Stochastic Predator-prey System with Hunting Cooperation and Regiem Switching and Its Dynamics

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Abstract: - In this paper, a random predator-prey system is established, based on which cooperative hunting and regional switching are considered. Firstly, the existence and uniqueness of the global positive solution of the model are proved. Secondly, the sufficient conditions for extinction and stationary distribution are obtained by using Lyapunov function. Finally, numerical simulation is used to demonstrate the correctness of the conclusion.

Key-Words: - hunting cooperation , regime-switching, stationary distribution, predator-prey system)

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1 Introduction

Studying the dynamic properties of predator-prey models, [1], [2], is an important topic in ecological research. However, in numerous studies, [3], proposes a classical model with densitydependent logical growth as follows

$$\begin{cases} du(t) = [u(t)(a - bu(t) - cv(t))]dt, \\ dv(t) = [v(t)(-h + fcu(t))]dt, \end{cases} \tag{1}$$

where $u(t)$ and $v(t)$ are the sizes of prey population and predator population at time t, respectively. a, b, c, h and f are positive constants. The prey carrying capacity of the environment is $\frac{a}{b}$ and the feeding efficiency of converting predators into new predators is f.

The system(1) was extensively studied by [4], [5], [6]. Many biological populations often engage in cooperative hunting, [7], to increase the probability of capturing and killing prey, thereby enhancing the birth and survival rates of the population, such as the mutual cooperation in wolf pack hunting. In 1999, [8], proposed a model with novel functional response that describes the aggregation behavior of predators when encountering a group of prey. In 2010, [9], established a model with Holling Type II response and a predator-prey model with cooperative hunting, finding that cooperative hunting disrupts the stability of predator-prey dynamics. Furthermore, [10], [11], also considered how cooperative hunting affects the dynamics of predator-prey populations modeled by ordinary differential equations. Based on the above research, we establish the following predator-prey model with cooperative predation:

$$\begin{cases} u(t) = u(t)(a - bu(t) - cv(t)) - (p + \alpha v(t))u(t)v(t), \\ v(t) = v(t)(-h + fcu(t)) + m(p + \alpha v(t))u(t)v(t), \end{cases} \tag{2}$$

where $p>$ 0 is the attack rate of predators on prey, α $>$ 0 represents the cooperative parameter of predators during hunting, [11], and m is a constant.

Since the environment is not static in real life, the above systems are usually affected by environmental noise. Generally, two types of environmental noises are considered to describe environmental disturbances, one is white noise, [12], the other is classical colored noise, such as telegraph noise, [13]. Among them, telegraph noise can be characterized as random switches between two or more environmental regimes. In most models, environmental factors such as nutrition, [14], or rainfall, [15], are considered to be different. Many scholars have introduced stochastic perturbations to elucidate the impact of environmental white noise, [16], on population dynamics. Therefore, we construct the following predator-prey stochastic model with cooperative predation, considering the interference of environmental noise to the model:

$$\begin{cases} du(t) = [u(t)(a - bu(t) - cv(t))]dt, \\ dv(t) = [v(t)(-h + fcu(t))]dt, \end{cases} \tag{3}$$

where $B_1(t)$ and $B_2(t)$ are one-dimensional standard Brownian motions, σ_1 and σ_2 denote the intensity of the white noise.

However, few researchers have studied the dynamic behavior of predator-predator models with state transitions, [17], and cooperative hunting. Therefore, based on system (1.3), we construct a random predation model with white noise linear perturbations of cooperative hunting, [11], and state transition, [18], as follows:

In this paper, we mainly study the relevant dynamic properties of system (4). In the second section, the existence and uniqueness of positive solutions are modeled by Itô's formula and the comparison theorem. And the suffcient conditions for the extinction of predators are obtained. In the third section, an appropriate Lyapunov function is constructed to prove the existence of stationary distributions. Finally, the correctness of the obtained results is verified by numerical simulation.

$$\begin{cases} du(t) =[u(t)(a(r(t)) - b(r(t))u(t) \\ \qquad - c(r(t))v(t)) - (p(r(t)) \\ \qquad + \alpha(r(t))v(t))u(t)v(t)]dt \\ \qquad + \sigma_1(r(t))u(t)dB_1(t), \\ dv(t) =[v(t)(-h(r(t)) \\ \qquad + f(r(t))c(r(t))u(t)) \\ \qquad + m(r(t))(p(r(t)) \\ \qquad + \alpha(r(t))v(t))u(t)v(t)]dt \\ \qquad + \sigma_2(r(t))v(t)dB_2(t), \end{cases} \tag{4}$$

2 The Existence and Uniqueness of Positive Solutions

In this section, we mainly prove the existence and uniqueness of the global positive solution, [12], of system (4), and obtain sufficient conditions for predator extinction.

Theorem 2.1. *For any initial value $(u(0), v(0)) \in R^2_+ \times S$, for t ≥ 0, system (1.4) has a unique solution (u (t), v (t)). The solution will remain in $R^2_+ \times S$ for all t ≥ 0 with probability one.*

Proof. Consider that the coefficients of system (1.4) satisfy the local Lipschitz conditions, there exists a unique local solution $(u(t), v(t))$ for any initial value $(u(0), v(0)) \in R^2_+ \times S$, for $t \in [0, \tau_e]$, where τ_e denoting the explosion time. Establishing global uniqueness requires confirming that $\tau_e = +\infty$ almost surely. To achieve this, a sufficiently large positive integer is selected to ensure that both $u(0)$ and $v(0)$ fall within the interval $[\frac{1}{n_0}, n_0]$. Each integer greater than or equal to this value defines a stopping time.

$$\tau_n = \inf\{\tau \in [0, \tau_e) : \min\{u(t), v(t)\} \leq \frac{1}{n} or \max\{u(t), v(t)\} \geq n\}, \tag{5}$$

where throughout this paper, we set $\inf \oslash = \infty$. Obviously, τ_n is increasing as $n \to \infty$. Let $\tau_\infty = \lim_{n\to+\infty} \tau_n$, where $\tau_\infty \leq \tau_e$ a.s. is true, then $\tau_e = \infty$ a.s. and $(u(t), v(t), r(t)) \in R^2_+ \times S$ a.s. for all $T \geq 0$. In other words, to complete the proof, we only need to prove $\tau_\infty = \infty$a.s. If this assertion is false, then there is a pair of constants t>0 and $\epsilon \in (0, 1)$such that

$$\mathcal{P}\{\tau_{+\infty} \leq \tau\} > \epsilon. \tag{6}$$

Thus there exists an integer $n_1 \geq n_0$ such that

$$\mathcal{P}\{\tau_n \leq \tau\} > \epsilon,$$

for $\forall\, n \geq n_1$ Take a C^2-function $V : \mathbb{R}^2_+ \to \mathbb{R}_+$ as

$$V = u - 1 - \ln u + v - 1 - \ln v + P. \tag{7}$$

From $v - 1 - \ln v \geq 0$, for $v > 0$, it follows that V is a nonnegative function. If $y \in \mathbb{R}^2_+$, then using the Itô's formula reduces to

$$\begin{aligned} dV =& (1 - \frac{1}{u})[u(a(r(t)) - b(r(t))u - c(r(t))v) \\ &- (p(r(t)) + \alpha((r(t)))v)uv]dt + u\sigma_1 dB_1(t) \\ &+ \frac{1}{2}\sigma_1^2(r(t))dt + (1 - \frac{1}{v})[(v(-h(r(t))) \\ &+ f(r(t))c(r(t))u) + m(r(t))(p(r(t)) \\ &+ \alpha(r(t))v)uv]dt + v\sigma_2(r(t))dB_2 t + \frac{1}{2}\sigma_2(r(t))dt \\ &= [(u - 1)[\alpha(r(t)) - b(r(t))u - c(r(t))u \\ &+ m(r(t))(p(r(t)) + \alpha(r(t))v)v] + \frac{1}{2}\sigma_1^2(v - 1) \\ &[-h(r(t)) + f(r(t))c(r(t))u + m(r(t)) \\ &(p(r(t)) + \alpha(r(t))v)u] + \frac{1}{2}\sigma_2^2(r(t))]dt \\ &+ (u - 1)\sigma_1(r(t))dB_1 t \\ &+ (v - 1)\sigma_2(r(t))dB_2(t) \\ &:= \mathcal{L}V dt + (u - 1)\sigma_1 dB_1 t + (v - 1)\sigma_2 dB_2(t). \end{aligned}$$

where

$$\begin{aligned} \mathcal{L}V =& -(a(r(t)) - b(r(t)))u - c(r(t))u - (p(r(t)) \\ &+ \alpha(r(t))v)v + u[a(r(t)) - b(r(t))u - c(r(t))v \\ &- (p((r(t)) + \alpha(r(t))v))v] + [h(r(t)) - f(r(t)) \\ &c(r(t))u - m(r(t))(p(r(t)) + \alpha(r(t))v)u] \\ &+ v[-h(r(t)) + f(r(t))c(r(t))u + m(r(t)) \\ &(p(r(t)) + \alpha(r(t))v)u] + \frac{1}{2}\sigma_1^2(r(t)) + \frac{1}{2}\sigma_2^2(r(t)) \\ &\leq -b(r(t))u^2 + (b(r(t)) - c(r(t)))u + (m - 1) \\ &(p(r(t)) + \alpha(r(t))uv^2 + (f(r(t)) - 1)c(r(t))uv \\ &+ \frac{1}{2}(\sigma_1^2(r(t)) + \sigma_2^2(r(t))) \\ &\leq -b(r(t))u^2 + (b(r(t)) - c(r(t)))u \\ &+ \frac{1}{2}(\sigma_1^2(r(t)) + \sigma_2^2(r(t))) \\ &\leq N_0, \end{aligned}$$

where N_0 is a positive constant which is independent of u, v and n. The remainder of the proof is similar to Theorem 2.1, [19]. so this part of the proof is omitted. Thus the proof of this theorem is completed.

3 Extinction

We discuss the extinction of both predator and prey now.

Theorem 3.1. *If $(u(t), v(t))$, is a solution of the system with initial values $(u(0), v(0)) \in R^2$ and if the condition is satisfied $a < \frac{\sigma_1^2}{2}$ then the predator becomes extinct.*

Proof. Consider the system on the boundary, using the strong number theorem for local control, it is known that

$$\sum_{k\in s\prod k}[a(k)-\frac{\sum_1^2(k)}{2}]<0, \quad (8)$$

$$\lim_{t\to\infty} u(t)=0.$$

Extinction of food bait assume:

$$\sum_{k\in s\prod k}[a(k)-\frac{\sigma_1^2(k)}{2}]>0, \rho(.,.):R\times S\to R.$$

Integrate from 0 to t and divide both sides simultaneously by t:

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\int_0^t\rho(u(t),r(t))dt\\&=\sum_{k\in s}\int_{R^+}\rho(u,k)d(du,k),\end{aligned} \quad (9)$$

Using the Itô's formula, it is possible to obtain:

$$\begin{aligned}&\frac{\ln u(t)-\ln u(0)}{t}=\frac{1}{t}\int_0^t[a(r(t))\\&-\frac{\sigma_1^2(r(t))}{2}]ds-\frac{1}{t}\int_0^t b(r(t))u(t)dt\\&+\frac{M_1(t)}{t},\end{aligned}$$

$$M_1(t)=\int_0^t=\sigma_1(r(t))dB_1(t). \quad (10)$$

Continuous local integration

$$<M_1,M_2>t=\int_0^t(r(t))dt. \quad (11)$$

Using the strong number theorem of local martingale, we can know that: $\lim_{t\to\infty}\frac{M_1}{t}=0$. Two-sided limit:

$$\begin{aligned}\lim_{t\to\infty}\frac{\ln u(t)}{t}&=\lim_{t\to\infty}\frac{1}{t}\int_0^t[a(r(t))-\frac{\sigma_1^2(r(t))}{2}]dt\\&-\lim_{t\to\infty}\frac{1}{t}\int_0^t b(r(t))u(t)dt.\end{aligned} \quad (12)$$

We make use of $\ln u(t)$ for using Itô's Eq:

$$\begin{aligned}d(\ln u(t))&=(a-bu(t)-cv(t)(p+\alpha v(t))v(t))dt\\&+u\sigma_1 dB_1(t)\\&\le(a-bu(t)-cv(t)-pv(t)-\frac{\sigma_1^2}{2})dt\\&+\sigma_1 dB_1 t.\end{aligned} \quad (13)$$

Integrate the above formula from 0 to 0 and divide both sides by t:

$$\begin{aligned}\frac{\ln u(t)-\ln u(0)}{t}&\le(a-\frac{\sigma_1^2}{2})-\frac{b}{t}\int_0^t u(t)dt\\&-\frac{c+\sigma_2}{t}\int_0^t v(t)dt+\frac{\sigma_1 B_1(t)}{t}\\&\le(a-\frac{\sigma_1^2}{2})+\frac{\sigma_1 B_1(t)}{t}.\end{aligned} \quad (14)$$

Take its upper bound and use the strong number theorem of the local martingale:

$$\lim_{t\to\infty}\frac{\sigma_1 B_1(t)}{t}=0, \quad (15)$$

$$\limsup_{t\to\infty}\frac{\ln u(t)}{t}\le a-\frac{\sigma_1^2}{2}\le 0. \quad (16)$$

The extinction of the prey has been proved.

Theorem 3.2. *Let* $\sum_{k\in S}\pi_k[a(k)-\frac{\sigma_1^2(k)}{2}]>0$ *hold. If* $\lambda<0$*, then for any initial value (u(0),v(0),r(s))∈* $R_+^2\times S$*, the predator populations goes to extinction almost surely.*

Proof. Then prove the predator extinction for the second equation of the system using Itô's formula:

$$dv=[v(-h+fcu)+m(p+\alpha v)uv]dt+\sigma_2 dB_2(t)$$

$$d(\ln v)\le[fcu+um\alpha(p+\alpha v)]dt+\sigma_2 dB_2(t)' \quad (17)$$

$$\begin{aligned}\frac{\ln v(t)-\ln v(0)}{t}&\le\frac{fc}{t}\int_0^t u(t)dt\\&+\frac{m\alpha}{t}\int_0^t u(t)\int_0^t v(t)dt\\&+\frac{\sigma_2}{t}dB_2(t)\\&\le\frac{fc+m\alpha p}{t}\int_0^t u(t)dt\\&+\frac{m\alpha}{t}\int_0^t u(s)v(t)dt\\&+\frac{\sigma_2}{t}dB_2(t).\end{aligned} \quad (18)$$

Because $\lim_{t\to\infty}\frac{\sigma_2}{t}dB_2(t)\to 0$,in the case,when $\lambda=\frac{fc+m\alpha p}{t}\int_0^t u(t)dt+\frac{m\alpha}{t}\int_0^t u(t)v(t)dt+\frac{\sigma_2}{t}dB_2(t)\le 0$,v(t) tends to zero a.s. This completes the proof of assertion.

4 Stationary Distribution

In this section, we will investigate the ergodic property of system(1.4) by using Lyapunov function method. For the purpose of proving our theorem, we will first introduce a transformation of system (4). Let $x(t) = \ln u(t)$,$y(t) = \ln v(t)$, for $\forall t \geq 0$. Applying Itô's formula to system (1.4), we have

$$\begin{cases} dx(t) = [a - \frac{\sigma_1^2}{2} - be^{x(t)} - e^{y(t)}(c + p + \alpha e^{y(t)})]dt + \sigma_1 dB_1(t), \\ dy(t) = [-h + \frac{\sigma_2^2}{2} + e^{x(t)}(fc + mp + me^{y(t)})]dt + \sigma_2 dB_2(t) \end{cases} \quad (19)$$

Now we impose the condition, [11]:

$$\begin{aligned}(H) : \overline{\lambda} = (fc + mp + me^{y(t)}) \sum_{k\in s} \pi_k [a(k) \\ - \frac{\sigma_1(k)^2}{2}] - b \sum_{k\in s} \pi_k [h(k) \\ - \frac{\sigma_2(k)^2}{2}] > 0\end{aligned} \quad (20)$$

and $min\{h(k) - \frac{\sigma_2(k)^2}{2}\} > 0$

Theorem 4.1. *Let us assume that hypothesis (H) holds. For any $k \in s$ and for any initial value $(u(0), v(0), r(0)) \in R_+^2 \times S$ be given, (u(t),v(t),r(t)) of system (4) is ergodic and has a unique stationary distribution $\in R_+^2 \times S$.*

Proof. We consider the bounded open subset

$$D = (x, y) : |x| \leq \ln \varepsilon^{-1} |y| \leq \ln \varepsilon^{-1} 0 < \varepsilon. \quad (21)$$

Define a C^2-function

$$\begin{aligned}\Phi(x, y) = M[-(fc + mp + e^y)x - by \\ + \frac{(fc + mp + e^y)(c + p + \alpha e^y)}{2} e^y] \\ + [\frac{e^x + qe^y}{2}]^2.\end{aligned} \quad (22)$$

where $q = \frac{c+p+\alpha e^y}{fc+mp+e^y}$ and $\frac{\lambda}{2} \max 2, \sup_{(u,v)\in k^2} -\frac{b}{2} e^{(3x)} - \frac{q^2}{4} \min_{k\in s} h(k) - \frac{\sigma_2(k)^2}{2} e^{2y} + ne^{2x}$ and $n = a + \frac{\sigma_1^2}{2} + \frac{a^2}{2 \min_{k\in s} h(k) - \frac{\sigma_2(k)^2}{2}}$.

By calculating the equation set of partial derivative functions of $\Phi(x, y)$, we know that

$$- \frac{M(fc + mp + me^y)}{e^x} + e^x + \frac{pbe^x}{q(fc + mp + me^y) + \frac{(fc+mp+me^y)(c+p+\alpha e^y)}{h} e^x} = 0 \quad (23)$$

has a unique solution, which can be seen from the monotonically property of the left function. The minimum point of $\Phi(x, y)$ is x_0, $y_0 = \ln(\frac{b}{(fc+mp+me_0^y)(\frac{c+p+\alpha e_0^y}{h} + pe^{-x_0})})$. So We assert that $\Phi(u, v) - \Phi(u_0, v_0) \geq 0$.

Define a $C^2 - fuctionV$:

$$\begin{aligned}V(x, y, k) = M[-FCx - by + \frac{FCC}{h} e^y] \\ + \frac{(e^x + pe^y)^2}{2} \\ - \Phi(u_0, v_0) + M(\varpi_k + \varpi_l) \\ = V_1(x, y) + V_2(x, y) \\ - \Phi(x_0, y_0) + V_3(k),\end{aligned} \quad (24)$$

where $\varpi = (\varpi_1, ..., \varpi_m)^T, |\varpi| = \sqrt{\sum_k = 1^m \varpi_k^2}$ and ϖ_k will be determined in the rest of the proof.

Note that we put $|\varpi_k|$ in order to make $|\varpi|k + |\varpi|$ non-negative. By Itô's formula, we have

$$\begin{aligned}\mathcal{L}_1(x, y) \leq M[-FC[a(k) - \frac{\sigma_1(k)^2}{2}] \\ + b[h(k) + \frac{\sigma_2(k)^2}{2}] \\ + \frac{FFCC^2}{h} e^{x+y}],\end{aligned} \quad (25)$$

and

$$\mathcal{L}V_3(k) = M \sum \gamma_{kl}(\varpi_l - \varpi_k), \quad (26)$$

where $FC = (fc + mp + me^y)$, $C = (c + p + \alpha e^y)$

Let us define the vector $\Lambda = (\Lambda_1, ...\Lambda_m)^T$ with $\Lambda_k = FC[a(k) - \frac{\sigma_1(k)^2}{2}] - b[h(k) + \frac{\sigma_2(k)^2}{2}]$. Since the generator matrix Γ is irreducible, then for Λ_k, there exists $\varpi = (\varpi_1, ..., \varpi_m)^T$ a solution of the poisson system, such that

$$\Gamma \varpi_k - \Lambda = -(\sum_{j=i}^{m} \pi_j \Lambda_j) \quad (27)$$

where $\overrightarrow{1}$ denotes the column vector with all its entries equal to 1. Thus, we have

$$\begin{aligned}\sum_{l\neq k, k\in s} \gamma_k l(\varpi_l - \varpi_k) - (FC[a(k) \\ - \frac{\sigma_1(k)^2}{2}] \\ - b[h(k) + \frac{\sigma_2(k)^2}{2}]) \\ - (FC \sum_{k\in s} \pi_k [a(k) - \frac{\sigma_1(k)^2}{2}] \\ - b \sum (h(k) + \frac{\sigma_2(k)^2}{2})) \\ = -\overline{\lambda}.\end{aligned} \quad (28)$$

Thereby, yields $\mathcal{L}(V_1 + V_3) \leq M(-\overline{\lambda} + \frac{FFCC^2}{h}e^{x+y})$

$$\begin{aligned}\mathcal{L}V_2(x,y) &\leq (e^x + qe^y)[e^x(a(k) - b(k)e^x) \\ &- Ce^{x+y} - ph(k)e^y + pFCe^{x+y}] \\ &+ \frac{\sigma_1(k)^2}{2} + q^2\frac{\sigma_2(k)^2}{2}e^{2y} \\ &\leq -be^{3x} - q^2\min_{k\in s}\{h(k) \\ &- \frac{\sigma_2(k)^2}{2}\}e^{2y} + (a \\ &+ \frac{\sigma_1^2}{2}e^{2x} + +qa)e^{x+y} \\ &\leq -be^{3x} - \frac{q^2}{2}\min_{k\in s}\{h(k) - \frac{\sigma_2(k)^2}{2}\}e^{2y} \\ &+ (a + \frac{\sigma_1^2}{2} + \frac{a^2}{2\min_{k\in s}\{h(k) - \frac{\sigma_2^2}{2}\}})e^{2x} \\ &= -be^{3x} - \frac{q^2}{2}\min h(k) - \frac{\sigma_2(k)^2}{2}e^{2y} + ne^{2x}\end{aligned} \tag{29}$$

Then, we have

$$\begin{aligned}\mathcal{L}V(x,y,k) &\leq -M\overline{\lambda} + M\frac{FFCC^2}{h}e^{x+y} \\ &- be^{3x} - \frac{q^2}{2}\min\{h(k) - \frac{\sigma_2(k)^2}{2}\}e^{2y} \\ &+ ne^{2y}.\end{aligned} \tag{30}$$

In the set $D^c \times S$, we choose sufficiently small ε such that

$$\begin{aligned}&0 < \varepsilon < \frac{h\overline{\lambda}}{4FFCC^2}, \\ &0 < \varepsilon < \frac{q^2h\min h(k) - \frac{\sigma_2(k)^2}{2}}{4MFFCC^2}, \\ &0 < \varepsilon < \frac{bh}{2MFFCC^2}, \\ &-M\overline{\lambda} - \frac{b}{2\varepsilon^2} + k_1 \leq -1, \\ &-M\overline{\lambda} - \frac{q^2}{4}\min\{h(k) - \frac{\sigma_2(k)^2}{2}\}e^{2y} + ne^{2x},\end{aligned}$$

where K_1 and K_2 are positive constants which can be found in. Denote that

$$D_\varepsilon^1 = (x,y) \in R^2 : -\infty \leq x \leq +\infty,$$

$$D_\varepsilon^2 = (x,y) \in R^2 : -\infty < y \leq \ln\varepsilon,$$

$$D_\varepsilon^3 = (x,y) \in R^2 : x \geq \ln\varepsilon^{-1},$$

$$D_\varepsilon^4 = (x,y) \in R^2 : y \geq \ln\varepsilon^{-1}.$$

Obviously, $D^c = D_1 \cup D_2 \cup D_3 \cup D_4$. In the following we prove $\mathcal{L}V(x,y,k) \leq -1$ on D^c.

case1: On $D_\varepsilon^1 \times S$, owing to$e^{x+y} \leq \varepsilon e^y \leq \varepsilon(1 + e^{2y})$, we have

$$\begin{aligned}\mathcal{L}V(x,y,k) &\leq -\frac{M\overline{\lambda}}{4} + (-\frac{M\overline{\lambda}}{4} + \frac{MFFCC^2}{h}) - \frac{b}{2}e^{3x} \\ &+ (-\frac{q^2}{4}\min h(k) - \frac{\sigma_2(k)^2}{2} + \frac{MFFCC^2}{h})e^{2y} \\ &+ [-\frac{M\overline{\lambda}}{2} - \frac{b}{2}e^{3x} - \frac{q^2}{4}\min\{h(k) - \frac{\sigma_2(k)^2}{2}\}e^{2y} \\ &+ ne^{2x}] \\ &\leq -\frac{M\overline{\lambda}}{4} + (-M\overline{\lambda}\frac{MFFCC^2}{h}) - \frac{b}{2}e^{3x} \\ &+ (-\frac{q^2}{4}\min\{h(k) - \frac{\sigma_2(k)^2}{2}\} + \frac{MFFCC^2}{h})e^{2y} \\ &+ [-\frac{M\overline{\lambda}}{2} + \sup -\frac{b}{2}e^{3x} - \frac{q^2}{4}\min h(k) - \frac{\sigma_2(k)^2}{2}e^{2y} \\ &+ ne^{2x}].\end{aligned}$$

We have

$$\begin{aligned}M =& \frac{\lambda}{2}\max\left\{2, \sup_{(u,v)\in k^2} -\frac{b}{2}e^(3x\right) \\ &-\frac{q^2}{4}\min_{k\in s}\left\{h(k) - \frac{\sigma_2(k)^2}{2}\right\}e^{2y} + ne^{2x}\Bigg\}.\end{aligned}$$

Then

$$-\frac{M\overline{\lambda}}{4} \leq -1.$$

case2: For any $(x,y,k) \in D_\varepsilon^2 \times S$, owing to $e^{u+v} \leq \varepsilon e^u \leq \varepsilon(1 + e^{3u})$, we have

$$\mathcal{L}(x,y,k) \leq -\frac{M\overline{\lambda}}{4} - \frac{b}{2}e^{3x} \leq -\frac{M\overline{\lambda}}{4} \leq -1,$$

$$\because e^{x+y} \leq \varepsilon e^x \leq \varepsilon\varepsilon(1 + e^{3x})$$

$$\begin{aligned}\mathcal{L}V(x,y,k) \leq& -\frac{M\overline{\lambda}}{4} + (-M\overline{\lambda}\frac{MFFCC^2\varepsilon}{h}) \\ &+ (-\frac{b}{2} + \frac{MFFCC^2\varepsilon}{h})e^{3x} \\ &- \frac{q^2}{4}\min h(k) - \frac{\sigma_2(k)^2}{2}e^{2y} \\ &+ (-\frac{M\overline{\lambda}}{2} + \sup -\frac{b}{2}e^{3x} \\ &- \frac{q^2}{4}\min_{k\in s}\{h(k) - \frac{\sigma_2(k)^2}{2}\}e^{2y} + ne^{2x}),\end{aligned}$$

$$\begin{aligned}\mathcal{L}V(x,y,k) \leq& [-\frac{M\overline{\lambda}}{4} - \frac{q^2}{4}\min_{k\in s}\{h(k) - \frac{\sigma_2(k)^2}{2}\}e^{2y} \\ &\leq] - \frac{M\overline{\lambda}}{4} \leq -1.\end{aligned}$$

case 3: when$(x, y, k) \in D_\varepsilon^3 \times S$

$$\begin{aligned}\mathcal{L}V(x,y,k) \le & -M\bar{\lambda} - \frac{b}{2}e^{3x} \\ & + (-\frac{b}{2}e^{3x} - \frac{q^2}{4}\min_{k\in s}\{h(k) - \frac{\sigma_2(k)^2}{2}\}e^{2y} \\ & + [n + \frac{(MFFCC^2)^2}{q^2h^2\min h(k) - \frac{\sigma_2(k)^2}{2}}]e^{2y}) \\ & \le -M\bar{\lambda} - \frac{b}{2\varepsilon^3} + k_1 \\ & \le -1.\end{aligned}$$

case4: when $(x, y, k) \in D_\varepsilon^4 \times S$

$$\begin{aligned}\mathcal{L}V(x,y,k) \le & -M\bar{\lambda} - \frac{p^2}{4}\min h(k) - \frac{\sigma_2(k)^2}{2}e^{2y} \\ & + (-be^{3x} + ((n + \frac{MFFCC^2}{q^2h^2\min h(k) - \frac{\sigma_2(k)^2}{2}}]e^{2y})) \\ & \le -M\bar{\lambda} - \frac{p^2}{4}\min h(k) - \frac{\sigma_2(k)^2}{2}\frac{1}{\varepsilon^2} + k_2 \\ & \le -1.\end{aligned}$$

To sum up, we deduce that $\mathcal{L}V(x, y, k) \le -1$,for all $(x, y, k) \in D^c \times S$.

5 Examples and Numerical Simulations

In this section, we use Milsteins Higher Order Method for discretization and the fourthorder RK4 techniques for iteration to perform numerical simulations. The following two examples show extinction and stationary distribution.

Example 1 When the selected parameters satisfy the conditions of Theorems 3.1 and 3.2, prey and predators will become extinct as shown in Figure 1.

Example 2 When the selected parameters satisfy the conditions of Theorem 4.1, prey and predators will appear as shown in Figure 2.

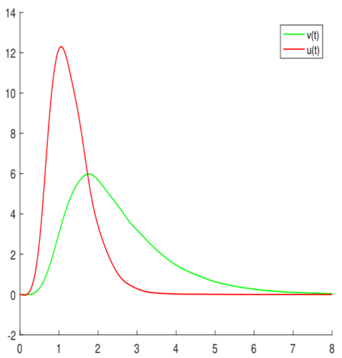

Fig. 1: Extinct

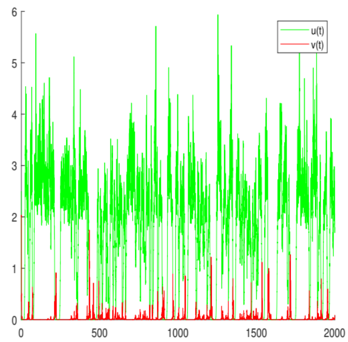

Fig. 2: Stationary distribution

6 Conclusions

In this paper, we establish a novel stochastic prey-predator model with hunting cooperation and regime switching. We establish the existence and uniqueness of the global positive solution, provide sufficient conditions for the extinction of system (1.4). Utilizing the Lyapunov function method, we demonstrate that the stochastic system with switching possesses a unique stationary distribution, marking the first attempt to address this issue. Based on our research findings, future studies can explore the following directions: further investigation into the stability and dynamical behavior of stochastic systems with switching; exploration of alternative methods to prove the unique stationary distribution of stochastic systems; consideration of more complex models on the model.

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Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.