New Right-truncated Shanker Distribution with Application to the Adaptive Multiple Dependent State Sampling Plan

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Abstract: - This study introduces and tests a new right-truncated Shanker distribution (RT-SD) that has specific statistical properties on four real datasets. The results showed that RT-SD is more consistent with the real dataset than the existing Shanker distribution. Furthermore, the adaptive multiple dependent state sampling plan (AMDSSP) implements the proposed new truncated distribution under the truncated life test. This sampling plan is constructed by integrating the principles of the double sampling plans (DSP) and the multiple dependent state sampling plans (MDSSP). The AMDSSP has greater inspection efficiency than the existing sampling plan. A nonlinear optimization is employed to calculate the optimal plan parameters for minimizing the average sample number (*ASN*) under various conditions for producers and consumers. A comparison between the AMDSSP under the RT-SD and the AMDSSP under the Weibull distribution (WD) was considered under the *ASN* based on the real dataset. The results showed that the proposed AMDSSP is more effective in terms of *ASN* when used with RT-SD.

Key-Words: - Lifetime distribution, Shanker distribution, Right-truncated distribution, Truncated lifetime test, Acceptance sampling plans, A adaptive multiple dependent state sampling plan.

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1 Introduction

Acceptance sampling plans (ASPs) improve product quality control by enabling inspection of a representative sample, reducing time and cost compared to checking every item. Additionally, acceptance sampling plans help identify trends in quality, supporting ongoing improvements in manufacturing processes. Therefore, ASPs are an important tool in product quality control. This methodology enables consumers to determine whether to accept or reject a product by inspecting a sample of items from a lot. Simultaneously, manufacturers can leverage ASPs to establish the minimum sample size and criteria for accepting or rejecting a given lot. Traditionally, single sampling plans (SSPs) have been the most widely used across industries. The decision to accept or reject a sample is based on a single sample. However, SSP often results in a higher average sample number (*ASN*) than other sampling plans. In some situations, accepting or rejecting a lot cannot be decided accurately using SSP. The Double Sampling Plan (DSP) [1] is one method that permits further sampling in cases where the first sample fails to yield conclusive results. In addition, the Multiple Dependent State Sampling Plan (MDSSP) is an efficient sampling plan that reduces the size of the sample, reduces inspection costs, and makes decisions about the current lot based on acceptance data from previous lots. [2] introduced the MDSSP for continuous manufacturing processes, incorporating serial inspections of lots. Based on data from previous lots, the MDSSP can determine whether to accept or reject the current lot, thereby reducing the sample size. Many researchers have studied the MDSSP in a variety of contexts. For example, [3] suggested designing an MDSSP to reduce the risks for both producers and consumers. [4] studied the MDSSP using variable sampling plans and the assumption of a normal distribution. [5] used Bayesian methods to investigate MDSSP. The studies, [6], [7], [8], [9] have combined the concept of MDSSP with control chart design. Additionally, [10] applied the MDSSP to COVID-19 outbreak data from China, assuming an Exponentiated Weibull Distribution for the lifetime data. [11] developed a modified MDSSP that they indicated was more adaptable and efficient than the current MDSSP. [12] presented an adaptive MDSSP for accelerated life tests by integrating the DSP concept and the existing MDSSP. They claimed that their proposed sample plan was more adaptable, efficient, and cost-effective than MDSSP and SSP.

Nowadays, product reliability is important, and testing the mean lifetime of each item is not possible. A time-truncated life test has become an important tool in determining product acceptability before exporting to consumers, which is a test of the lifetime of samples based on ASPs over a predetermined time. After that, a decision is made to accept or reject the lot according to the specified criteria. To ensure an optimal product lifetime, numerous researchers have suggested a variety of lifetime distributions for the design of ASP under a time-truncated life test. For example, [13] examined SSPs and DSPs by analyzing variable data under the WD. Their method employed sudden mortality testing to decrease the *ASN* and inspection time. Likewise, [14] examined the group ASP to reduce the inspection time by employing the generalized exponential distribution and WD. [15] suggested the MDSSP when the lifetimes of products are cut short using an exponentiated half-logistic distribution. Furthermore, [16] looked into the median life of products for the MDSSP concerning the generalized inverted exponential distribution. [17] proposed a group MDSSP, which demonstrated that the actual mean lifetime exceeded the specified mean lifetime based on the WD. [18] used the gamma distribution, the Burr type XII distribution, and the Birnbaum-Saunders distribution to determine the mean lifetime for a generalized MDSSP. [19] also employed a truncated life test to develop a new ASP for the Length-Biased Weighted Lomax distribution. [20] introduced a group ASP that uses a truncated life test and an alpha power transformation-inverted benefits distribution to determine a product's lifetime. Selecting a lifetime distribution that is consistent with the real dataset, is one of the ways to increase the efficiency of the ASP. Lifetime distribution is often used for ASP such as Exponential distribution, WD, Lindley distribution, etc. [21]. However, there are some limitations to the parameters of the lifetime distribution that do not correspond to the real dataset. Many researchers have developed the distribution for flexibility and correspondence with the real dataset by mixing distributions ; such as Shanker distribution (SD), a mixture of exponential (θ) and gamma distribution $(2,\theta)$ [22] it is used to mimic real lifetime datasets from various disciplines of knowledge. Shanker distribution fits better than both the Lindley and exponential distributions. The probability density function (PDF) and the cumulative distribution function (CDF) are given as Equations (1) and (2), respectively, [22].

$$f(x;\theta)=\frac{\theta^2}{\theta^2+1}(\theta+x)e^{-\theta x};\ x>0,\theta>0 \tag{1}$$

$$F(x;\theta)=1-\frac{(\theta^2+1)+\theta x}{\theta^2+1}e^{-\theta x};\ x>0,\theta>0 \tag{2}$$

where θ is the scale parameter. The first moment about origin (the expected value) is presented in Equation (3).

$$M_1(X)=E(X)=\frac{\theta^2+2}{\theta(\theta^2+1)} \tag{3}$$

Developing a lifetime distribution for more consistency with real data will further enhance the effectiveness of ASP. One of the improvements to the lifetime distribution is truncation. A conditional distribution known as a truncated distribution is produced when the domain of baseline distribution is constrained, [23]. Truncated distributions can be applied to expedite the asymptotic theory of robust estimators. A conditional distribution that arises from limiting the domain of another probability distribution is known as a truncated distribution in statistics. In practical statistics, truncated distributions emerge when the range of values that fall inside or beyond a specific threshold are the only ones that may be used to record or even know about occurrences. Consequently, values outside the data set are not considered, making the model more consistent with real data. In real-world statistics, the lifetime distributions are truncated on both left and right sides, or they may only be truncated on the left or right side. SD is one alternative distribution that is more consistent with lifetime data. If SD is developed by truncation, it will be more consistent with the actual data. A truncated lifetime distribution has been applied for ASP and the results showed that it improved the efficiency of the sampling plan. For example, [24] proposed the application of a truncated Weibull-X family for ASP based on run lengths of conforming items. The effectiveness of the developed plan was established by comparison with the existing plan, based on the average number of checklists. The results showed that the developed plan could help to save inspection costs and effort while protecting from producer and consumer risks. [25] presented a continuous ASP for truncated Lomax distribution based on cumulative sum (CUSUM) schemes. The product lifetime is distributed according to Lomax distribution. When there were restrictions on the lower and upper bounds of the variable being studied, truncated distributions were utilized in numerous real-world scenarios. They used the Gauss-Chebyshev integration method to optimize CASP-CUSUM schemes through the truncated Lomax distribution, based on these understandings. The suggested plan offered the optimal CASP-CUSUM schemes, allowing the average run length (ARL) and the probability of acceptance (P(A)) values to reach their maximums. For these reasons, we propose developing AMDSSP under a truncated Shanker distribution. Since the lifetime of the product starts from zero to t, this research presents the right truncation of the Shanker distribution to develop AMDSSP. The rest of this article is organized as follows: A brief description of the theory of the right-truncated Shanker distribution and the Adaptive Multiple Dependent State Sampling Plan under right-truncated Shanker distribution are discussed in Section 2. In Section 3, numerical experiments are discussed. Finally, the discussion and conclusion are given in Section 4.

2 Theory

2.1 Right-truncated Distribution

Let X, a continuous random variable, $0 < x < \infty$, be a baseline distribution with the parameter θ. The PDF and CDF of X are $f(x;\theta)$ and $F(x;\theta)$, respectively. If $a = 0$ and $b < \infty$, it is called the right truncated distribution on the interval [0, b]. Then, the PDF of x is [26], [27],

$$t(x;\theta) = \frac{f(x,\theta)}{F(b,\theta)}\,;\, 0 \le x \le b < \infty \tag{4}$$

2.2 Right-truncated Shanker Distribution with Specific Statistical Properties

Equations (5) and (6) provide the PDF and CDF of the right truncated distribution (RT-SD) of X on the interval [0, b], thereby verifying the properties of RT-SD, as detailed in Appendices A and B. Figure 1 (Appendix D) displays the PDF and CDF plots of the RT-SD.

$$t(x;\theta) = \frac{\theta^2(\theta + x)e^{-\theta x}}{\left[(1+\theta^2) - (1+\theta b+\theta^2)e^{-\theta b}\right]} \tag{5}$$

$$T(x;\theta) = \frac{\left[(1+\theta^2) - (1+\theta x+\theta^2)e^{-\theta x}\right]}{\left[(1+\theta^2) - (1+\theta b+\theta^2)e^{-\theta b}\right]} \tag{6}$$

2.2.1 Moments

Moments at the origin of order rth can be used to originate the most important characteristics of the distribution (e.g., mean, variance, skewness, kurtosis, etc.). Calculations of the moment regarding the origin of RT-SD can be found in Appendix C. The expected value or the first moment is as Equation (7).

$$\begin{aligned} M_1(X) &= E(X) \\ &= \frac{(\theta^2 + 2)}{\theta\left[(1+\theta^2) - (1+\theta b+\theta^2)e^{-\theta b}\right]} \end{aligned} \tag{7}$$

2.2.2 Parameter Estimation

The maximum likelihood estimate (MLE) is used for parameter estimation of the RT-SD based on the random sample $\tilde{x} = (x_1, x_2, \ldots, x_n)$ of n observation. Let $x_i \sim RT-SD(\theta, b),\ i = 1,2,\ldots,n$ be an independent and identically distributed (i.i.d.) random variable. The likelihood function and log-likelihood function of x_i are shown in Equation (8) and (9), respectively.

$$L(\Theta|\tilde{x}) = \frac{\theta^2}{\left[(1+\theta^2) - (1+\theta b+\theta^2)e^{-\theta b}\right]}\prod_{i=1}^{n}(\theta + x_i)e^{-\theta x_i}, \tag{8}$$

$$\begin{aligned} LL(\Theta|\tilde{x}) &= n\log\left(\frac{\theta^2}{\left[(1+\theta^2) - (1+\theta b+\theta^2)e^{-\theta b}\right]}\right) \\ &+ \sum_{i=1}^{n}\log(\theta + x_i) - \theta\sum_{i=1}^{n}x_i \end{aligned} \tag{9}$$

The value of b in equations (8) and (9) is the maximum value of the datasets. The maximum likelihood estimate (MLE) ($\hat{\theta}$) of θ can be solved by the equation $\frac{d\left(LL\left(\Theta|\tilde{x}\right)\right)}{d\theta}=0$. It is a nonlinear equation, so it is solved using the Newton-Raphson method.

2.3 Adaptive Multiple Dependent State Sampling Plan under RT-SD

The MDSSP requires a continuous sample from both the current and previous lots to decide whether to accept or reject the current lot under a single sampling. This approach leads to a smaller sample size. The MDSSP is commonly used in situations where manufacturing occurs continuously in multiple lots, and each lot is inspected one after the other. A single sampling may not be enough to accept or reject the current lot, increasing the producer's risk and reducing the consumer's risk. The AMDSSP, an adaptive MDSSP, was introduced in [12]. The ideas of MDSSP and DSP are used in this sample plan. It is suggested that if the quality of the first sample is uncertain, the second sample should be examined before deciding whether to accept or reject the current lot. For inspection, the AMDSSP uses a smaller sample size than the existing MDSSP because current lots will be accepted if they are of good or moderate quality. The AMDSSP follows the same criteria as the existing MDSSP, performing continuous serial lot-by-lot sample inspections. The current lot has the same quality, and consumers trust the producer.

This research proposes an AMDSSP to design an optimal plan parameter when the product lifetime follows the RT-SD under a truncated life test. The proposed AMDSSP involves five parameters: n_1, n_2, c_1, c_2, and m, where n_1 and n_2 represent the first sample size and the second sample size. The number of previous lots required for the disposition of the current lot is m, and the maximum number of unconditionally accepted nonconforming items is c_1 $\left(c_1 \geq 0\right)$, while the maximum number of conditionally accepted nonconforming items is c_2 $\left(c_2 > c_1\right)$. Unconditional acceptance occurs when d_1 £ c_1, whereas conditional acceptance occurs when $d_1 + d_2$ £ c_2 and the previous m lots.

The operational steps of the AMDSSP are as follows:

Step 1. Select the first random sample of size n_1 from the current lot through a life test. Define d_1 as the number of nonconforming items that fail before the predetermined trial time t_0.

Step 2. If d_1 £ c_1, the current lot is considered acceptable for good quality, which is called unconditionally accepted. Conversely, the lot is rejected if $d_1 > c_2$ or if the trial time t_0 is reached, whichever occurs first. Otherwise, go to Step 3.

Step 3 . If $c_1 < d_1$ £ c_2, select a second random sample of size n_2 from the current lot and test its lifetime. Define d_2 as the number of nonconforming items that fail before t_0. If $d_1 + d_2$ £ c_2 and the previous m lots are of good quality, accept the current lot as being of moderate quality called conditionally accepted. Otherwise, the current lot is rejected.

Figure 2 (Appendix D) presents a flow chart that summarizes the above steps. The following Equation (10) denotes the probability of accepting the current lot $\left(P_{1a}\left(p\right)\right)$ when it is of good quality, disregarding the quality of m previous lots:

$$P_{1a}\left(p\right)=P\left(d_1 \leq c_1\right). \tag{10}$$

The probability of accepting the current lot $\left(P_{2a}\left(p\right)\right)$ when the current lot is of moderate quality $(d_1 + d_2$ £ $c_2)$ and the m previous lots were good quality $\left(d_1 \leq c_1\right)$. Such probability is obtained using Equation (11):

$$\begin{aligned} P_{2a}\left(p\right) &= P\left(c_1 < d_1 \leq c_2\right) \\ &\times P\left(d_1 + d_2 \leq c_2\right) \\ &\times \left[P\left(d_1 \leq c_1\right)\right]^m \end{aligned} \tag{11}$$

The operating characteristic (OC) function of the AMDSSP is the function that is provided by Equation (12).

$$\begin{aligned} P_a\left(p\right) &= P_{1a}\left(p\right)+P_{2a}\left(p\right) \\ &= P\left(d_1 \leq c_1\right) \\ &+ P\left(c_1 < d_1 \leq c_2\right) \\ &\times P\left(d_1 + d_2 \leq c_2\right) \\ &\times \left[P\left(d_1 \leq c_1\right)\right]^m. \end{aligned} \tag{12}$$

Let the random variable d_i follows the binomial distribution, $d_i = 0,1,\ldots, n_i$, where $i =$ 1,2. The probability mass function (PMF) of d_i is

$P(d_i = k) = \binom{n_i}{k} p^k (1-p)^{n_i - k}$ and CDF of d_i is $P(d_i \le k) = \sum_{d_i=0}^{k} \binom{n_i}{d_i} p^{d_i} (1-p)^{n_i - d_i}$, respectively.

From Equation (12), the OC function in term of the binomial distribution as shown in Equation (13):

$$
\begin{aligned}
P_{\mathrm{a}}(p) = & \sum_{d_1=0}^{c_1} \binom{n_1}{d_1} p^{d_1} (1-p)^{n_1-d_1} \\
& + \left(\sum_{d_1=c_1+1}^{c_2} \binom{n_1}{d_1} p^{d_1} (1-p)^{n_1-d_1} \right) \\
& \times \left(\sum_{d_2=0}^{c_2-d_1} \binom{n_2}{d_2} p^{d_2} (1-p)^{n_2-d_2} \right) \\
& \times \left(\sum_{d_1=0}^{c_1} \binom{n_1}{d_1} p^{d_1} (1-p)^{n_1-d_1} \right)^m .
\end{aligned} \tag{13}
$$

Note that specific conditions can reduce the AMDSSP to either a single sampling plan (SSP) or a DSP. If m® ¥ , the AMDSSP reduces to an SSP with an acceptance number of c_1. m® 0 reduces the AMDSSP to the DSP with acceptance numbers c_1 and c_2. This flexibility allows AMDSSPs to encompass the characteristics of both SSPs and DSPs, depending on the values assigned to the acceptance probabilities.

The average sample number (*ASN*) of the AMDSSP is a crucial metric that determines the efficiency of the sampling plan. The *ASN* is derived using Equation (14):

$$
\begin{aligned}
ASN &= n_1 P_I + (n_1 + n_2)(1 - P_I) \\
&= n_1 P_I + n_1 + n_2 - n_1 P_I - n_2 P_I \\
&= n_1 + n_2 (1 - P_I) \\
&= n_1 + n_2 \sum_{d_1=c_1+1}^{c_2} \binom{n_1}{d_1} p^{d_1} (1-p)^{n_1-d_1}
\end{aligned} \tag{14}
$$

where P_I is the probability of deciding on acceptance or rejection of the lot on the first sample. $P_I = P(d_1 \le c_1) + P(d_1 > c_2)$ and $1 - P_I = P(c_1 < d_1 \le c_2)$.

In the context of quality control, sample size plays a critical role in the inspection process. It is well-established that effective economic sampling plans can reduce the sample size required for inspection, thereby improving efficiency and reducing costs. The AMDSSP has been designed to minimize the average sample number compared to existing sampling plans, while maintaining a desired level of quality assurance.

The proposed sampling plan aims to ensure that the mean ratio (μ/μ_0) is critical to the quality of the product. If μ/μ_0 is increased, the actual mean lifetime exceeds the specified mean lifetime $(\mu/\mu_0 > 1)$. Assuming that the mean lifetime of the product is represented by Equation (15) following the RT-SD:

$$
\mu = \frac{(\theta^2 + 2)}{\theta\left[(1+\theta^2) - (1 + \theta b + \theta^2)e^{-\theta b}\right]} \tag{15}
$$

The failure probability of a product before t_0 under the RT-SD is expressed by the following formula (16):

$$
p = \frac{(1+\theta^2) - (1 + \theta t_0 + \theta^2)e^{-\theta t_0}}{(1+\theta^2) - (1 + \theta b + \theta^2)e^{-\theta b}} \tag{16}
$$

The value of t_0 can be expressed in terms of μ_0 and an experiment termination ratio (a) by $t_0 = a\mu_0$. Consequently, the failure probability of the product before t_0 can be rewritten as Equation (17):

$$
p = \frac{(1+\theta^2) - (1 + \theta m + \theta^2)e^{-\theta m}}{(1+\theta^2) - (1 + \theta b + \theta^2)e^{-\theta b}} \tag{17}
$$

where $m = \left(\frac{\theta^2 + 2}{\theta\left((1+\theta^2) - (1 + \theta b + \theta^2)e^{-\theta b}\right)} \right) \frac{a}{\mu/\mu_0}$.

Equation (17) determines the failure probability of the product in terms of a, θ, b and μ/μ_0 based on RT-SD.

In terms of quality control, ASP cannot guarantee the quality of all products. Therefore, the choice of ASP should be considered in accordance with both the producer's risk(α) and the consumer's risk (β). The mean ratio of a product is influenced by its failure probability, which in turn affects its quality level. The requirements for α and β are determined by taking into account the acceptable quality level (AQL or p_1) and the limiting quality level (LQL or p_2). The AMDSSP is practical and will be considered for modifications to the OC curve due to two points (p_1 ,$1-\alpha$) and (p_2 , β). A producer considers that at p_1, there is a higher probability of current lot acceptance than at $1-\alpha$. However, a consumer also expects that at p_2, the probability of current lot acceptance should be lower than β . Figure 2 (Appendix D) displays the AMDSSP algorithm.

3 Numerical Experiments

3.1 Parameter Estimation and Goodness-Of-Fit Test

The developed distribution is applied for the three datasets a long with parameter estimation and the goodness-of-fit test. To compare WD, SD and RT-SD, Akaike Information Criterion (AIC) and Kolmogorov–Smirnov test (K–S test) for the four datasets are presented in Table 1 (Appendix D). The formula for computing the AIC and K-S test include Equation (18) and (19), respectively.

$$AIC = -2LL + 2k \ , \quad (18)$$

$$K - S = \underset{x}{Sup}\left|F_n\left(x\right) - F_0\left(x\right)\right| \ , \quad (19)$$

where $F_n\left(x\right)$ is the empirical distribution function F_n for n independent and identically distributed (i.i.d.) ordered observations x_i , $F_0\left(x\right)$ is the null distribution, n is the sample size and k is the number of parameters. The distribution with minimum AIC and K-S values is chosen as the best distribution to fit the data. The details of the four datasets are as follows:

Dataset 1: Failures can occur in microcircuits because of the movement of atoms in the conductors in the circuit; this is referred to as electromigration. The data below are from an accelerated life test of 59 conductors [28]. Failure times are expressed in hours: 6.545 9.289 7.543 6.956 6.492
5.459
8.12 4.706 8.687 2.997 8.591 6.129
11.038 5.381 6.958 4.288 6.522 4.13
7.459 7.495 6.573 6.538 5.589 6.087
5.807 6.725 8.532 9.663 6.369 7.024
8.336 9.218 7.945 6.869 6.352 4.7
6.948 9.254 5.009 7.489 7.398 6.033
10.092 7.496 4.531 7.974 8.799 7.683
7.224 7.365 6.923 5.64 5.434 7.93
6.515 6.476 6.071 10.491 5.923.

Dataset 2: The data consists of 30 observations of March precipitation (in inches) in Minneapolis/St [29]:
0.77 1.74 0.81 1.20 1.95 1.20
0.47 1.43 3.37 2.20 3.00 3.09
1.51 2.10 0.52 1.62 1.31 0.32
0.59 0.81 2.81 1.87 1.18 1.35
4.75 2.48 0.96 1.89 0.9 2.05.

Dataset 3: The number of million revolutions to failure for 23 ball bearings is considered to illustrate the proposed GASIP [30]. The details of the information are as follows: 17.88, 28.92, 33, 41.52, 42.12, 45.60, 48.40, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64,68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.

Dataset 4: The failure times of 63 aircraft windshield [31] : The data are measured in 1000 hours for ready reference as follows : 0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 0.280, 1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900, 2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483, 1.003, 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015, 1.183, 2.341,4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881, 1.262, 2.543, 5.140.

Table 1 (Appendix D) shows parameter estimations and goodness- of- fit tests for WD, SD, and RT-SD for the four real datasets. The results show that the RT-SD offers a better fit than SD for modelling real lifetime datasets, because it provides K-S tests with higher p-values. In addition, both distributions are compared with the Weibull distribution, which is a distribution regularly used for lifetime data. All three distributions are consistent with the real datasets, except for dataset 1, which is consistent with the Weibull distribution but not with the SD and RT-SD. For datasets 3 and 4, the RT-SD is found to be slightly more consistent with the real datasets than the SD due to the lower K-S values.

3.2 Numerical Examples of the AMDSS under the RT-SD

In this section, the AMDSS is applied for RT-SD with $\theta = 0.3, b = 50\left(\theta < 1\right)$ and $\theta = 1.5, b = 50\left(\theta > 1\right)$ which represent unimodal and decreasing functions (Figure 1, Appendix D), respectively. The determination of plan parameters for the AMDSSP is undertaken by considering varying levels of the mean ratio of the producer's risk μ/μ_0 = 2, 4, 6, 8,10, while assuming a constant mean ratio for the consumer's risk μ/μ_0 =1. The failure probability of product (p) corresponding to the mean ratios μ/μ_0 = 2, 4, 6, 8,10 is considered as p_1 and the probability of failure at the ratio μ/μ_0 =1 is taken as p_2. The optimal plan parameters (n_1,n_2,c_1,c_2,m) of the AMDSSP are determined based on the RT-SD criterion to effectively manage both producer and customer risks while minimizing the *ASN*. Specifically, the producer's risk is set at α = 0.05, with different consumer' risk values of β = 0.25, 0.10, and 0.05. Two cases of the experiment termination ratio are set as a = 0.5 and 1.0. The values for parameters under the RT-SD are (θ,b) = (0.3, 50), and (1.5, 50). We use the following nonlinear optimization problem to determine the optimal plan parameters of the AMDSSP for minimization:

Objective function: Minimize

$$ASN(p_1) = n_1 + n_2 \sum_{d_1=c_1+1}^{c_2} \binom{n_1}{d_1} p_1^{d_1} (1-p_1)^{n_1-d_1}$$

Subject to:

$P_a(p_1) \geq 1-\alpha$ and $P_a(p_2) \leq \beta$.

$n_1 > n_2 > 1,\ m \geq 1,\ c_2 > c_1 \geq 0$.

The steps to determine the optimal parameters for minimizing *ASN* are outlined below:

1) Input parameters: $\theta, b, a, \mu/\mu_0, \alpha$ and β.
2) Compute p_1 and p_2 using Equation (17) for different values μ/μ_0.
3) Formulate constraints $P_a(p_1) \geq 1-\alpha$ and $P_a(p_2) \leq \beta$ using Equation (13) based on two points (p_1, $1-\alpha$) and (p_2, β).
4) Set up bounds of (n_1,n_2,c_1,c_2,m) and formulate objective function using Equation (14) with p_1.
5) The nonlinear optimization method was used to determine the optimal parameters for minimizing *ASN* under the given constraints.

In Appendix D, Table 2 and Table 3 represent the optimal parameters (n_1, n_2, c_1, c_2, m), *ASN* and the probability of current lot acceptance at p_1 and p_2 for the AMDSSP under the RT-SD. We observe a decrease in *ASN* as μ/μ_0 rises if a and β remain fixed. If a and μ/μ_0 fixed, we observe an increase in *ASN* as β decreases. Manufacturers typically set the α at 0.05, whereas consumers have varying values for β

We found that, decrease in β results in an increase in *ASN*, indicating that consumers are less likely to encounter low-quality products. This is because manufacturers are increasingly using *ASNs* for inspection. Furthermore, when we fix a, β and μ/μ_0, the RT-SD with $\theta = 0.3$ yields smaller *ASN* values than $\theta = 1.5$.

The OC curves with m = 1, 2, 3, and 4 propose a range of m values based on the probability of current lot acceptance with the same values n_1, n_2, c_1 and c_2. From Table 2 (Appendix D), for fixed a = 0.05, β = 0.10 and μ/μ_0 = 6, the optimal parameter is (n_1,n_2,c_1,c_2,m) = (18, 7, 2, 3, 1) and the OC function for the AMDSSP with m = 1, 2, 3 and 4 are shown in Figure 3 (Appendix D). The results show that the probability of current lot acceptance provided by m = 2, 3, and 4 is equal to or less than m = 1. As a result, there is a high probability of accepting the current lot based on the acceptance of only one previous lot.

As an example, assume that the producer plans to apply the AMDSSP throughout the inspection process where the lifetime is based on the RT-SD with (θ,b) = (1.5, 50). Let $t_0 = 500$ h and $\mu_0 = 1{,}000$ h; then $a = 0.5$. We assume that $\alpha = 0.05$, $\beta = 0.05$ and μ/μ_0 = 10. Table 3 (Appendix D) gives the optimal plan parameters for the AMDSSP as $n_1 = 19,\ n_2 = 8,\ c_1 = 3, c_2 = 5,\ m = 2$ with a probability of current lot acceptance of 0.9991 and an *ASN* of 19.07. The process for the inspection is as follows:

Step 1: Select the first random sample of 19 items and deliver it to the life test. Count d_1 that occur before t_0 = 500 h.

Step 2: Accept the current lot as $d_1 \leq 3$, regardless of the quality of the previous lot. $d_1 > 5$ rejects the current lot. If not, proceed to Step 3.

Step 3: Select a second sample size of 8 items and put it on a life test. Then, d_2 is count that fail before t_0 = 500 h. The current lot is accepted as being of moderate quality. If $d_1 + d_2 \leq 5$ and two previous lot $(m=2)$ are of good quality $(d_1 \leq 3)$. If not, the current lot is rejected**.**

In this study, we use the third real dataset to compare the performance of the AMDSSP under the WD and the AMDSSP under the RT-SD. Table 1 (Appendix D) shows parameter estimates for the third real dataset over both distributions. The results indicate that the parameter estimate for RT-SD is $(\hat{\theta},\hat{b})$ = (0.02,173.40), while the parameter estimate for WD is $(\hat{\alpha},\hat{\beta})$ = (81.87, 2.10). To compare the efficiency of the two sampling plans, set α = 0.05, β = 0.25, 0.10, 0.05, and a = 0.5 and 1.0, respectively. We determine the optimal plan parameters and *ASN* of the AMDSSP under RT-SD and the AMDSSP under WD by following the previously mentioned steps to minimize *ASN*, as shown in Table 4 (Appendix D). For additional details regarding the procedures for identifying the optimal parameters for minimizing ASN, as determined by the AMDSSP in WD, [12].

The result indicates that the AMDSSP under the RT-SD has a smaller *ASN* than the AMDSSP under WD. For example, for fixed a = 0.5, β = 0.25 and μ/μ_0 = 6, the *ASN* of the AMDSSP under the WD is 12.34 while the *ASN* of the AMDSSP under the RT-SD is 9.41. Because of its smaller *ASN* value, the AMDSSP under RT-SD performed better than the AMDSSP under WD.

It was also found that the optimal parameters of the AMDSSP under RT-SD are $n_1=9$, $n_2=3$, $c_1=0, c_2=2$, $m=3$ with a probability of current lot acceptance of 0.9501. Substituting $(\hat{\theta},\hat{b})$ = (0.02,173.40) into Equation (15) then we get $\hat{\mu}$ as 116.16. Given that μ_0 is 116.16 and a is 0.5 then t_0 is 58.08. To illustrate, we select the first random sample from a set of 9 items and then conduct a life test on the samples from the current lot in the trial time as follows:

17.88 41.52 **68.88** 45.60 48.40
51.96 54.12 55.56 **84.12**

Before the trial time of 58.08, we record 2 failures $(d_1=2)$ out of 9 items, and the results indicate an $0<d_1\leq 2$. Therefore, the second sample was chosen with 3 items, put on the life test, and counted for the nonconforming item during the trial time of 58.08 as follows:

28.92 42.12 51.84

Upon the second inspection, we found no failures $(d_2=0)$ in the current lot. The results show that $d_1+d_2\leq 2$. Then, the current lot is accepted if three previous lot $(m=3)$ are of good quality $(d_1\leq 0)$.

4 Discussion and Conclusion

This study developed the AMDSSP with truncated lifetime distributions, which focus on the lifetime of the product of interest in the inspection. This allows the proposed acceptance sampling plan to focus on the relevant quality range, allowing for more targeted, reliable, and cost-effective decision-making. A new right-truncated Shanker distribution and specific statistical properties are presented with the application of four real datasets. The goodness-of-fit tests show that RT-SD is more consistent with the real dataset than the Shanker distribution, a baseline distribution, and WD. Further, the AMDSSP was developed under right -truncated Shanker distribution. The optimal values of (n_1,n_2,c_1,c_2,m) and *ASN* were calculated using a nonlinear optimization method under the difference a, β and μ/μ_0. Moreover, a performance comparison between the AMDSSP under the RT-SD and the AMDSSP under the WD was considered based on *ASN* under the real datasets. From the study, the AMDSSP under RT-SD provided a smaller *ASN* than the AMDSSP under WD. Thus, it can be concluded that the AMDSSP under RT-SD is more flexible and efficient than the AMDSSP under WD. Future research should consider the extension of the proposed sampling plan in other truncated distribution cases.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work, the authors used Quillbot to check grammar. After using this tool/service, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

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The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

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Conflict of Interest

The authors have no conflicts of interest to declare.

APPENDICES

Appendix A: Verifying the properties of RT-SD.

$$\int_0^b t(x;\theta)\,dx = \frac{\theta^2}{\left[\left(1+\theta^2\right)-\left(1+\theta b+\theta^2\right)e^{-\theta b}\right]} \times \int_0^b (\theta+x)e^{-\theta x}dx$$

$$= \frac{\theta}{\left[\left(1+\theta^2\right)-\left(1+\theta b+\theta^2\right)e^{-\theta b}\right]} \times \left[-(\theta+x)e^{-\theta x}-\frac{1}{\theta}e^{-\theta x}\right]_{x=0}^{x=b}$$

$$= \frac{\theta}{\left[\left(1+\theta^2\right)-\left(1+\theta b+\theta^2\right)e^{-\theta b}\right]} \times \left[\left(\theta+\frac{1}{\theta}\right)-\left(\theta+b+\frac{1}{\theta}\right)e^{-\theta b}\right]$$

$$= \frac{\theta}{\left[\left(1+\theta^2\right)-\left(1+\theta b+\theta^2\right)e^{-\theta b}\right]} \times \left[\left(\frac{\theta^2+1}{\theta}\right)-\left(\frac{\theta^2+\theta b+1}{\theta}\right)e^{-\theta b}\right]$$

$$= \frac{\left[\left(1+\theta^2\right)-\left(1+\theta b+\theta^2\right)e^{-\theta b}\right]}{\left[\left(1+\theta^2\right)-\left(1+\theta b+\theta^2\right)e^{-\theta b}\right]} = 1$$

Appendix B: Finding the CDF value of RT-SD.

$$\int_0^b t(x;\theta)\,dx = \frac{\theta^2}{\left[\left(1+\theta^2\right)-\left(1+\theta b+\theta^2\right)e^{-\theta b}\right]} \times \int_0^x (\theta+x)e^{-\theta x}dx$$

$$= \frac{\theta}{\left[\left(1+\theta^2\right)-\left(1+\theta b+\theta^2\right)e^{-\theta b}\right]} \times \left[-(\theta+x)e^{-\theta x}-\frac{1}{\theta}e^{-\theta x}\right]_0^x$$

$$= \frac{\left[\left(1+\theta^2\right)-\left(1+\theta x+\theta^2\right)e^{-\theta x}\right]}{\left[\left(1+\theta^2\right)-\left(1+\theta b+\theta^2\right)e^{-\theta b}\right]}$$

Appendix C: Moment r^th of RT-SD.

$$M_r(X) = \int_0^b x^r t(x;\theta)\,dx$$

$$= \frac{\theta^2}{\left[\left(1+\theta^2\right)-\left(1+\theta b+\theta^2\right)e^{-\theta b}\right]} \times \int_0^x x^r(\theta+x)e^{-\theta x}dx$$

$$= \frac{\theta^2}{\left[\left(1+\theta^2\right)-\left(1+\theta b+\theta^2\right)e^{-\theta b}\right]} \times \left[\begin{array}{l}\left(\frac{1}{\theta^r}\int_0^\infty (\theta x)^r e^{-\theta x} d(\theta x)\right) \\ +\left(\frac{1}{\theta^{r+2}}\int_0^\infty (\theta x)^{r+1} e^{-\theta x} d(\theta x)\right)\end{array}\right]$$

$$= \frac{\theta^2}{\left[\left(1+\theta^2\right)-\left(1+\theta b+\theta^2\right)e^{-\theta b}\right]} \times \left[\frac{\Gamma(r+1)}{\theta^r}+\frac{\Gamma(r+2)}{\theta^{r+2}}\right]$$

Appendix D

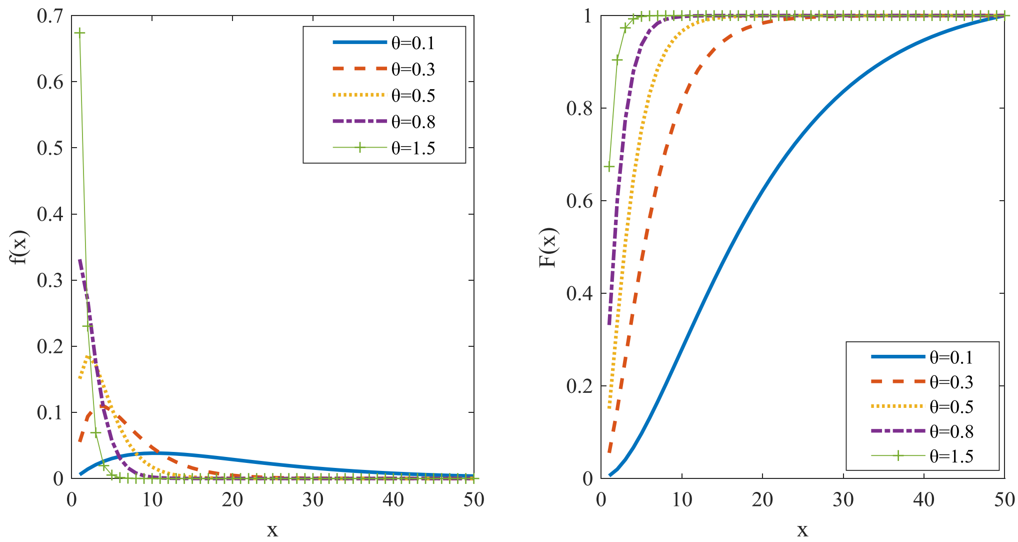

Fig. 1: PDF and CDF plots of RT-SD at b =50

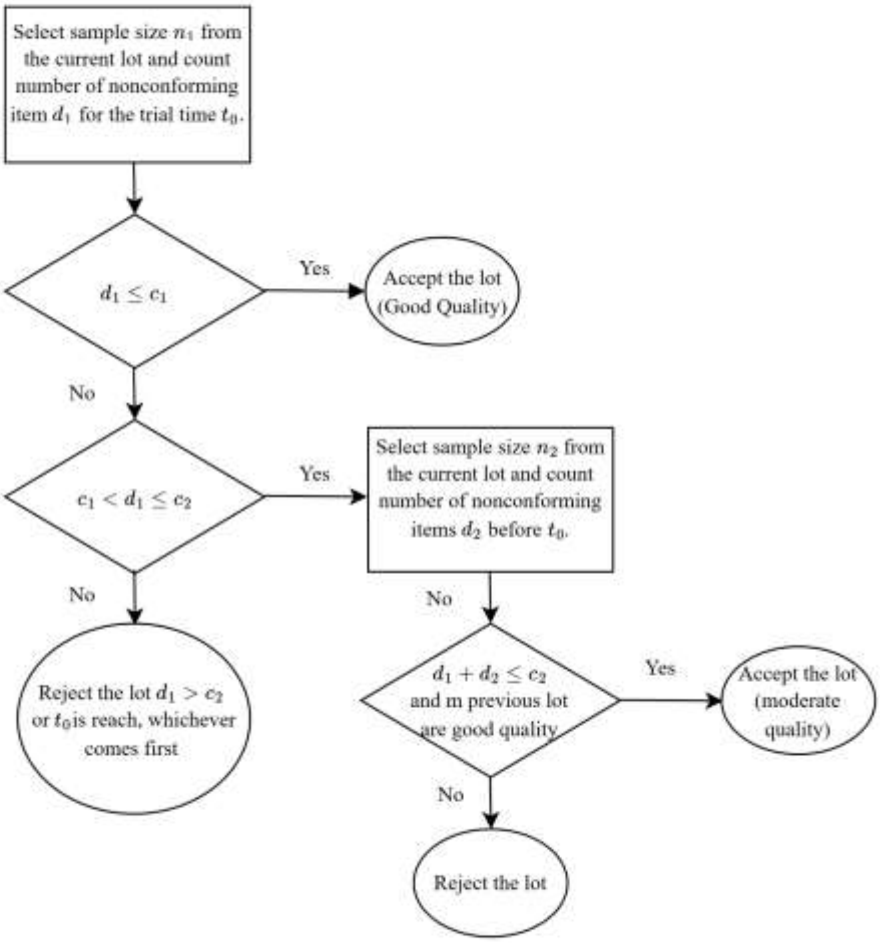

Fig. 2: Flow diagram of the AMDSSP

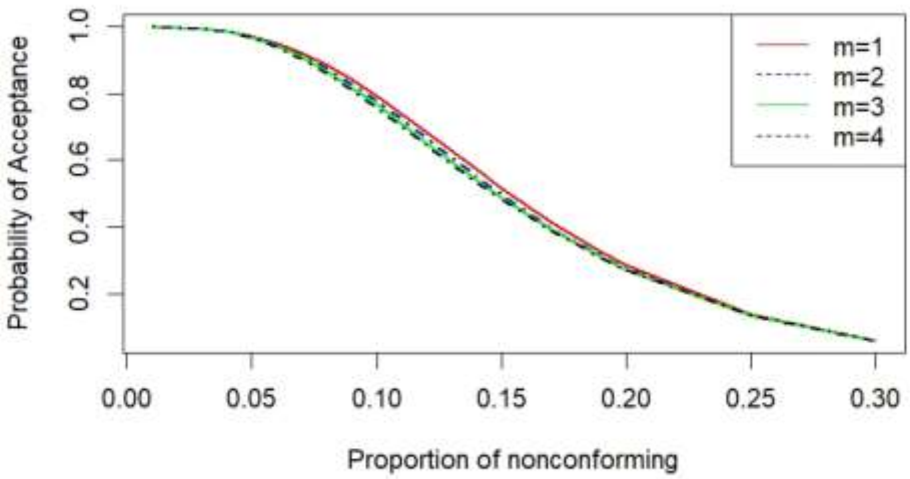

Fig. 3: OC curves for the AMDSSP under the RT-SD with different m values

Table 1. MLE estimates and goodness-of-fit tests of WD, SD and RT-SD for the four datasets

Data	Para-meter	WD			Para-meter	SD			RT-SD		
		Parameter Estimation	AIC	K-S (p-value)		Parameter Estimation	AIC	K-S (p-value)	Parameter Estimation	AIC	K-S (p-value)
1	α β	7.6130 4.6988	228.9946	0.0957 (0.6176)	θ b	0.2723 -	310.95	0.3163 (<0.01)	0.045257 11.038	261.99	0.2074 (0.0105)
2	α β	1.8418 1.7926	81.3510	0.0753 (0.9923)	θ b	0.8984 -	87.9748	0.1595 (0.3888)	0.7747 4.75	86.2028	0.1328 (0.6181)
3	α β	81.8746 2.1018	231.3839	0.1510 (0.6170)	θ b	0.0277 -	233.0601	0.1891 (0.3395)	0.02333 173.4	232.0805	0.1470 (0.6518)
4	α β	2.3098 1.6290	204.6354	0.1087 (0.4167)	θ b	0.7666 -	209.9488	0.1439 (0.1332)	0.6186 5.14	199.7206	0.08813 (0.6788)

Note: α and β are scale and shape parameters

Table 2. Optimal plan parameters of the AMDSSP under RT-SD with $\theta = 0.3$ and $b = 50$

a	β	μ/μ_0	n_1	n_2	c_1	c_2	m	ASN	$P_a(p_1)$	$P_a(p_2)$
0.5	0.25	2	26	13	5	7	2	26.49	0.9512	0.2265
		4	18	7	2	5	3	18.23	0.9964	0.2497
		6	16	7	1	4	1	16.35	0.9973	0.1831
		8	13	5	1	3	1	13.08	0.9996	0.2233
		10	8	5	0	2	2	8.44	0.9847	0.2291
	0.10	2	44	19	8	9	2	44.53	0.9583	0.0979
		4	23	11	3	5	1	23.13	0.9983	0.0999
		6	18	7	2	3	1	18.05	0.9983	0.0981
		8	16	8	1	3	1	16.20	0.9990	0.0780
		10	13	5	0	2	1	13.70	0.9796	0.0526
	0.05	2	51	30	9	10	2	51.78	0.9555	0.0446
		4	26	15	3	5	2	26.30	0.9955	0.0367
		6	21	15	2	5	1	21.16	0.9991	0.0484
		8	17	10	1	3	1	17.28	0.9986	0.0484
		10	15	7	0	1	1	15.04	0.9762	0.0496
1	0.25	2	*	*	*	*	*	*	*	*
		4	11	5	2	4	1	12.66	0.9616	0.0094
		6	9	5	1	2	2	9.43	0.9512	0.0044
		8	7	4	1	2	1	7.11	0.9932	0.0225
		10	5	3	1	2	1	5.04	0.9976	0.1070
	0.10	2	*	*	*	*	*	*	*	*
		4	12	6	2	4	1	12.76	0.9536	0.0039
		6	11	3	2	3	1	11.41	0.9733	0.0041
		8	9	5	1	2	1	9.22	0.9857	0.0045
		10	7	4	1	2	1	7.06	0.9972	0.0225
	0.05	2	*	*	*	*	*	*	*	*
		4	14	6	3	5	1	14.33	0.9841	0.0052
		6	12	6	2	3	1	12.17	0.9862	0.0032
		8	9	5	2	3	1	9.02	0.9989	0.0280
		10	7	4	0	2	1	7.75	0.9620	0.0064

*Note: *There is no optimal plan parameter*

Table 3. Optimal plan parameters of the AMDSSP under RT-SD with $\theta = 1.5$ and $b = 50$

a	β	μ/μ_0	n_1	n_2	c_1	c_2	m	ASN	$P_a(p_1)$	$P_a(p_2)$
0.5	0.25	2	*	*	*	*	*	*	*	*
		4	20	12	4	5	1	20.49	0.9525	0.0789
		6	18	8	2	4	1	19.06	0.9517	0.0200
		8	18	8	2	4	1	18.57	0.9842	0.0205
		10	12	6	2	3	1	12.08	0.9952	0.1228
	0.10	2	*	*	*	*	*	*	*	*
		4	23	15	2	4	1	23.38	0.9965	0.0022
		6	19	6	2	4	1	19.74	0.9525	0.0238
		8	18	8	2	3	1	18.47	0.9604	0.0149
		10	15	6	2	4	1	15.23	0.9797	0.0460
	0.05	2	*	*	*	*	*	*	*	*
		4	28	15	5	7	1	29.00	0.9509	0.0216
		6	24	12	3	5	1	25.14	0.9584	0.0073
		8	23	10	3	5	3	23.34	0.9902	0.0115
		10	19	8	3	5	2	19.07	0.9991	0.0487
1	0.25	2	*	*	*	*	*	*	*	*
		4	15	7	3	4	1	15.26	0.9699	0.0011
		6	12	6	2	3	1	12.09	0.9942	0.0018
		8	11	5	1	2	1	11.17	0.9914	0.0005
		10	10	6	1	2	1	10.17	0.9926	0.0011
	0.10	2	*	*	*	*	*	*	*	*
		4	15	5	4	5	2	15.32	0.9627	0.0262
		6	13	8	3	5	3	13.45	0.9603	0.0055
		8	12	6	2	4	1	12.76	0.9543	0.0021
		10	12	6	2	3	1	12.47	0.9806	0.0021
	0.05	2	*	*	*	*	*	*	*	*
		4	16	10	5	7	1	16.84	0.9504	0.0015
		6	15	8	4	5	2	15.30	0.9596	0.0059
		8	13	7	4	5	2	13.05	0.9943	0.0223
		10	12	8	2	4	2	12.63	0.9713	0.0018

*Note: *There is no optimal plan parameter.*

Table 4. Optimal plan parameters of the AMDSSP under RT-SD and the AMDSSP under WD.

a	β	μ/μ_0	WD				RT-SD			
			Optimal parameter (n_1,n_2,c_1,c_2,m)	ASN	$P_a(p_1)$	$P_a(p_2)$	Optimal parameter (n_1,n_2,c_1,c_2,m)	ASN	$P_a(p_1)$	$P_a(p_2)$
0.5	0.25	2	(27,15,2,4,3)	28.41	0.9567	0.1962	(15,9,3,5,4)	15.73	0.9501	0.2446
		4	(15,8,0,2,3)	16.09	0.9500	0.1851	(13,12,1,2,3)	13.76	0.9608	0.0420
		6	(12,7,0,1,5)	12.34	0.9871	0.1738	(9,3,0,2,3)	9.41	0.9501	0.1322
		8	(12,10,0,1,5)	12.27	0.9957	0.1531	(9,4,0,1,5)	9.32	0.9674	0.0437
		10	(10,9,0,1,2)	10.13	0.9993	0.2261	(8,4,0,1,5)	8.19	0.9876	0.0666
	0.05	2	(44,22,3,5,1)	46.16	0.9518	0.0500	(34,8,6,8,2)	34.62	0.9512	0.0493
		4	(30,7,1,2,5)	30.22	0.9900	0.0485	(16,8,1,3,5)	16.82	0.9515	0.0233
		6	(27,22,1,2,6)	27.22	0.9991	0.0498	(14,5,0,2,1)	15.02	0.9549	0.0222
		8	(22,18,0,1,5)	22.86	0.9866	0.0212	(12,11,0,1,4)	13.13	0.9504	0.0087
		10	(18,15,0,1,5)	18.38	0.9961	0.0467	(10,2,0,1,4)	10.23	0.9843	0.0329
1	0.25	2	(16,4,4,6,2)	16.40	0.9507	0.0296	*	*	*	*
		4	(13,3,1,2,3)	13.25	0.9556	0.0010	(11,5,2,4,2)	11.60	0.9501	0.0006
		6	(10,3,0,1,1)	10.46	0.9553	0.0008	(7,5,1,2,6)	7.26	0.9693	0.0034
		8	(9,6,0,1,8)	9.49	0.9531	0.0010	(6,4,0,2,1)	6.76	0.9606	0.0016
		10	(8,6,0,1,7)	8.28	0.9841	0.0022	(6,4,0,1,2)	6.49	0.9546	0.0007
	0.05	2	*	*	*	*	*	*	*	*
		4	(28,8,2,4,4)	28.82	0.9515	0.0001	(14,7,3,4,2)	14.37	0.9501	0.0002
		6	(18,7,1,2,4)	18.26	0.9870	0.0002	(10,7,1,4,5)	10.77	0.9500	0.0002
		8	(14,5,0,2,3)	14.64	0.9559	0.0001	(10,5,1,2,10)	10.20	0.9769	0.0001
		10	(11,8,0,1,6)	11.51	0.9748	0.0002	(8,4,0,2,1)	8.67	0.9695	0.0002

*Note: *There is no optimal plan parameter*