0.9393	0.9480	0.9430	**0.7061**	0.7098	0.1106	0.1106
		50,100	**0.9593**	0.9490	0.9473	0.9450	**0.5673**	0.5662	0.0874	0.0873
		100,100	**0.9547**	0.9450	**0.9537**	**0.9530**	0.4938	0.4912	0.0701	**0.0700**
	0.5,0.5	30,30	**0.9623**	0.9380	0.9383	0.9417	**1.0819**	1.1100	0.1871	0.1869
		30,50	**0.9707**	**0.9527**	0.9427	0.9453	**0.9657**	0.9839	0.1543	0.1542
		30,100	**0.9577**	0.9463	0.9473	**0.9503**	0.8660	0.8784	0.1357	**0.1356**
		50,50	**0.9637**	0.9467	0.9490	**0.9507**	0.8242	0.8313	0.1210	**0.1208**
		50,100	**0.9597**	0.9457	0.9427	0.9393	**0.7109**	0.7132	0.0989	0.0990
		100,100	**0.9603**	**0.9527**	0.9473	0.9473	0.5760	**0.5736**	0.0754	0.0753
0.5,0.5	0.1,0.1	30,30	**0.9500**	0.9370	0.9433	0.9460	**0.5256**	0.5239	0.3944	0.3937
		30,50	**0.9593**	**0.9510**	0.9490	0.9477	0.4703	**0.4666**	0.3442	0.3437
		30,100	0.9460	0.9480	**0.9530**	**0.9543**	0.4221	0.4184	**0.3065**	0.3067
		50,50	**0.9503**	0.9403	0.9427	0.9413	**0.4071**	0.4011	0.2903	0.2899
		50,100	**0.9580**	**0.9530**	**0.9507**	**0.9527**	0.3525	0.3464	0.2474	**0.2472**
		100,100	**0.9610**	0.9493	0.9473	0.9453	**0.2888**	0.2820	0.1984	0.1983
	0.1,0.3	30,30	**0.9607**	0.9400	0.9493	0.9467	**0.6764**	0.6702	0.4166	0.4160
		30,50	**0.9560**	0.9393	0.9447	0.9440	**0.5732**	0.5644	0.3545	0.3549
		30,100	**0.9570**	**0.9510**	**0.9510**	**0.9513**	0.4825	0.4746	0.3112	**0.3111**
		50,50	**0.9643**	**0.9507**	0.9413	0.9410	0.5240	**0.5127**	0.3018	0.3023
		50,100	**0.9533**	0.9373	0.9440	0.9457	**0.4214**	0.4108	0.2523	0.2525
		100,100	**0.9623**	**0.9520**	**0.9520**	**0.9523**	0.3690	0.3577	**0.2032**	0.2034

Table 1. *Continued*

α_1,α_2	γ_1,γ_2	m_1,m_2	Coverage probabilities				Average widths			
			MOVER	BCI	G.VST	G.WS	MOVER	BCI	G.VST	G.WS
0.5.0.5	0.1,0.5	30,30	**0.9663**	0.9433	0.9483	**0.9533**	0.9337	0.9324	0.4956	**0.4956**
		30,50	**0.9660**	0.9470	**0.9553**	**0.9553**	0.7530	0.7408	**0.3960**	0.3968
		30,100	**0.9663**	**0.9503**	**0.9530**	**0.9523**	0.5897	0.5773	0.3303	**0.3300**
		50,50	**0.9553**	0.9387	0.9470	0.9470	**0.7160**	0.7009	0.3508	0.3511
		50,100	**0.9653**	0.9473	0.9443	0.9480	**0.5426**	0.5270	0.2760	0.2759
		100,100	**0.9580**	0.9450	**0.9503**	**0.9503**	0.5025	0.4860	0.2296	**0.2294**
	0.3,0.3	30,30	**0.9697**	0.9483	0.9477	0.9447	**0.8055**	0.7952	0.4363	0.4361
		30,50	**0.9630**	0.9453	**0.9523**	**0.9550**	0.7129	0.6996	0.3760	**0.3756**
		30,100	**0.9633**	**0.9513**	0.9440	0.9443	0.6446	**0.6310**	0.3335	0.3332
		50,50	**0.9647**	0.9463	0.9487	0.9453	**0.6176**	0.6020	0.3120	0.3116
		50,100	**0.9607**	0.9453	**0.9533**	**0.9557**	0.5358	0.5197	0.2649	**0.2648**
		100,100	**0.9577**	0.9450	**0.9540**	**0.9540**	0.4358	0.4202	**0.2090**	0.2091
	0.3,0.5	30,30	**0.9693**	0.9473	0.9417	0.9427	**1.0306**	1.0285	0.5150	0.5148
		30,50	**0.9740**	0.9487	**0.9500**	**0.9507**	0.8703	0.8574	0.4182	**0.4181**
		30,100	**0.9613**	0.9433	**0.9527**	**0.9520**	0.7303	0.7140	0.3545	**0.3543**
		50,50	**0.9703**	0.9487	0.9423	0.9433	**0.7888**	0.7726	0.3584	0.3576
		50,100	**0.9677**	0.9477	**0.9517**	**0.9507**	0.6346	0.6157	0.2860	**0.2859**
		100,100	**0.9637**	0.9467	**0.9500**	**0.9510**	0.5520	0.5328	**0.2349**	**0.2349**
	0.5,0.5	30,30	**0.9773**	0.9417	0.9440	0.9467	**1.2189**	1.2231	0.5887	0.5890
		30,50	**0.9780**	0.9447	0.9460	0.9460	**1.0854**	1.0799	0.5010	0.5005
		30,100	**0.9703**	**0.9550**	**0.9503**	**0.9517**	0.9697	0.9596	**0.4391**	0.4396
		50,50	**0.9690**	0.9443	0.9443	0.9433	**0.9308**	0.9141	0.4014	0.4012
		50,100	**0.9683**	0.9483	**0.9510**	**0.9503**	0.8027	0.7833	**0.3351**	0.3355
		100,100	**0.9673**	0.9477	**0.9517**	**0.9513**	0.6528	0.6312	**0.2605**	**0.2605**
1.0,1.0	0.1,0.1	30,30	**0.9510**	0.9467	**0.9503**	**0.9530**	0.7419	0.7408	0.7044	**0.7043**
		30,50	0.9430	0.9430	0.9497	**0.9503**	0.6646	0.6632	0.6255	**0.6257**
		30,100	0.9373	0.9410	0.9470	0.9473	0.5992	0.5979	0.5614	0.5620
		50,50	**0.9520**	0.9463	0.9470	0.9457	**0.5777**	0.5760	0.5356	0.5354
		50,100	0.9457	0.9473	0.9517	**0.9507**	0.5008	0.4993	0.4609	**0.4614**
		100,100	0.9397	0.9373	0.9417	0.9417	0.4113	0.4103	0.3744	0.3741
	0.1,0.3	30,30	**0.9540**	0.9443	**0.9530**	**0.9530**	0.9122	0.9155	0.7833	**0.7828**
		30,50	0.9453	0.9380	0.9417	0.9403	0.7783	0.7781	0.6734	0.6730
		30,100	0.9423	0.9407	0.9490	0.9493	0.6659	0.6648	0.5880	0.5872
		50,50	0.9423	0.9343	0.9413	0.9420	**0.7095**	0.7105	0.5925	0.5925
		50,100	**0.9510**	0.9463	0.9487	0.9487	**0.5768**	0.5761	0.4921	0.4924
		100,100	0.9487	0.9460	0.9450	0.9457	0.5018	0.5014	0.4112	0.4112
	0.1,0.5	30,30	0.9433	0.9363	0.9430	0.9480	1.2248	1.2454	0.9517	0.9514
		30,50	**0.9560**	0.9477	0.9487	0.9470	**1.0059**	1.0141	0.7864	0.7857
		30,100	**0.9520**	0.9450	0.9523	**0.9510**	0.8016	0.8025	0.6479	**0.6476**
		50,50	0.9480	0.9427	0.9490	**0.9517**	0.9529	0.9620	0.7148	**0.7149**
		50,100	**0.9557**	0.9480	0.9410	0.9443	**0.7332**	0.7348	0.5626	0.5623
		100,100	0.9463	0.9460	0.9463	0.9483	0.6720	0.6734	0.4915	0.4912
	0.3,0.3	30,30	0.9443	0.9323	0.9433	0.9430	**1.0528**	1.0611	0.8529	0.8500
		30,50	0.9443	0.9313	0.9393	0.9427	**0.9408**	0.9457	0.7523	0.7529
		30,100	0.9377	0.9440	0.9483	0.9487	0.8460	0.8490	0.6679	0.6694
		50,50	**0.9530**	0.9450	0.9443	0.9440	**0.8166**	0.8201	0.6435	0.6433
		50,100	0.9497	0.9430	**0.9573**	**0.9567**	0.7077	0.7098	**0.5516**	0.5520
		100,100	0.9480	0.9417	0.9493	0.9477	0.5774	0.5788	0.4450	0.4452
	0.3,0.5	30,30	**0.9547**	0.9377	0.9487	0.9480	**1.3467**	1.3734	1.0197	1.0189
		30,50	**0.9563**	0.9387	0.9493	**0.9517**	1.1358	1.1484	0.8572	**0.8577**
		30,100	0.9467	0.9350	0.9430	0.9443	0.9576	0.9631	0.7275	0.7276

Table 1. *Continued*

α_1,α_2	γ_1,γ_2	m_1,m_2	Coverage probabilities				Average widths			
			MOVER	BCI	G.VST	G.WS	MOVER	BCI	G.VST	G.WS
1.0,1.0	0.3,0.5	50,50	**0.9540**	0.9433	0.9490	**0.9507**	1.0361	1.0462	0.7578	**0.7577**
		50,100	**0.9559**	**0.9530**	**0.9560**	**0.9530**	0.8387	0.8419	0.6162	**0.6155**
		100,100	**0.9547**	**0.9537**	**0.9540**	**0.9537**	0.7311	0.7350	0.5219	**0.5217**
	0.5,0.5	30,30	**0.9613**	0.9343	0.9467	0.9457	**1.5859**	1.6296	1.1630	1.1614
		30,50	**0.9540**	0.9407	0.9423	0.9427	**1.4067**	1.4350	1.0148	1.0153
		30,100	0.9493	0.9430	0.9433	0.9403	1.2625	1.2835	0.9048	0.9055
		50,50	**0.9527**	0.9377	0.9397	0.9410	**1.2190**	1.2351	0.8612	0.8605
		50,100	**0.9563**	0.9473	**0.9503**	0.9487	1.0537	1.0635	**0.7333**	0.7332
		100,100	**0.9517**	0.9410	**0.9533**	**0.9543**	0.8592	0.8645	**0.5880**	0.5883
2.0,2.0	0.1,0.1	30,30	0.9233	0.9460	**0.9520**	**0.9533**	0.7024	0.7714	**0.6625**	0.6631
		30,50	0.9180	0.9397	**0.9520**	**0.9517**	0.6262	0.6885	**0.5884**	0.5892
		30,100	0.9230	0.9457	0.9460	0.9493	0.5634	0.6196	0.5298	0.5299
		50,50	0.9177	0.9407	**0.9503**	**0.9517**	0.5397	0.5927	**0.5055**	0.5061
		50,100	0.9213	0.9413	0.9487	**0.9500**	0.4657	0.5112	0.4352	**0.4359**
		100,100	0.9230	0.9453	0.9467	0.9473	0.3783	0.4140	0.3534	0.3536
	0.1,0.3	30,30	0.8987	0.9380	0.9483	0.9497	0.8632	1.0126	0.7526	0.7517
		30,50	0.9110	0.9460	**0.9500**	**0.9517**	0.7329	0.8507	**0.6472**	0.6477
		30,100	0.9097	0.9437	**0.9530**	**0.9520**	0.6228	0.7104	0.5617	**0.5612**
		50,50	0.9053	0.9430	**0.9517**	**0.9513**	0.6588	0.7738	**0.5711**	0.5713
		50,100	0.9163	**0.9513**	**0.9563**	**0.9570**	0.5364	0.6183	**0.4748**	**0.4748**
		100,100	0.9093	**0.9550**	**0.9513**	0.9497	0.4607	0.5398	**0.3989**	0.3984
	0.1,0.5	30,30	0.8920	0.9477	**0.9550**	**0.9533**	1.1890	1.4712	**0.9377**	0.9387
		30,50	0.8867	0.9477	**0.9517**	**0.9513**	0.9570	1.1733	**0.7729**	0.7730
		30,100	0.8870	0.9397	0.9480	0.9463	0.7528	0.9047	0.6321	0.6320
		50,50	0.8787	0.9403	**0.9527**	**0.9500**	0.8995	1.1193	**0.7090**	0.7095
		50,100	0.8853	0.9443	0.9453	0.9483	0.6816	0.8346	0.5553	0.5547
		100,100	0.8777	0.9420	**0.9537**	**0.9527**	0.6247	0.7781	**0.4920**	0.4921
	0.3,0.3	30,30	0.8990	0.9483	0.9477	**0.9503**	1.0008	1.2057	0.8344	**0.8342**
		30,50	0.8863	0.9363	0.9453	0.9493	0.8899	1.0757	0.7395	0.7391
		30,100	0.8890	0.9463	**0.9507**	**0.9520**	0.7995	0.9651	0.6646	**0.6642**
		50,50	0.8857	0.9413	**0.9537**	**0.9513**	0.7627	0.9219	0.6327	**0.6326**
		50,100	0.8930	0.9440	**0.9513**	**0.9553**	0.6570	0.7951	0.5447	**0.5439**
		100,100	0.8970	0.9473	**0.9520**	**0.9507**	0.5310	0.6427	**0.4400**	**0.4400**
	0.3,0.5	30,30	0.8917	0.9387	**0.9530**	**0.9513**	1.2900	1.6161	**1.0020**	1.0029
		30,50	0.8923	0.9483	**0.9577**	**0.9597**	1.0814	1.3486	**0.8497**	0.8509
		30,100	0.8977	0.9477	**0.9510**	**0.9520**	0.9033	1.1145	**0.7251**	0.7256
		50,50	0.8780	0.9373	0.9477	0.9497	0.9794	1.2295	0.7602	0.7608
		50,100	0.8883	0.9430	0.9433	0.9423	0.7813	0.9717	0.6166	0.6162
		100,100	0.8813	0.9427	0.9483	0.9467	0.6787	0.8528	0.5267	0.5266
	0.5,0.5	30,30	0.8907	0.9483	**0.9520**	**0.9507**	1.5408	1.9604	1.1538	**1.1536**
		30,50	0.8747	0.9380	**0.9520**	**0.9517**	1.3725	1.7411	**1.0252**	1.0253
		30,100	0.8660	0.9363	0.9473	0.9483	1.2207	1.5482	0.9164	0.9164
		50,50	0.8947	**0.9553**	0.9477	0.9480	1.1560	1.4736	**0.8693**	0.8690
		50,100	0.8837	0.9450	0.9457	0.9480	0.9974	1.2702	0.7479	0.7483
		100,100	0.8637	0.9457	0.9460	0.9457	0.8002	1.0226	**0.6001**	0.6010

Note: Bold text indicates coverage probabilities greater than or equal to 0.95 and the most appropriate average widths.

Table 2. Data on the wind speed (knot) from the Chachoengsao Agrometeorological Station and Lamphun Weather Observing Station, Thailand

Chachoengsao Agrometeorological Station						Lamphun Weather Observing Station					
October		November		December		October		November		December	
0.4	0.0	0.0	1.1	1.1	0.9	0.5	0.3	0.5	0.3	0.0	0.2
0.2	0.1	0.0	1.2	0.8	1.0	0.7	0.0	0.1	0.8	0.4	0.7
0.6	0.0	0.0	1.2	1.0	0.9	0.2	0.1	0.1	0.6	0.3	0.1
0.2	0.1	0.1	0.9	0.9	1.3	0.6	0.4	0.2	0.4	0.3	0.1
0.0	0.1	0.1	0.8	0.6	2.0	0.5	0.4	0.3	0.4	0.0	0.3
0.4	0.1	0.2	0.6	0.7	2.5	1.0	0.0	0.1	0.0	0.2	1.3
0.4	0.0	0.6	0.5	1.0	2.6	0.1	0.0	0.1	0.0	0.9	1.3
0.0	0.0	0.6	0.6	0.6	2.4	0.5	0.0	0.0	0.0	0.0	0.4
0.0	0.0	0.2	0.9	0.5	3.1	0.7	0.1	0.1	0.1	0.0	0.3
0.0	0.0	0.5	0.9	0.5	1.2	0.1	0.3	0.0	0.0	0.4	0.1
0.1	0.0	0.2	0.8	0.5	1.2	0.6	0.2	0.0	0.1	0.1	0.0
0.0	0.0	0.5	1.2	0.4	1.2	0.3	0.1	0.0	0.2	0.3	0.1
0.0	0.0	0.5	1.3	0.9	1.1	0.5	0.4	0.3	0.3	0.3	0.0
0.0	0.0	0.9	0.9	0.9	1.0	0.3	0.5	0.9	0.0	0.2	0.1
0.0	0.0	0.7	1.1	0.6	0.8	0.2	0.4	0.4	0.1	0.1	0.1
0.0				0.7		0.5				0.3	

Table 3. The AIC and BIC values of each model for the wind speed data

Data	Model	Normal	Logistic	Exponential	Cauchy	**Birnbaum-Saunders**
Chachoengsao Agrometeorological Station	AIC	127.852	116.125	112.906	123.838	**107.942**
	BIC	132.321	120.593	115.140	128.307	**114.644**
Lamphun Weather Observing Station	AIC	19.727	11.750	-3.852	23.393	**-23.326**
	BIC	24.308	16.331	-1.462	27.974	**-16.455**

Table 4. Summary statistics for the wind speed data

Data	m_i	$m_{i(0)}$	$m_{i(1)}$	$\hat{\gamma}_i$	$\hat{\alpha}_i$	$\hat{\beta}_i$	$\hat{\zeta}_i$
Chachoengsao Agrometeorological Station	92	23	69	0.2500	0.9148	0.5793	1.2116
Lamphun Weather Observing Station	92	19	73	0.2065	0.7870	0.2699	1.0333

Table 5. The 95% confidence intervals for Ψ of the wind speed data

Point estimation	Methods	Interval [L, U]	Widths
$\hat{\zeta}_1 - \hat{\zeta}_2 = 0.1783$	MOVER	[-0.0726, 0.4292]	0.5018
	BCI	[-0.0445, 0.4453]	0.4898
	G.VST	[-0.0215, 0.3504]	0.3719
	G.WS	[-0.0308, 0.3653]	0.3961