Dirichlet Functions Generated by Blaschke Products

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Abstract: The continuation of general Dirichlet series to meromorphic functions in the complex plane remains an outstanding problem. It has been completely solved only for Dirichlet L-series. A sufficient condition for the general case exists, however it is impossible to verify that it is fulfilled. We solve this problem here for another class of general Dirichlet series, namely those series which are obtained from infinite Blaschke products by a particular change of variable. This is a source of examples of general Dirichlet series with infinitely many poles. An interesting new case is now revealed, in which the singular points of the extended function form a continuum. We take a closer look at the case of Dirichlet series with natural boundary and give examples of such series. Some figures illustrate the theory.

Key-Words: Dirichlet series, Blaschke product, Dirichlet function, conformal mapping, fundamental domains, natural boundary.

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1 Introduction

A *general Dirichlet series* is an expression of the form

$$\zeta_{A,\Lambda}(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \tag{1}$$

where $A = (a_n)$ is an arbitrary sequence of complex numbers, $a_n \neq 0$ infinitely many times and

$$\Lambda = \{0 = \lambda_1 \leq \lambda_2 \leq ...\}$$

is a nondecreasing sequence of nonnegative numbers such that $\lim_{n\to\infty} \lambda_n = +\infty$. We will deal only with normalized Dirichlet series for which $a_1 = 1$. It is obvious that for such a series we have

$$\lim_{\sigma\to+\infty} \zeta_{A,\Lambda}(\sigma + it) = 1. \tag{2}$$

If $\zeta_{A,\Lambda}(s)$ does not converge for $s = 0$, then [1], [2], [3], [4], [5], [6], [7],

$$\sigma_c = \limsup \ln \left| \sum_{k=1}^{n} a_k \right|^{\frac{1}{\lambda_n}},$$

is called the *abscissa of convergence* of the series (1). The series converges for every $s = \sigma+it$ with $\sigma > \sigma_c$ and it diverges for every $s = \sigma + it$ with $\sigma < \sigma_c$. Obviously, if $\sigma_c = +\infty$, the series does not converge anywhere. If the series converges at $s = 0$, then the abscissa of convergence is

$$\sigma_c = \limsup_{n\to\infty} \frac{1}{\lambda_{n+1}} \ln \left| \zeta_{A,\Lambda}(0) - \sum_{k=1}^{n} a_k \right| < 0.$$

To every series (1), a series

$$\zeta_{A,e^{\Lambda}}(s) = \sum_{n=1}^{\infty} a_n e^{-e^{\lambda_n} s} \tag{3}$$

is associated, where the exponents λ_n are replaced by e^{λ_n}. It is known, [8], that if the abscissa of convergence σ_c of the series (1) is a finite number, then the abscissa of convergence of the series (3) is zero. Moreover, if the series (3) converges everywhere on the imaginary axis, except for some isolated points, then the series (1) can be extended to a meromorphic function in the whole complex plane.

We call the extended function *Dirichlet function*. When $\lambda_n = \ln n$ the series (1) becomes

$$\zeta_A(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

and it is called *ordinary Dirichlet series*. The abscissa of convergence of this series is

$$\sigma_c = \frac{\limsup \left[\ln \left| \sum_{k=1}^{n} a_k \right|\right]}{\ln n}.$$

The ordinary Dirichlet series with

$$a_n = 1, \quad n = 1, 2, ...,$$

has the abscissa of convergence $\sigma_c = 1$ and it admits an analytic continuation in $\mathbb{C}\backslash\{1\}$, which is the famous *Riemann Zeta function,* $\zeta(s)$. A lot of studies have been devoted to this function, [9], [10], due to its applications in number theory, statistics and physics. One of the most unbelievable properties of $\zeta(s)$ is that its all nonreal zeros are located on the line $\Re s = 1/2$ (the Riemann Hypothesis). It has been shown, [11], that this property is common to all Dirichlet functions satisfying a Riemann type of functional equation and which can be written as an Euler product. The so-called *Dirichlet L-series* satisfy to these conditions. These are series whose coefficients a_n are *Dirichlet characters* $\chi(n)$, i.e. periodic functions of some period q, whose values are zero or some roots of order q of the unity. When these values are just 0 and 1, the abscissa of convergence of the series is 1, otherwise, it is 0. Every one of these series admits an analytic continuation to the whole complex plane, except for one simple pole at 0 or 1. The continued meromorphic function is called *Dirichlet L-function.*

In this paper we deal with general Dirichlet series which have infinitely many poles or do not admit analytic continuation across the convergence line.

Example 1 and Example 2 below display Dirichlet functions having infinitely many poles, all located on the convergence line. In our knowledge, this is something new. These functions are obtained from some Blaschke products by a change of variable. Those Blaschke products are meromorphic functions in the complex plane and so are the Dirichlet functions generated by them. Then, the original Dirichlet series are automatically extended to meromorphic functions in the whole complex plane. This way of generating Dirichlet functions can be used starting from any Blaschke product, finite or infinite, which does not cancel at the origin. In the case of infinite Blaschke products, the abscissa of convergence of the Dirichlet series is zero. Beside the poles on the line of convergence, the Dirichlet function obtained by a change of variable has also non isolated essential singular points corresponding to the cluster points of poles of the original Blaschke product. Again, this has not been encountered in the literature until now. Moreover, these essential singular points may be found almost everywhere on the line of convergence, making impossible the continuation of the Dirichlet series across that line.

We have shown in [12], that the series

$$\sum_{n=0}^{\infty} e^{-2^n s}$$

is such a series, for which the abscissa of convergence is 0. Indeed, this is a Dirichlet series with $a_k = 0$ if $k \neq 2^n$, $a_{2^n} = 1$ and $\lambda_{2^n} = 2^n$. It is known that the Hadamard series

$$h(z) = \sum_{n=0}^{\infty} z^{2^n}$$

has the unit circle as natural boundary. We obtain the Dirichlet series

$$\sum_{n=0}^{\infty} e^{-2^n s}$$

by changing the variable $z = re^{i\theta}$ in the Hadamard series with $e^{-s} = e^{-\sigma}e^{-it}$. We notice that $r \to 1$ if and only if $\sigma \to 0$ and this is true for any t. We have seen in [12], that

$$\lim_{r\to 1} h\left(re^{\frac{2k\pi i}{2^n}}\right) = \infty$$

for every $n \in \mathbb{N}$ and every $k = 1, 2, ..., 2^n$, i.e.,

$$\lim_{r\to 1} h(re^{i\theta}) = \infty$$

for almost every $\theta \in \mathbb{R}$. This implies

$$\lim_{\sigma\to 0} \sum_{n=0}^{\infty} e^{-2^n(\sigma+it)} = \infty$$

for almost every $t \in \mathbb{R}$.

This is a clear example of Dirichlet series for which the convergence line is a natural boundary. Any power series

$$\sum_{n=1}^{\infty} a_n(z - z_0)^n,$$

having the convergence radius R and the circle of convergence $|z - z_0| = R$ as natural boundary, can be converted by the change of variable $z - z_0 = e^{-s}$ into a Dirichlet series with the natural boundary $\Re s = \ln \frac{1}{R}$.

The geometry of the mappings by the Dirichlet L-functions is relatively well known, [13]. Its description is based on the local behavior, [14], of an analytic function. Namely, any complex function $f(s)$ which is analytic in a neighborhood of a point s_0 and such that $f'(s_0) \neq 0$ is locally injective at s_0, which means that it maps conformally, hence bijectively, a certain neighborhood of s_0 onto an open disk. If s_0 is a zero of order m of $f'(s)$, then m Jordan arcs starting at s_0 divide that neighborhood into m domains which are mapped conformally onto the same disk with a radial slit, Fig. 1. Something similar happens when s_0 is an isolated pole of order m of $f(s)$, except that instead of a disk we have the exterior of a disk, Fig. 2.

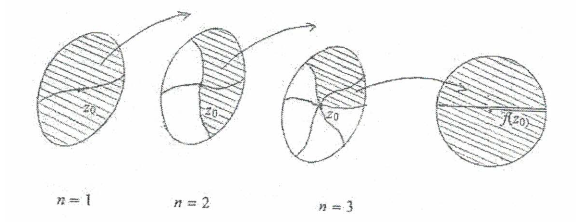

Fig. 1: Local conformal mapping by an analytic function in the neighborhood of a zero of order n

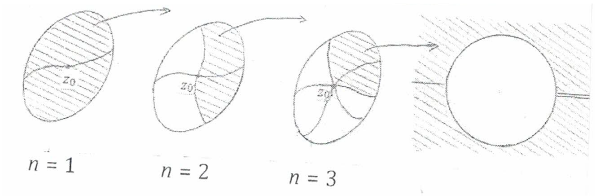

Fig. 2: Local conformal mapping by an analytic function in the neighborhood of a pole of order n

Since any Dirichlet function $\zeta_{A,\Lambda}(s)$ is analytic in the complex plane except for some poles its local behavior is that illustrated by Fig. 1 and Fig. 2. Obviously, any pole of $\zeta_{A,\Lambda}(s)$ cannot exist for $\Re s > \sigma_c$. However, if $\zeta_{A,\Lambda}(s)$ admits a meromorphic continuation in the whole plane, at least one pole of the extended function must appear. In our knowledge, there is no concrete example in the literature where more than one pole exists.

The purpose of this paper is to give such examples and also to study the geometry of the mappings by Dirichlet functions with natural boundary.

The applications in science are for the moment unclear for us. However, having in view the applications in science of the Riemann Zeta function, which is a particular example of Dirichlet function, such applications are easily conceivable.

When studying Dirichlet series (1) it is useful to compare them with Dirichlet series whose coefficients are $\overline{a}_n$. Let us denote by $\zeta_{\overline{A},\Lambda}(s)$ such a series. When a_n are all real we have

$$\zeta_{\overline{A},\Lambda}(s) = \zeta_{A,\Lambda}(s).$$

Since

$$\overline{\sum_{k=1}^{n} a_k} = \sum_{k=1}^{n} \overline{a_k},$$

the abscissa of convergence of the two series is the same. Let us notice also that

$$\overline{\zeta_{A,\Lambda}(s)} = \zeta_{\overline{A},\Lambda}(\overline{s}).$$

2 Dirichlet Functions with Several Singular Points

Let

$$B(z) = \prod_{k=1}^{m\leq\infty} e^{-i\alpha_k} \frac{z - z_k}{1 - \overline{z}_k z}$$

be a Blaschke product with the zeros $z_k = r_k e^{i\alpha_k}$, $0 \leq r_k < 1$, which in the case $m = \infty$ satisfies the Blaschke condition of convergence, [15].

$$\sum_{k=1}^{\infty}(1 - r_k) < \infty.$$

Theorem 1. The Taylor series

$$B(z) = \sum_{n=0}^{\infty} a_n z^n,$$

where $a_n = \frac{B^{(n)}(0)}{n!}$, converges for $|z| < 1$ and it can be converted into the Dirichlet series

$$\zeta_{A,\mathbb{N}}(s) = \sum_{n=0}^{\infty} a_n e^{-ns} \tag{4}$$

by the change of variable $z = e^{-s}$. The abscissa of convergence of this Dirichlet series is $\sigma_c = 0$ when $m = \infty$ and the unit circle from the z-plane is carried by this change of variable into the imaginary axis of the s-plane. For every singular point $\zeta_0 = e^{i\alpha_0}$ of $B(z)$ on the unit circle there are infinitely many points $s_m = i(m\pi - \alpha_0)$, $m \in \mathbb{Z}$ on the imaginary axis which are singular for $\zeta_{A,\mathbb{N}}(s)$.

Proof: Indeed, $B(z)$ is analytic in the unit disk and if $m = \infty$, it has at least one singular point on the unit circle, which is a cluster point of the zeros z_k. The Taylor series of $B(z)$ coincides with $B(z)$ in a neighborhood of 0, and its convergence radius is at least 1, since there is no singular point of $B(z)$ in the open unit disk. On the other hand, when $m = \infty$, since there are singular points of $B(z)$ on the unit circle, this radius cannot be greater than 1. Thus, the Taylor series of $B(z)$ converges exactly in the unit disk. By the change of variable $z = e^{-s}$ we get

$$B(e^{-s}) = \sum_{n=0}^{\infty} a_n e^{-ns} = \zeta_{A,\mathbb{N}}(s)$$

and this series converges for

$$|e^{-s}| = |e^{-\sigma-it}| = e^{-\sigma} < 1,$$

i.e., for $\sigma > 0$, and diverges for $\sigma < 0$, hence the abscissa of convergence of $\zeta_{A,\mathbb{N}}(s)$ is $\sigma_c = 0$. For any real number α_0 we have that $s_j = i(2j\pi - \alpha_0)$ is a singular point for $\zeta_{A,\mathbb{N}}(s)$ if an only if $e^{i\alpha_0}$ is a singular point for $B(z)$.

When m is finite, the abscissa of convergence of $\zeta_{A,\mathbb{N}}(s)$ is $\sigma_c < 0$ and the only singular points of $\zeta_{A,\mathbb{N}}(s)$ are those satisfying the equation

$$e^{-s} = \frac{1}{\overline{z_k}}.$$

□

Example 1.
Let

$$B(z) = \left(\frac{pz-1}{p-z}\right)^2,$$

where $p > 1$ be a Blaschke product of degree 2.

Differentiating $B(z)$ for $z \neq p$, we get

$$B'(z) = 2(p^2-1)\frac{pz-1}{(p-z)^3}.$$

Suppose that

$$B^{(k)}(z) = \frac{k!(p^2-1)[2pz + (k-1)p^2 - (k+1)]}{(p-z)^{k+2}}. \tag{5}$$

It can be easily checked that this is true for $k = 1$. Then

$$B^{(k+1)}(z) = \frac{k!(p^2-1)}{(p-z)^{2k+4}}\{2p(p-z)^{k+2} + (k+2)(p-z)^{k+1}[2pz + (k-1)p^2 - (k+1)]\}$$

After simplification with $(p-z)^{k+1}$ we get

$$\begin{aligned}B^{(k+1)}(z) &= \frac{k!(p^2-1)}{(p-z)^{k+3}}[2p(p-z) + 2p(k+2)z + (k-1)(k+2)p^2 - (k+1)(k+2)] \\ &= \frac{k!(p^2-1)}{(p-z)^{k+3}}[2p(k+1)z + k(k+1)p^2 - (k+1)(k+2)] \\ &= \frac{(k+1)![2pz + kp^2 - (k+2)]}{(p-z)^{k+3}}\end{aligned}$$

which is (5) with k replaced by $k+1$, and this shows that indeed (5) is true for every $k \geq 1$.

In particular,

$$B^{(k)}(0) = \frac{k!(p^2-1)[(k-1)p^2 - (k+1)]}{p^{k+2}},$$

thus the Taylor series of $B(z)$ is

$$B(z) = \frac{1}{p^2} - \frac{2(p^2-1)}{p^3}z + (p^2-1)\sum_{k=2}^{\infty}[(k-1)p^2 - (k+1)]\frac{z^k}{p^{k+2}}$$

and it can be easily checked that its radius of convergence is p. The function $B(z)$ is a meromorphic function in the complex plane having the unique double pole $z = p$ and the unique double zero $z = \frac{1}{p}$.

We note that

$$B(e^{-s}) = \frac{1}{p^2} - \frac{2(p^2-1)}{p^3}e^{-s} + (p^2-1) \cdot \sum_{k=2}^{\infty}[(k-1)p^2 - (k+1)]\frac{1}{p^{k+2}}e^{-ks}$$

is a Dirichlet series with $a_1 = \frac{1}{p^2}$.

Multiplying $B(e^{-s})$ by p^2 we obtain a normalized Dirichlet series

$$\zeta_{A,\mathbb{N}}(s) = \sum_{n=1}^{\infty} a_n e^{-ns}$$

where $a_1 = 1$, $a_2 = -2\left(p - \frac{1}{p}\right)$ and for $n \geq 3$,

$$a_n = \frac{(p^2-1)[(n-2)p^2 - n]}{p^{n-1}}.$$

For $\zeta_{A,\mathbb{N}}(s)$ we have

$$\lim \sigma_{\sigma\to+\infty}\zeta_{A,\mathbb{N}}(\sigma + it) = 1$$

for every $t \in \mathbb{R}$. Moreover, $\zeta_{A,\mathbb{N}}(s) = -1$ implies

$$p\frac{pe^{-s}-1}{p-e^{-s}} = \pm i$$

or

$$\begin{aligned}&p^2e^{-s} - p = \pm i(p - e^{-s}), \\ &(p^2 \pm i)e^{-s} = p(1 \pm i), \\ &e^s = \frac{p^2 \pm i}{p(1 \pm i)}.\end{aligned}$$

In both cases

$$\sigma = \ln\frac{p^2+1}{2p} > 0.$$

Thus, every component of the preimage by $\zeta_{A,\mathbb{N}}(s)$ of the unit circle is a parabola-shaped curve having the vertex on the line

$$\sigma = \ln\frac{p^2+1}{2p}$$

and the branches tend asymptotically to the lines $t = (2k+1)\pi i$.

The fundamental domains of $B(z)$ are the upper and the lower half-planes, which are mapped conformally by $B(z)$ onto the complex plane with a slit alongside the interval $(0, +\infty)$ of the real axis. On the other hand, the function e^{-s} maps conformally every strip

$$\{s|s = \sigma + it, \sigma \in \mathbb{R}, t \in (2k\pi, (2k+1)\pi)\}$$

onto the upper half-plane and every strip

$$\{s|s = \sigma + it, \sigma \in \mathbb{R}, t \in ((2k-1)\pi, 2k\pi)\}$$

onto the lower half-plane. Summing up, the fundamental domains of $\zeta_{A,\mathbb{N}}(s)$, Fig. 3, are horizontal strips of height π which are mapped conformally by $\zeta_{A,\mathbb{N}}(s)$ onto the complex plane with a slit alongside some interval of the real axis. A little computation shows that the lines

$$s = \sigma + (2k+1)\pi i$$

are mapped by $\zeta_{A,\mathbb{N}}(s)$ onto the interval $(1, p^4)$, while the image of the lines

$$s = \sigma + 2k\pi i$$

is the whole real half-axis and the intervals $(0, 1)$ and $(p^4, +\infty)$ are covered twice. Thus every strip

$$\{s = \sigma + it | k\pi < t < (k+1)\pi\}$$

is mapped conformally by $\zeta_{A,\mathbb{N}}(s)$ onto the whole complex plane with a slit alongside the positive real half-axis. The boundaries of these strips are components of the preimage by $\zeta_{A,\mathbb{N}}(s)$ of the real axis.

The zeros and the poles of $\zeta_{A,\mathbb{N}}(s)$ belong to these boundaries and they are double zeros and respectively double poles. A small half-disk centered at $\ln p + 2k\pi i$ is mapped conformally by $\zeta_{A,\mathbb{N}}(s)$ onto a neighborhood of the origin with a slit alongside the positive real half-axis, while a small half-disk centred at $-\ln p$ is mapped conformally onto the exterior of a closed curve containing the origin with a slit alongside the positive real half-axis.

The preimage by $\zeta_{A,\mathbb{N}}(s)$ of the circle of radius p^2 centered at the origin is the imaginary axis. Indeed, every segment $(2k\pi, (2k+1)\pi]$ and $((2k-1)\pi, 2k\pi]$ of the imaginary axis is mapped by e^{-s} onto the half unit circle, which is mapped by $B(z)$ onto the full unit circle and by $p^2 B(z)$ onto the circle centered at the origin and of radius p^2.

It is expected that a similar configuration will be produced by the Dirichlet functions $p^q B(e^{-s})$ generated by any Blaschke product of the form

$$B(z) = \left(\frac{pz-1}{p-z}\right)^q$$

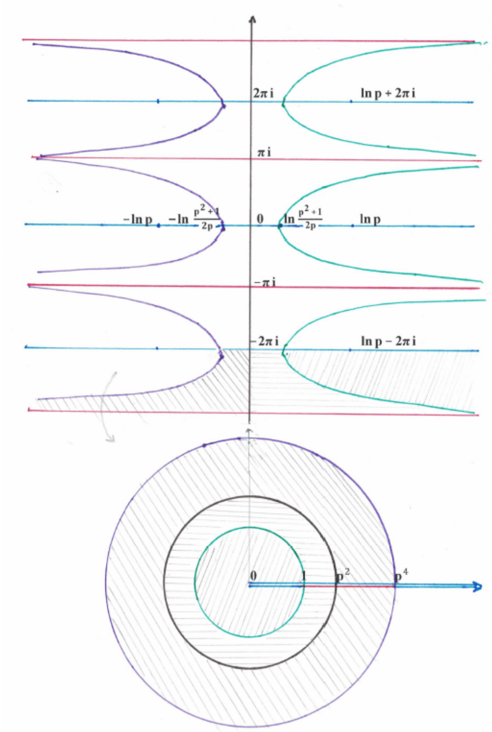

Fig. 3: An illustration of the fundamental domains of the Blaschke product generated Dirichlet function and their conformal mappings

for arbitrary integer $q > 2$. The points $\ln p + 2k\pi i$ are this time zeros of order q of the function, while the points $-\ln p + 2k\pi i$ are poles of order q. The geometry of the mapping in the neighborhood of these points is that shown in Fig. 1 and Fig. 2. This implies a change in the preimage of the real axis in the sense that q components of that preimage will pass through the points $\pm \ln p + 2k\pi i$. These are all components of the preimage of the positive real half-axis when q is even and of both, positive and negative real half axis when q is odd. We illustrate this affirmation for the case $q = 3$ in Fig. 4 and of the case $q = 4$ in Fig. 5.

These preimages border q fundamental domains of $\zeta_{A,\mathbb{N}}(s)$ in every horizontal strip of width 2π. The conformal mapping of these domains by $\zeta_{A,\mathbb{N}}(s)$ is illustrated in Fig. 4 and Fig. 5 below.

Example 2.

Let us study the Dirichlet function defined by the Blaschke product

$$B(z) = \left(\frac{az-1}{\overline{a}-z}\right)^2,$$

where $a = re^{i\alpha}$, $r > 1$, $\alpha \in \mathbb{R}$.

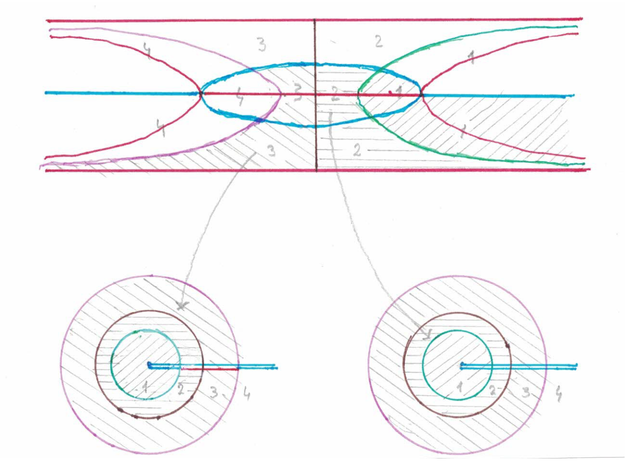

Fig. 4: The fundamental domains of $p^{3}\left(\frac{pe^{-s}-1}{p-e^{-s}}\right)^{3}$ in the strip $\{s|s=\sigma+it, \sigma\in\mathbb{R}, -\pi\le t\le\pi\}$ and their conformal mapping

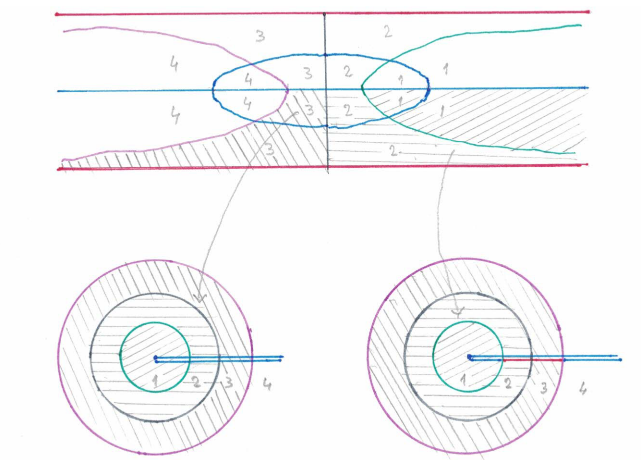

Fig. 5: The fundamental domains of $p^{4}\left(\frac{pe^{-s}-1}{p-e^{-s}}\right)^{4}$ in the strip $\{s|s=\sigma+it, \sigma\in\mathbb{R}, -\pi\le t\le\pi\}$ and their conformal mapping

This is the function

$$\zeta_{A,\mathbb{N}}(s) = \overline{a}^2\left(\frac{ae^{-s}-1}{\overline{a}-e^{-s}}\right)^2$$
$$= 1+\sum_{n=1}^{\infty} a_n e^{-ns}, s=\sigma+it,$$

where

$$a_n = \frac{\overline{a}^2}{n!}B^{(n)}(0)$$

Differentiating $B(z)$ for $z\neq\overline{a}$ we obtain

$$B'(z) = 2(r^2-1)\frac{az-1}{(\overline{a}-z)^3}$$

and an induction argument shows that

$$B^{(n)}(z) = \frac{n!(r^2-1)[2az+(n-1)r^2-(n+1)]}{(\overline{a}-z)^{n+2}},$$

thus,

$$a_n = \frac{(r^2-1)[(n-1)r^2-(n+1)]}{\overline{a}^n},$$

for $n=1,2,3,....$

Let us notice that

$$\lim_{\sigma\to+\infty}\zeta_{A,\mathbb{N}}(\sigma+it) = 1$$

for every $t\in\mathbb{R}$. Moreover, $\zeta_{A,\mathbb{N}}(s)=0$ if and only if $ae^{-s}=1$, i.e. $e^{\sigma+it}=re^{i\alpha}$, which is $\sigma=\ln r$ and $t=\alpha+2k\pi$. The points

$$s_k = \ln r+(\alpha+2k\pi)i$$

are all double zeros of $\zeta_{A,\mathbb{N}}(s)$. The symmetric points with respect to the imaginary axis, which are

$$s'_k = -\ln r+(\alpha+2k\pi)i$$

are double poles of $\zeta_{A,\mathbb{N}}(s)$. A small half circle around s_k is mapped by $\zeta_{A,\mathbb{N}}(s)$ one to one onto a closed curve around the origin, while a small half circle around s'_k is mapped by $\zeta_{A,\mathbb{N}}(s)$ one to one onto a closed curve around ∞. The line through s_k and s'_k is mapped by $\zeta_{A,\mathbb{N}}(s)$ onto the positive real half axis. Indeed,

$$\zeta_{A,\mathbb{N}}(\sigma+i(\alpha+2k\pi)) =$$
$$= r^2e^{-2i\alpha}\left(\frac{re^{i\alpha}e^{-\sigma}e^{-i(\alpha+2k\pi)}-1}{re^{-i\alpha}-e^{-\sigma}e^{-i(\alpha+2k\pi)}}\right)^2$$
$$= r^2\left(\frac{re^{-\sigma}-1}{r-e^{-\sigma}}\right)^2 \geq 0$$

and the equation

$$\zeta_{A,\mathbb{N}}(\sigma+i(\alpha+2k\pi))=h$$

has two solutions for every $h\geq 0, h\neq r^2$, and no solution for $h<0$.

The lines

$$L_{2k}:s=\sigma+2k\pi i, k\in\mathbb{Z},\sigma\in\mathbb{R}$$

are mapped one to one by $w=\zeta_{A,\mathbb{N}}(s)$ onto a Jordan arc having the end points at $w=1$ and $w=r^4$.

Indeed,

$$w=\zeta_{A,\mathbb{N}}(\sigma+2k\pi i)=r^2e^{-2i\alpha}\left(\frac{re^{i\alpha}e^{-\sigma}-1}{re^{-i\alpha}-e^{-\sigma}}\right)^2$$

and for $\sigma=0$ we have

$$\zeta_{A,\mathbb{N}}(2k\pi i)=r^2e^{-2i\alpha}\left(\frac{re^{i\alpha}-1}{re^{-i\alpha}-1}\right)^2,$$

which is for every $k\in\mathbb{Z}$ the same point w_0 on the circle $|w|=r^2$. Performing continuations along every line L_k starting from w_0 and keeping in mind that

$$\lim_{\sigma\to+\infty}\zeta_{A,\mathbb{N}}(\sigma+2k\pi i)=1$$

and

$$\lim_{\sigma\to-\infty}\zeta_{A,\mathbb{N}}(\sigma+2k\pi i)=r^4,$$

we obtain the respective Jordan arc.

The lines

$$L_{2k+1}:s=\sigma+(2k+1)\pi i$$

are mapped one to one by $w=\zeta_{A,\mathbb{N}}(s)$ onto another arc with the end points at $w=1$ and $w=r^4$. Indeed,

$$\begin{aligned}w&=\zeta_{A,\mathbb{N}}(\sigma+(2k+1)\pi i)\\&=r^2e^{-2i\alpha}\left(\frac{-re^{i\alpha}e^{-\sigma}-1}{re^{-i\alpha}+e^{-\sigma}}\right)^2\end{aligned}$$

and for $\sigma=0$ we have

$$\begin{aligned}\zeta_{A,\mathbb{N}}((2k+1)\pi i)&=r^2e^{-2i\alpha}\left(\frac{re^{i\alpha}+1}{re^{-i\alpha}+1}\right)^2\\&=r^2\left(\frac{re^{i\alpha}+1}{re^{-i\alpha}+1}\right)^2,\end{aligned}$$

which is, again, for every $k\in\mathbb{Z}$ the same point w_1 on the circle $|w|=r^2$. Performing continuations along L_{2k+1} starting from w_1 we obtain the respective Jordan arc.

The horizontal lines through $\ln r+(\alpha+(2k+1)\pi)i$ are mapped one to one by $\zeta_{A,\mathbb{N}}(s)$ onto the interval $(1,r^4)$ of the real axis. Indeed,

$$\begin{aligned}&\zeta_{A,\mathbb{N}}(\sigma+(\alpha+(2k+1)\pi)i)=\\&=r^2e^{-2i\alpha}\left(\frac{re^{i\alpha}e^{-(\sigma+(\alpha+(2k+1)\pi)i}-1}{re^{-i\alpha}-e^{-(\sigma+(\alpha+(2k+1)\pi)i}}\right)^2\\&=r^2\left(\frac{re^{-\sigma}+1}{r+e^{-\sigma}}\right)^2>0\end{aligned}$$

and for $\sigma=0$ we have

$$\zeta_{A,\mathbb{N}}((\alpha+(2k+1)\pi)i)=r^2.$$

Performing continuations along those lines starting from $w=r^2$ and taking into account that

$$\lim_{\sigma\to+\infty}\zeta_{A,\mathbb{N}}(\sigma+2k\pi i)=1$$

and

$$\lim_{\sigma\to-\infty}\zeta_{A,\mathbb{N}}(\sigma+2k\pi i)=r^4,$$

we obtain the desired result.

The fundamental domains of $\zeta_{A,\mathbb{N}}(s)$ are the horizontal strips between consecutive lines L_k. They are mapped conformally by $\zeta_{A,\mathbb{N}}(s)$ onto the complex plane with a slit alongside the positive real half axis. It is easier to visualize the fundamental domains between consecutive lines

$$L_k^{(\alpha)}:s=\sigma+(k\pi+\alpha)i.$$

Fig. 6 below portrays the conformal mapping of the domain between $L_{-3}^{(\alpha)}$ and $L_{-2}^{(\alpha)}$. It shows the components belonging to that domain of the preimage of the unit disk and of the disks centered at the origin and having the radius r^2 and r^4. The component of the preimage of the exterior of this last disk is also visible. We can distinguish as well the image by $\zeta_{A,\mathbb{N}}(s)$ of the line L_{-2}.

The Dirichlet function generated by a Blaschke product of the form

$$B(z)=\left(\frac{az-1}{\overline{a}-z}\right)^q$$

for an integer $q>2$, where $a=re^{i\alpha}$, $r>1$, $\alpha\in\mathbb{R}$, can be obtained in a similar way. It is

$$\zeta_{A,\mathbb{N}}(s)=\overline{a}^qB(e^{-s}).$$

Every point

$$s_k=\ln r+(2k\pi+\alpha)i$$

is a zero of order q of $\zeta_{A,\mathbb{N}}(s)$, while the points

$$s_k'=-\ln r+(2k\pi+\alpha)i$$

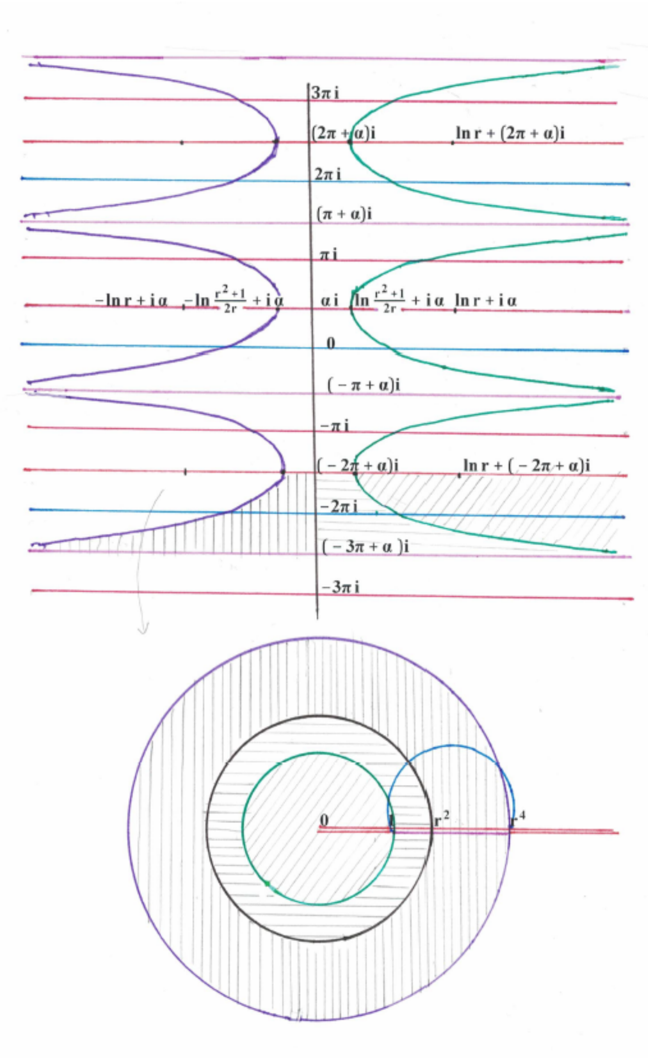

Fig. 6: The fundamental domains of the Dirichlet function $\overline{a}^2\left(\frac{ae^{-s}-1}{\overline{a}-e^{-s}}\right)^2$ and their conformal mapping

are poles of order q. The configurations in a neighborhood of these points are those described in Fig. 1 and Fig. 2 and the global mapping properties of $\zeta_{A,\mathbb{N}}(s)$ are those illustrated in Fig. 4 and Fig. 5.

Theorem 2. Let $B(z)$ be an infinite Blaschke product satisfying the Blaschke condition and let E be the set of cluster points of the zeros z_k of $B(z)$. If $\mathbb{C}\backslash E$ contains an arc of the unit circle, then there is a unique function $\widetilde{B}(z)$ meromorphic in $\overline{\mathbb{C}}\backslash E$, which coincides with $B(z)$ in the unit disk and such that the points $\frac{1}{\overline{z}_k}$ are poles of $\widetilde{B}(z)$ of the same order as the zeros z_k of $B(z)$.

Proof: Let D be the open unit disk and ∂D be the unit circle. The function defined by

$$\widetilde{B}(z) = \frac{1}{\overline{B\left(\frac{1}{\overline{z}}\right)}}$$

for $|z| > 1$ and

$$\widetilde{B}(z) = B(z)$$

for $|z| < 1$, satisfies to these conditions in $\mathbb{C}\backslash\partial D$. Indeed, it is meromorphic for $|z| \neq 1$ since its only singular points for $|z| \neq 1$ are the points $\frac{1}{\overline{z}_k}$ for which we get

$$\widetilde{B}\left(\frac{1}{\overline{z}_k}\right) = \frac{1}{\overline{B(z_k)}} = \frac{1}{B(z_k)}.$$

This last equality is due to the fact that

$$B(z_k) = 0 = \overline{B(z_k)}.$$

This shows that $\frac{1}{\overline{z}_k}$ is a pole of $\widetilde{B}(z)$ of the same order as the zero z_k of $B(z)$.

We need to define $B(z)$ on $\partial D\backslash E$. We notice that for every n, the function

$$B_n(z) = \prod_{k=1}^{n} e^{-i\alpha_k}\frac{z - z_k}{1 - \overline{z}_k z}$$

is defined in the whole complex plane as a meromorphic function having the poles $\frac{1}{\overline{z}_k}$, $k = 1, 2, ..., n$ and

$$B_n(z) = \frac{1}{\overline{B_n\left(\frac{1}{\overline{z}}\right)}} = \widetilde{B_n}(z).$$

Indeed, this is straightforward since for every factor

$$e^{-i\alpha_k}\frac{z - z_k}{1 - \overline{z}_k z}$$

we have

$$\frac{1}{\overline{e^{-i\alpha_k \frac{\frac{1}{\overline{z}}-z_k}{1-\frac{\overline{z}_k}{\overline{z}}}}}} = e^{-i\alpha_k}\frac{z-z_k}{1-\overline{z}_k z}.$$

Moreover, for every $\zeta = e^{i\theta}$ for which $e^{i\varphi} = B_n(\zeta)$ we have

$$\widetilde{B}_n(\zeta) = \frac{1}{\overline{B_n\left(\frac{1}{\overline{\zeta}}\right)}} = \frac{1}{\overline{B_n\left(\frac{1}{e^{-i\theta}}\right)}} = \frac{1}{\overline{B_n(e^{i\theta})}} = \frac{1}{\overline{e^{i\varphi}}} = e^{i\varphi} = B_n(\zeta),$$

thus

$$B_n(z) = \widetilde{B_n}(z)$$

everywhere. Since

$$B(z) = \lim_{n\to\infty} B_n(z)$$

for $|z| < 1$, we have also

$$\widetilde{B}(z) = \lim_{n\to\infty} \widetilde{B}_n(z)$$

for $|z| \neq 1$.

To define $B(z)$ on $\partial D\backslash E$, let us notice that by Cauchy integral formula, for every $\zeta \in \partial D\backslash E$ we have

$$B_n(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{\widetilde{B_n}(z)}{z-\zeta}dz,$$

where γ is a circle centered at ζ and included in that neighborhood. Letting $n \to \infty$ under the integral sign shows that there is $\lim_{n\to\infty} B_n(\zeta)$, which is by definition $B(\zeta)$ and we have

$$B(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{\widetilde{B}(z)}{z-\zeta}dz.$$

Moreover, $B_n'(\zeta)$ exists and

$$B_n'(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{B_n(z)}{(z-\zeta)^2}dz.$$

Letting $n \to \infty$ in this formula, we get

$$\widetilde{B}'(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{\widetilde{B}(z)}{(z-\zeta)^2}dz,$$

thus the function $\widetilde{B}(z)$ is analytic on $\partial D\backslash E$ and meromorphic in $\mathbb{C}\backslash E$. Regarding its behavior at ∞, we have

$$\lim_{z\to\infty} \widetilde{B}(z) = \lim_{z\to\infty} \frac{1}{\overline{B\left(\frac{1}{\overline{z}}\right)}} = \frac{1}{\prod_{n=1}^{\infty} r_n} = \frac{1}{\overline{B(0)}},$$

which shows that $\widetilde{B}(z)$ is a meromorphic function in $\overline{\mathbb{C}}\backslash E$. □

Theorem 3. Let $\zeta_{A,\mathbb{N}}(s)$ be a Dirichlet function obtained from a Blaschke product $B(z)$ by the change of variable $z = \phi(s) = e^{-s}$. Let E be the set of cluster points of zeros of $B(z)$. If $\mathbb{C}\backslash E$ contains an arc of the unit circle on which $B(z)$ is defined, then $\zeta_{A,\mathbb{N}}(s)$ admits a meromorphic continuation to $\overline{\mathbb{C}}\backslash\phi^{-1}(E)$.

Proof: Indeed, under this hypothesis, by Theorem 2, $B(z)$ admits a meromorphic continuation $\widetilde{B}(z)$ to $\overline{\mathbb{C}}\backslash E$. If $B(z)$ is a finite Blaschke product, $E = \varnothing$ and $\widetilde{B}(z)$ is a meromorphic function in $\overline{\mathbb{C}}$. In any case,

$$\zeta_{A,\mathbb{N}}(s) = \widetilde{B}(e^{-s})$$

is the function we are looking for. The case where $\phi^{-1}(E)$ is a continuum is obtained from any infinite Blaschke product for which E is an arc of the unit circle

$$E = \{z|z = e^{i\theta}, 0 \le \alpha < \theta < \beta \le 2\pi\}.$$

Such an example has been given in [12]. □

Theorem 4. Let $\zeta_{A,\mathbb{N}}(s)$ be a Dirichlet function obtained from a Blaschke product $B(z)$ by the change of variable $z = \phi(s) = e^{-s}$. If z_0 is a pole of order p of $B(z)$, then there are infinitely many poles of order p of $\zeta_{A,\mathbb{N}}(s)$ which are carried by $\phi(s)$ into z_0. These are the only singular points of $\zeta_{A,\mathbb{N}}(s)$.

Proof: Indeed,

$$B(z) = \frac{\Phi(z)}{(z-z_0)^p},$$

where $\Phi(z)$ is analytic in a neighborhood of z_0 and $\Phi(z_0) \neq 0$. Thus,

$$\zeta_{A,\mathbb{N}}(s) = B(\phi(s)) = \frac{\Phi(\phi(s))}{[\phi(s)-z_0]^p}.$$

The equation

$$e^{-s} = e^{-\sigma-it} = z_0 = r_0e^{i\alpha_0}$$

is equivalent to $e^{-\sigma} = r_0$ and $t = 2k\pi - \alpha_0$, $k \in \mathbb{Z}$. It is obvious that all the points

$$s_k = -\ln r_0 + i(2k\pi - \alpha_0)$$

are poles of $\zeta_{A,\mathbb{N}}(s)$ of the order p and they are carried by $\phi(s)$ into z_0 and these are the only ones. □

3 Dirichlet Series with Natural Boundary

Examples of Dirichlet series with natural boundary have been given in [12]. They were obtained either from Hadamard type of series, or from infinite Blaschke products with natural boundary. We will revisit them in Section 4.

Let us notice that for a series (1) with real coefficients, we have

$$\overline{\zeta_{A,\Lambda}(s)} = \zeta_{A,\Lambda}(\overline{s}),$$

hence, if it converges at $s = \sigma_c + it$, where σ_c is its abscissa of convergence, then it converges also at $\overline{s} = \sigma_c - it$. Moreover, if we denote

$$\widetilde{\zeta_{A,\Lambda}}(s) = \overline{\zeta_{A,\Lambda}}(2\sigma_c - \overline{s}) = \zeta_{A,\Lambda}(2\sigma_c - s)$$

for $s = \sigma + it$, $\sigma < \sigma_c$, since we have

$$2\sigma_c - \sigma > \sigma_c,$$

the function $\widetilde{\zeta_{A,\Lambda}}(s)$ is well defined and it is an analytic function in the half-plane $s = \sigma + it, \sigma < \sigma_c$.

Thus, to any Dirichlet series (1) defined for $\Re s > \sigma_c$ we can associate a function (4) which is analytic in the half-plane $\Re s < \sigma_c$. If the series (1) has the natural boundary the line $\Re s = \sigma_c$, then the function (4) has the same natural boundary. If the series (1) converges at a point $\overline{s} = \sigma_c - it$, then

$$\begin{aligned}\widetilde{\zeta_{A,\Lambda}}(\sigma_c + it) &= \zeta_{A,\Lambda}(2\sigma_c - \sigma_c - it)\\ &= \zeta_{A,\Lambda}(\sigma_c - it)\\ &= \zeta_{A,\Lambda}(\overline{s}).\end{aligned}$$

Thus, $\widetilde{\zeta_{A,\Lambda}}(s)$ converges at $s = \sigma_c + it$ and

$$\widetilde{\zeta_{A,\Lambda}}(s) = \zeta_{A,\Lambda}(\overline{s}).$$

The two functions do not coincide on the line separating their domains, therefore they are not continuations one of each other.

It is known, [11], that if a Dirichlet function satisfies a Riemann type of functional equation, then it admits an analytic continuation across the convergence line giving rise to a meromorphic function in the whole complex plane. Such an analytic continuation does not coincide with the function (4). However, with the notation

$$\begin{aligned}\widehat{\zeta_{A,\Lambda}(s)} &= \overline{\zeta_{A,\Lambda}}(2\sigma_c - s)\\ &= \zeta_{A,\Lambda}(2\sigma_c - \overline{s})\\ &= \widetilde{\zeta_{A,\Lambda}}(\overline{s})\end{aligned}$$

when $s = \sigma_c + it$ we have

$$\begin{aligned}\widehat{\zeta_{A,\Lambda}(s)} &= \zeta_{A,\Lambda}(2\sigma_c - (\sigma_c - it))\\ &= \zeta_{A,\Lambda}(\sigma_c + it)\\ &= \zeta_{A,\Lambda}(s),\end{aligned}$$

hence the two functions coincide at the points of the line $\Re s = \sigma_c$ where the series (1) is convergent. The function $\widehat{\zeta_{A,\Lambda}(s)}$ is not an analytic function for $\sigma < \sigma_c$, yet its complex conjugate is analytic. Such a function is called *antianalytic* and it preserves some of the properties of analytic functions, for example that of having the real and the imaginary parts harmonic functions. These functions satisfy equations similar (but not identical) to Cauchy-Riemann equations. Also, as mappings, they preserve the absolute value of the angles, yet they reverse the sense of these angles. When the domain of the two functions contains a segment of the convergence line of the series (1), we will call $\widehat{\zeta_{A,\Lambda}(s)}$ the antianalytic continuation of $\zeta_{A,\Lambda}(s)$ and the function equal to each one of them in their domains will be called a *dianalytic function*. We keep the notation $\widehat{\zeta_{A,\Lambda}(s)}$ for this function. It is a continuous function in its domain and it is analytic for $\Re s > \sigma_c$ and antianalytic for $\Re s < \sigma_c$.

4 The Preimage of the Real Axis by a General Dirichlet Series

The geometry of the mappings by a general Dirichlet series can be revealed by studying the preimage by such a series of the real axis and of some circles centered at the origin. The technique of *continuation along an arc* or *lifting of an arc*, [16], p. 23, and the uniqueness theorem related to it is frequently used.

The function $z = \zeta_{A,\Lambda}(s)$ is an analytic function in the half-plane $\Re s > \sigma_c$. The point $s = \infty$ is an essential singular point for every term $a_n e^{-\lambda_n s}$ of the series (1) and therefore for $\zeta_{A,\Lambda}(s)$.

By Picard Theorem, there are infinitely many points s_n for which $\zeta_{A,\Lambda}(s)$ assumes any given value z, except perhaps for a lacunary one. Then there are infinitely many components γ of the preimage by $\zeta_{A,\Lambda}(s)$ of the unit circle. We have proved in [11], that at least one of them must be unbounded. Since

$$\lim_{\sigma\to\infty} \zeta_{A,\Lambda}(\sigma + it) = 1,$$

infinitely many of them are unbounded and since $\zeta_{A,\Lambda}(s) = 1$ infinitely many times, infinitely many of them are bounded.

Let us examine an unbounded component γ of the preimage of the unit circle. If $s = \sigma + it \in \gamma$ then

$$\zeta_{A,\Lambda}(s) = e^{i\theta}, \quad 0 < \theta < 2\pi,$$

and $\sigma \to +\infty$ if and only if $\theta \to 0$, or $\theta \to 2\pi$. We take a ray

$$L_\alpha : z = \rho e^{i\alpha}, \quad \rho \geq 0,$$

through the origin of the z-plane making a small angle $\alpha \neq 0$ with the positive real half axis and let $s = \sigma + it$ be on the component Ψ_α of the preimage by $\zeta_{A,\Lambda}(s)$ of L_α intersecting γ. There is a unique point

$$s_\alpha = \sigma_\alpha + it_\alpha \in \gamma$$

such that

$$\zeta_{A,\Lambda}(s_\alpha) \in L_\alpha.$$

The component Ψ_α is obtained by doing continuation along L_α from s_α. Since L_α starts in the origin, the curve Ψ_α starts in a zero of $\zeta_{A,\Lambda}(s)$. Since L_α does not pass through $z = 1$, we have that σ does not tend to $+\infty$ on Ψ_α, hence Ψ_α remains in a left half-plane. When $\alpha \to 0$ we have $\sigma_\alpha \to +\infty$ and Ψ_α becomes the union of two components, one inside γ, belonging to the preimage of the interval $(0, 1)$ and the other one exterior to γ, belonging to the preimage of the interval $(1, +\infty)$. This last component is above γ if α was positive and below it if α was negative.

Since there are in finitely many unbounded components γ of the preimage by $\zeta_{A,\Lambda}(s)$ of the unit circle, there will be infinitely many components Γ'_n, $n \in \mathbb{Z}$, of the preimage by $\zeta_{A,\Lambda}(s)$ of the interval $(1, +\infty)$ of the real axis such that in the open strip S_n between two consecutive components Γ'_n and Γ'_{n+1} there is a unique unbounded component of the preimage of the unit circle. We count these strips such that S_{n+1} is above S_n and S_0 is the strip containing the point $s = 0$.

Theorem 5. The open strips S_n are disjoint.

Proof: It is enough to deal with adjacent open strips S_n and S_{n+1}. If they were not disjoint the curves Γ'_n and Γ'_{n+1} would intersect each other at a point

$$s_0 = \sigma_0 + it_0, \quad \sigma_0 > \sigma_c.$$

Let $x_0 = \zeta_{A,\Lambda}(s_0)$. The parts of Γ'_n and Γ'_{n+1} with $\sigma \geq \sigma_0$ are both mapped by $\zeta_{A,\Lambda}(s)$ one to one onto the interval $(1, x_0]$ and therefore the domain bounded by them is conformally mapped by $\zeta_{A,\Lambda}(s)$ onto the complex plane with the slit $(1, x_0]$. This domain should contain a pole $s = \sigma + it$ of $\zeta_{A,\Lambda}(s)$, with $\sigma > \sigma_c$, which is not possible since $\zeta_{A,\Lambda}(s)$ is analytic in that domain. Thus any two open strips are disjoint. □

Theorem 6. The set of the strips S_n fills the whole plane when they are unbounded at the left and at the right and it fills the half-plane $\Re s > \sigma_c$ when they are bounded on the left by $\Re s = \sigma_c$.

Proof: What we need to prove is that for every strip S_n there is a strip above it and one below it. Suppose there is no strip above S_n and let s_1 be a point in S_n and s_2 be a point above S_n. Their images $\zeta_{A,\Lambda}(s_1)$ and $\zeta_{A,\Lambda}(s_2)$ can always be connected by a Jordan arc γ not intersecting the interval $(1, +\infty)$. Continuation η along γ from s_1 must end in s_2 and cannot intersect Γ'_{n+1} since γ does not intersect

$$\zeta_{A,\Lambda}(\Gamma'_{n+1}) = (1, +\infty)$$

and this is a contradiction. A similar contradiction is found if we suppose that there is no strip below some S_n with negative n, which completely prove the theorem. □

Theorem 7. Let $\sigma_0 \in \mathbb{R}$ be arbitrary and

$$\sigma_0 + it_k \in \Gamma'_k, \quad k = n, n+1,$$

where Γ'_n and Γ'_{n+1} are two adjacent components of the preimage by $\zeta_{A,\Lambda}(s)$ of the interval $(1, +\infty)$ of the real axis, as previously defined. Let us denote

$$L_n = \{\sigma_0 + it | t_n \leq t \leq t_{n+1}\}.$$

Then the image by $\zeta_{A,\Lambda}(s)$ of the strip S'_n bounded by Γ'_n, Γ'_{n+1} and L_n is the whole complex plane.

Proof: Indeed, the image of L_n by $\zeta_{A,\Lambda}(s)$ is a Jordan arc γ connecting

$$\zeta_{A,\Lambda}(\sigma_0 + it_k), \quad k = n, n+1.$$

Suppose

$$\zeta_{A,\Lambda}(\sigma_0 + it_n) < \zeta_{A,\Lambda}(\sigma_0 + it_{n+1}).$$

Since the intervals

$$(1, \zeta_{A,\Lambda}(\sigma_0 + it_n)) \text{ and } (1, \zeta_{A,\Lambda}(\sigma_0 + it_{n+1}))$$

are the images of Γ'_n and Γ'_{n+1}, there must be a point s_0 on Γ'_{n+1} such that

$$\zeta_{A,\Lambda}(s_0) = \zeta_{A,\Lambda}(\sigma_0 + it_n).$$

When s describes Γ'_k with σ varying from $+\infty$ to σ_0, then $\zeta_{A,\Lambda}(s)$ goes on the real axis from 1 to $\zeta_{A,\Lambda}(\sigma_0 + it_k)$. When s describes L_n, then $\zeta_{A,\Lambda}(s)$ goes on γ from

$$\zeta_{A,\Lambda}(\sigma_0 + it_n) \text{ to } \zeta_{A,\Lambda}(\sigma_0 + it_{n+1}).$$

The interval $(1, \zeta_{A,\Lambda}(\sigma_0 + it_n))$ is travelled twice in both directions. Let z be any point in the complex plane not belonging to the interval $(1, \zeta_{A,\Lambda}(\sigma_0 + it_n))$ or to the arc γ. There is an arc η connecting z with $\zeta_{A,\Lambda}(\sigma_0 + it_n)$ which does not intersect the interval $(1, \zeta_{A,\Lambda}(\sigma_0 + it_{n+1}))$ or the arc γ. Continuation over η from $\sigma_0 + it_n$ ends up in a point $s \in S'_n$ such that $\zeta_{A,\Lambda}(s) = z$. This shows that the image by $\zeta_{A,\Lambda}(s)$ of the closed strip S'_n is indeed the whole complex plane. □

This result does not mean that $\zeta_{A,\Lambda}(s)$ maps conformally S'_n onto the complex plane with a slit, since S'_n might contain branch points of $\zeta_{A,\Lambda}(s)$. This is a known fact in the case of Dirichlet L-functions.

The bounded components of the preimage by $\zeta_{A,\Lambda}(s)$ of the closed unit disk contain each one a point $s_k^{(1)}$ such that $\zeta_{A,\Lambda}(s_k^{(1)}) = 1$ and a point $s_k^{(0)}$ such that $\zeta_{A,\Lambda}(s_k^{(0)}) = 0$. Every component $\Gamma_{n,k}$ of the preimage by $\zeta_{A,\Lambda}(s)$ of the real axis which is mapped one to one onto the whole real axis must contain both of these points. The components $\Gamma_{n,0}$ which are mapped one to one onto the interval $(-\infty, 1)$ contains just a zero of $\zeta_{A,\Lambda}(s)$.

Theorem 8. Every curve $\Gamma_{n,k}$ is included in an open strip S_n.

Proof: It is useful to color differently the preimage of the negative real half axis and that of the positive real half axis, say red and blue. The joint points of the two colors can only be zeros or poles of $\zeta_{A,\Lambda}(s)$. The curves Γ'_n are all blue, while the curves $\Gamma_{n,k}$ are part red and part blue, and the two colors join at points $s_k^{(0)}$ where $\zeta_{A,\Lambda}(s_k^{(0)}) = 0$. Obviously, the preimage of the negative real half axis and that of the interval $(1, +\infty)$ are disjoint and therefore the red part of $\Gamma_{n,k}$ cannot meet any curve Γ'_n. Also, the intervals $(0, 1)$ and $(1, +\infty)$ being disjoint, no Γ'_n can meet any $\Gamma_{n,k}$ between $s_k^{(0)}$ and $s_k^{(1)}$. Suppose that $\Gamma_{n,k}$ and Γ'_n meet at a point s_0 with $x_0 = \zeta_{A,\Lambda}(s_0) > 1$. The uniqueness theorem of continuation along a curve tells us that continuation along the interval $(1, x_0)$ from s_0 cannot produce two distinct curves $\Gamma_{n,k}$ and Γ'_n, therefore such a point s_0 does not exist. □

It is known that for the Dirichlet L-functions the number of the curves $\Gamma_{n,k}$ included in S_n, $n \neq 0$ increases approximately logarithmically with $|n|$. The strip S_0 contains infinitely many curves $\Gamma_{0,k}$.

Every circle centered at the origin of the z-plane intersects orthogonally both, the positive and the negative real half-axis and therefore a point moving in the same direction on a component of the preimage of such a circle will meet alternatively components of the preimage of the real axis colored red and blue, intersecting them orthogonally. This simple topological fact has been called the *color alternating rule*. This is true in particular for an unbounded component of the unit circle. It intersects orthogonally every curve $\Gamma_{n,k}$ from the respective strip S_n, with alternating colors red and blue.

Increasing the radius of the unit circle its unbounded components from all the strips S_n fuse into a unique unbounded curve intersecting all the curves Γ'_n and $\Gamma_{n,k}$ and the color alternating rule remains true. This means, among other things, that the components $\Gamma_{n,k}$ neighboring Γ'_n must face them with the red part and then the alternation of colors in the same component $\Gamma_{n,k}$ must change somewhere in S_n without violating the color alternating rule. This is possible if and only if one of the components $\Gamma_{n,k}$ is intersected just once on the red side. Such a component exists in every strip S_n and it is unique. This is $\Gamma_{n,0}$, which is mapped one to one by $\zeta_{A,\Lambda}(s)$ onto the interval $(-\infty, 1)$. Indeed, the interval $(0, 1)$ meets the unit circle in $z = 1$ and when

$$z = \zeta_{A,\Lambda}(\sigma + it) \to 1$$

we have that

$$\sigma + it \to \infty$$

on the unbounded component of the preimage of the unit circle. Then this component does not meet the preimage of the interval $(0, 1)$. Therefore, this component meets only the preimage of the interval $(-\infty, 0)$, i.e. it meets $\Gamma_{n,0}$ only on the red part.

The study of the derivative $\zeta'_{A,\Lambda}(s)$ of a Dirichlet L-function, [13], allowed a precise description of the mapping by such a function. We have shown that if $\zeta_{A,\Lambda}(s)$ has m_n points $s_k^{(1)}$ in the strip S_n for which $\zeta_{A,\Lambda}(s_k^{(1)}) = 1$, then $\zeta'_{A,\Lambda}(s)$ has exactly m_n zeros $v_{n,k}$ in S_n counted with multiplicities. Connecting $\zeta_{A,\Lambda}(v_{n,k})$ with $z = 1$ by a segment of line $L_{n,k}$ and performing continuation along $L_{n,k}$ from every point $v_{n,k}$ we obtain curves $\gamma_{n,k}$ which together with the preimage of the interval $(1, +\infty)$ of the real axis bound fundamental domains of $\zeta_{A,\Lambda}(s)$. These domains are mapped conformally by $\zeta_{A,\Lambda}(s)$ onto the whole complex plane with a slit along side the interval $(1, +\infty)$ and $L_{n,k}$.

For arbitrary general Dirichlet functions, the geometry of the mappings is more complicated. While a Dirichlet L-function has just one pole and this is in the strip S_0, Theorem 4 exhibits examples of general Dirichlet series with infinitely many poles. However, there is a regularity of the location of those poles, since they are of the form

$$s_k = -\ln r_0 + i(2k\pi - \alpha_0),$$

hence located each one in a horizontal strip of height 2π. The behavior of $\zeta_{A,\Lambda}(s)$ in such a strip is expected to be similar to that of Dirichlet L-functions in the whole plane.

5 Conclusions

There are many studies devoted to Dirichlet functions, most of which focus on the local behavior of these functions. We dealt with global mapping properties of Dirichlet functions in some of our previous publications, yet limiting our research to Dirichlet L-functions. These are functions obtained by analytic continuation of Dirichlet L-series. When studying general Dirichlet series the researchers are confronted with unsolved problems related to the possibility of analytic continuation of these series and the boundary behavior of those series which cannot be analytically continued across the convergence line. The purpose of this paper was to tackle these problems.

For the Dirichlet series obtained from finite Blaschke products, the analytic continuation to the whole complex plane except for some poles is automatic, since these Blaschke products are meromorphic functions in the complex plane and so are the Dirichlet functions obtained by a change of variable.

When the Blaschke product generating a Dirichlet function is infinite, non isolated singular points of that product appear on the unit circle. To every one of these points correspond infinitely many singular points of that Dirichlet function all located on the convergence line.

We succeeded to prove that for those Dirichlet series obtained from infinite Blaschke products, the continuation is possible as long as some arcs of the unit circle do not contain singular points of the respective Blaschke products. Even in the case when the continuation was not possible, we were able to find twin functions of those series outside of the half-plane of convergence whose behavior imitate that of the original functions.

Further research directions would be to study other classes of Dirichlet series which admit analytic continuation across the convergence line. It is known that any power series can be converted into a Dirichlet series by a change of variable. If the circle of convergence of that power series is not a natural boundary, then the Dirichlet series can be analytically continued across the convergence line. Many open questions remain about these series. Can they always be extended to the whole complex plane? What are their fundamental domains? Is there a way to classify these functions?

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