New Vector Fields and Planes Obtained by the Extended Darboux Frame Apparatus of the Second Kind

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Abstract: - In this paper, new vector fields are defined along a curve lying on an orientable hypersurface with nonvanishing extended Darboux curvatures of the second kind, and some new planes and curves are introduced using these vector fields in Euclidean 4-space. It is also shown that in 4-dimensional space, these new planes play the same role that the Darboux vector $\mathrm{W} = \tau_g \mathrm{T} - \kappa_n \mathrm{V} + \kappa_g \mathrm{N}$ plays in Euclidean 3-space. Besides, developable and non-developable ruled hypersurfaces associated with these new vector fields are defined.

Key-Words: - Darboux vector, extended Darboux frame of the second kind, geodesic torsion of order 1, geodesic curvature of order 2, normal curvature, developable ruled hypersurface, non-developable ruled hypersurface.

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1 Introduction

In differential geometry, vector fields are used in three-dimensional and four-dimensional spaces and higher-dimensional spaces to determine the geometric properties of curves and surfaces. The best-known and most widely used vector fields are the Frenet vector fields, the Darboux vector fields, tangent vector fields, and normal vector fields. Frenet vector fields form an orthonormal frame along a space curve. This frame field is called the Frenet frame which includes important knowledge about the curve. Frenet formulas, which consist of the derivative equations of the Frenet frame, can be rewritten using the Darboux vector field in Euclidean 3-space $\mathbb{E}^3$, [1], [2]. If a curve lies on a surface, then by using the tangent vector field of the curve and the normal vector field of the surface, we can construct a different frame called the Darboux frame. Using the vector fields of the Frenet frame, and the Darboux frame, some new frames, curves, and surfaces have been defined and characterized in different spaces, [3], [4], [5], [6], [7], [8]. Among these surfaces, ruled surfaces have many applications in CAGD, architecture, and physics. Developable surfaces which are special kinds of ruled surfaces have also been studied in many spaces, [4], [5], [9], [10]. Also, since special surface curves such as lines of curvature, and geodesic curves play an important role in surface analysis and geometric design, several methods have been given to construct a developable surface or hypersurface possessing a given curve as the line of curvature or a geodesic curve of it, [4], [5], [6], [11].

The Darboux frame field has been extended into Euclidean 4-space $\mathbb{E}^4$ and the extended Darboux frame field of the first kind, and the extended Darboux frame field of the second kind have been defined in [12]. Some special curves according to this extended Darboux frame field have been defined in [13].

Defining some new vector fields along a space curve with nonvanishing curvatures in $\mathbb{E}^4$, the Frenet formulas have been rewritten as ternary products of Frenet vectors in [14]. Later, some new planes, curves, and ruled hypersurfaces have been introduced and then some characterizations related to these planes, curves, and ruled hypersurfaces have been given [14]. The results of [14] have been studied in Minkowski 4-space by [15].

This paper aims to define new vector fields along a geodesic curve on an orientable hypersurface with nonvanishing curvatures of extended Darboux frame of the second kind in $\mathbb{E}^4$ and to rewrite the derivatives of the extended Darboux frame field vectors of the second kind as ternary products of these vector fields. This study also aims to define new planes and curves using these new vector fields. In addition, we construct some ruled hypersurfaces associated with these vector fields.

2 Preliminaries

Let α be a unit speed curve in $\mathbb{E}^3$, and $\{\mathrm{T},\mathrm{N},\mathrm{B},\kappa,\tau\}$ denote the Frenet apparatus of α. By using the Darboux vector field $\mathrm{D}=\tau\mathrm{T}+\kappa\mathrm{B}$ of α, the Frenet formulas:

$$\mathrm{T}'=\kappa\mathrm{N},\quad \mathrm{N}'=-\kappa\mathrm{T}+\tau\mathrm{B},\quad \mathrm{B}'=-\tau\mathrm{N}$$

can be rewritten as [1]:

$$\mathrm{T}'=\mathrm{D}\times\mathrm{T},\quad \mathrm{N}'=\mathrm{D}\times\mathrm{N},\quad \mathrm{B}'=\mathrm{D}\times\mathrm{B}.$$

Let $S\subset\mathbb{E}^3$ be an oriented surface and α be a unit speed curve on S. Let T be the unit tangent vector field of α, and N be the surface unit normal vector field restricted to α. Then, the Darboux frame field along the curve α is given by $\{\mathrm{T},\mathrm{V},\mathrm{N}\}$, where $\mathrm{V}=\mathrm{N}\times\mathrm{T}$. Then, the derivative equations of the Darboux frame field are given by [1].

$$\begin{cases}\mathrm{T}' &= \kappa_g\mathrm{V}+\kappa_n\mathrm{N},\\ \mathrm{V}' &= -\kappa_g\mathrm{T}+\tau_g\mathrm{N},\\ \mathrm{N}' &= -\kappa_n\mathrm{T}-\tau_g\mathrm{V},\end{cases}$$

where κ_g,κ_n, and τ_g denote the geodesic curvature, the normal curvature, and the geodesic torsion of α, respectively. By using the Darboux vector field:

$$\mathrm{W}=\tau_g\mathrm{T}-\kappa_n\mathrm{V}+\kappa_g\mathrm{N}$$

of α, we can rewrite the above equations as:

$$\mathrm{T}'=\mathrm{W}\times\mathrm{T},\quad \mathrm{V}'=\mathrm{W}\times\mathrm{V},\quad \mathrm{N}'=\mathrm{W}\times\mathrm{N}.$$

Definition 2.1 Let $\{\mathrm{e}_1,\mathrm{e}_2,\mathrm{e}_3,\mathrm{e}_4\}$ be the standard basis of $\mathbb{R}^4$. The ternary product of the vectors $\mathrm{X}=\sum_{\mathrm{i}=1}^4 \mathrm{x_i e_i}$, $\mathrm{Y}=\sum_{\mathrm{i}=1}^4 \mathrm{y_i e_i}$, and $\mathrm{Z}=\sum_{\mathrm{i}=1}^4 \mathrm{z_i e_i}$ is defined by [16]:

$$\mathrm{X}\otimes\mathrm{Y}\otimes\mathrm{Z}=\begin{vmatrix}\mathrm{e}_1 & \mathrm{e}_2 & \mathrm{e}_3 & \mathrm{e}_4\\ x_1 & x_2 & x_3 & x_4\\ y_1 & y_2 & y_3 & y_4\\ z_1 & z_2 & z_3 & z_4\end{vmatrix}.$$

Let δ be a unit speed curve in $\mathbb{E}^4$, and $\{\mathrm{T},\mathrm{N},\mathrm{B}_1,\mathrm{B}_2,\kappa_1,\kappa_2,\kappa_3\}$ denote the Frenet apparatus of δ. By using the vector fields [14]:

$$\mathrm{D}_1=\mathrm{B}_2,\quad \mathrm{D}_2=\kappa_2\mathrm{T}+\kappa_1\mathrm{B}_1,$$
$$\mathrm{D}_3=\kappa_3\mathrm{N}+\kappa_2\mathrm{B}_2,\quad \mathrm{D}_4=\mathrm{T}$$

along δ, the Frenet formulas:

$$\mathrm{T}'=\kappa_1\mathrm{N},\quad \mathrm{N}'=-\kappa_1\mathrm{T}+\kappa_2\mathrm{B}_1,$$
$$\mathrm{B}_1'=-\kappa_2\mathrm{N}+\kappa_3\mathrm{B}_2,\quad \mathrm{B}_2'=-\kappa_3\mathrm{B}_1$$

can be rewritten as [14]:

$$\begin{aligned}\mathrm{T}' &= \mathrm{D}_1\otimes\mathrm{D}_2\otimes\mathrm{T},\\ \mathrm{N}' &= \mathrm{D}_1\otimes\mathrm{D}_2\otimes\mathrm{N},\\ \mathrm{B}_1' &= \mathrm{D}_3\otimes\mathrm{D}_4\otimes\mathrm{B}_1,\\ \mathrm{B}_2' &= \mathrm{D}_3\otimes\mathrm{D}_4\otimes\mathrm{B}_2.\end{aligned}$$

Let $\mathcal{M}\subset\mathbb{E}^4$ be an orientable hypersurface oriented by the unit normal vector field $\mathcal{N}$, and δ be a unit speed Frenet curve of class C^n $(n\geq 4)$ on $\mathcal{M}$. Let T denote the unit tangent vector field of δ, and N denote the hypersurface unit normal vector field restricted to δ. Then, if the set $\{\mathrm{N},\mathrm{T},\delta''\}$ is linearly independent, one can construct the extended Darboux frame field $\{\mathrm{T},\mathrm{E},\mathrm{D},\mathrm{N}\}$ of the first kind along δ [12], where:

$$\mathrm{E}=\frac{\delta''-\langle\delta'',\mathrm{N}\rangle\mathrm{N}}{||\delta''-\langle\delta'',\mathrm{N}\rangle\mathrm{N}||},\quad \mathrm{D}=\mathrm{N}\otimes\mathrm{T}\otimes\mathrm{E}.$$

In this case, the differential equation of this frame is given by [12]:

$$\begin{cases}\mathrm{T}' &= \kappa_g^1\mathrm{E}+\kappa_n\mathrm{N},\\ \mathrm{E}' &= -\kappa_g^1\mathrm{T}+\kappa_g^2\mathrm{D}+\tau_g^1\mathrm{N},\\ \mathrm{D}' &= -\kappa_g^2\mathrm{E}+\tau_g^2\mathrm{N},\\ \mathrm{N}' &= -\kappa_n\mathrm{T}-\tau_g^1\mathrm{E}-\tau_g^2\mathrm{D}.\end{cases}$$

If the set $\{\mathrm{N},\mathrm{T},\delta''\}$ is linearly dependent, then one can construct the extended Darboux frame field $\{\mathrm{T},\mathrm{E},\mathrm{D},\mathrm{N}\}$ of the second kind along δ [12], where:

$$\mathrm{E}=\frac{\delta'''-\langle\delta''',\mathrm{N}\rangle\mathrm{N}-\langle\delta''',\mathrm{T}\rangle\mathrm{T}}{||\delta'''-\langle\delta''',\mathrm{N}\rangle\mathrm{N}-\langle\delta''',\mathrm{T}\rangle\mathrm{T}||},$$

and $\mathrm{D}=\mathrm{N}\otimes\mathrm{T}\otimes\mathrm{E}$. In this case, the differential equation of this frame is given by [12]:

$$\begin{cases}\mathrm{T}' &= \kappa_n\mathrm{N},\\ \mathrm{E}' &= \kappa_g^2\mathrm{D}+\tau_g^1\mathrm{N},\\ \mathrm{D}' &= -\kappa_g^2\mathrm{E},\\ \mathrm{N}' &= -\kappa_n\mathrm{T}-\tau_g^1\mathrm{E},\end{cases}\tag{1}$$

where κ_n denotes the normal curvature of the hypersurface, κ_g^2 denotes the geodesic curvature of order 2 and τ_g^1 denotes the geodesic torsion of order 1.

Definition 2.2 Let $\delta: \mathrm{I}\subset\mathbb{R}\to\mathbb{E}^4$ be a curve with unit tangent vector e_0, and $\{e_1(t),e_2(t)\}$ denotes an orthonormal basis of generating plane along δ. Then the hypersurface:

$$\varphi(t,v_1,v_2)=\delta(t)+v_1e_1(t)+v_2e_2(t),$$

is called a ruled hypersurface represented by the map $\varphi: I\times\mathbb{R}^2\to\mathbb{E}^4$ in $\mathbb{E}^4$. If

$$rank[e_0,e_1,e_2,e_1',e_2']=4,\tag{2}$$

then the ruled hypersurface is called non-developable. If

$$rank[e_0,e_1,e_2,e_1',e_2']=3,\tag{3}$$

then the ruled hypersurface is called developable, [17].

3 New Vector Fields and Curves in $\mathbb{E}^4$

Let $\mathcal{M} \subset \mathbb{E}^4$ be an orientable hypersurface, δ be a unit speed geodesic curve on $\mathcal{M}$ with nonzero curvatures $\kappa_n, \kappa_g^2, \tau_g^1$, and $\{\mathrm{T}, \mathrm{E}, \mathrm{D}, \mathrm{N}\}$ denotes its extended Darboux frame of the second kind. Let us define the following vector fields along δ:

$$\begin{aligned} &\mathrm{W}_1 = \mathrm{D}, \quad \mathrm{W}_2 = -\tau_g^1 \mathrm{T} + \kappa_n \mathrm{E}, \\ &\mathrm{W}_3 = \kappa_g^2 \mathrm{N} - \tau_g^1 \mathrm{D}, \quad \mathrm{W}_4 = \mathrm{T}. \end{aligned} \tag{4}$$

Note that the set $\{\mathrm{W}_1, \mathrm{W}_2, \mathrm{W}_3, \mathrm{W}_4\}$ is linearly independent along δ and $\{\mathrm{W}_1, \mathrm{W}_2\}$, $\{\mathrm{W}_3, \mathrm{W}_4\}$ and $\{\mathrm{W}_2, \mathrm{W}_3\}$ are orthogonal sets. Let us denote the subspaces spanned by $\{\mathrm{W}_1, \mathrm{W}_2\}$, $\{\mathrm{W}_3, \mathrm{W}_4\}$ and $\{\mathrm{W}_2, \mathrm{W}_3\}$ as $\mathrm{W}_1\mathrm{W}_2$-plane, $\mathrm{W}_3\mathrm{W}_4$-plane, and $\mathrm{W}_2\mathrm{W}_3$-plane, respectively. We may rewrite (1) by using the vector fields W_i $(1 \le i \le 4)$ as follows:

$$\begin{aligned} \mathrm{T}' &= \mathrm{W}_1 \otimes \mathrm{W}_2 \otimes \mathrm{T}, \\ \mathrm{E}' &= \mathrm{W}_3 \otimes \mathrm{W}_4 \otimes \mathrm{E}, \\ \mathrm{D}' &= \mathrm{W}_3 \otimes \mathrm{W}_4 \otimes \mathrm{D}, \\ \mathrm{N}' &= \mathrm{W}_1 \otimes \mathrm{W}_2 \otimes \mathrm{N}. \end{aligned} \tag{5}$$

So, according to (5), T and N rotate around the $\mathrm{W}_1\mathrm{W}_2$-plane, and E and D rotate around the $\mathrm{W}_3\mathrm{W}_4$-plane. This means the $\mathrm{W}_1\mathrm{W}_2$-plane and the $\mathrm{W}_3\mathrm{W}_4$-plane play the role that the Darboux vector $\mathrm{W} = \tau_g \mathrm{T} - \kappa_n \mathrm{V} + \kappa_g \mathrm{N}$ plays in 3-space.

Now, we will define some new curves and call them $\mathrm{W}_1\mathrm{W}_2$-curve, $\mathrm{W}_3\mathrm{W}_4$-curve, and $\mathrm{W}_2\mathrm{W}_3$-curve, respectively.

Definition 3.1 If the position vector of a geodesic curve on an orientable hypersurface $\mathcal{M} \subset \mathbb{E}^4$ always lies in its $\mathrm{W}_1\mathrm{W}_2$-plane, it is called a $\mathrm{W}_1\mathrm{W}_2$-curve.

Theorem 3.1 Let δ be a geodesic curve with arc-length parameter s lying on an orientable hypersurface $\mathcal{M}$ with nonvanishing curvatures κ_n, κ_g^2, and τ_g^1. Then, δ is congruent to a $\mathrm{W}_1\mathrm{W}_2$-curve if and only if the curvatures of δ satisfy the equation:

$$\left\{\frac{1}{\kappa_g^2(s)}\left(\frac{\kappa_n(s)(c-s)}{\tau_g^1(s)}\right)'\right\}' + \frac{\kappa_n(s)\kappa_g^2(s)(c-s)}{\tau_g^1(s)} = 0, \tag{6}$$

where $c =$ constant.

Proof. $(\Rightarrow)$ Let δ be a $\mathrm{W}_1\mathrm{W}_2$-curve with non-vanishing curvatures κ_n, κ_g^2, and τ_g^1 on $\mathcal{M}$. Then, we can write:

$$\delta(s) = \lambda_1(s)\mathrm{W}_1(s) + \lambda_2(s)\mathrm{W}_2(s) \tag{7}$$

for some functions $\lambda_i(s)$, $(i = 1,2)$. Taking the derivative of (7) according to s and using (1) gives us:

$$\begin{cases} (\lambda_2(s)\tau_g^1(s))' + 1 = 0, \\ (\lambda_2(s)\kappa_n(s))' - \lambda_1(s)\kappa_g^2(s) = 0, \\ \lambda_1'(s) + \lambda_2(s)\kappa_n(s)\kappa_g^2(s) = 0. \end{cases} \tag{8}$$

If we use the first equation of (8), we obtain:

$$\lambda_2(s) = \frac{c-s}{\tau_g^1(s)},$$

where c denotes an integration constant. The second equation of (8) gives:

$$\lambda_1(s) = \frac{1}{\kappa_g^2(s)}\left(\frac{\kappa_n(s)(c-s)}{\tau_g^1(s)}\right)'.$$

Thus, we obtain equation (6) by substituting the obtained results into the third equation of (8).

$(\Leftarrow)$ Let us assume that the curvatures of δ satisfy the equation (6). Differentiating the vector:

$$A(s) = \delta(s) - \frac{1}{\kappa_g^2(s)}\left(\frac{\kappa_n(s)(c-s)}{\tau_g^1(s)}\right)' \mathrm{W}_1(s) - \frac{c-s}{\tau_g^1(s)}\mathrm{W}_2(s)$$

according to s yields zero vector. This means $A(s)$ is a constant vector. Then, $\delta(s)$ is congruent to a $\mathrm{W}_1\mathrm{W}_2$-curve.

Definition 3.2 If the position vector of a geodesic curve on an orientable hypersurface $\mathcal{M} \subset \mathbb{E}^4$ always lies in its $\mathrm{W}_3\mathrm{W}_4$-plane, it is called a $\mathrm{W}_3\mathrm{W}_4$-curve.

Theorem 3.2 Let ζ be a geodesic curve with arc-length parameter s lying on an orientable hypersurface $\mathcal{M}$ with nonvanishing curvatures κ_n, κ_g^2, and τ_g^1. Then, ζ is congruent to a $\mathrm{W}_3\mathrm{W}_4$-curve if and only if the curvatures of ζ satisfy the equation:

$$c\left\{\frac{1}{\kappa_n(s)}\left(\frac{\kappa_g^2(s)}{\tau_g^1(s)}\right)'\right\}' + \frac{c\kappa_n(s)\kappa_g^2(s)}{\tau_g^1(s)} + 1 = 0, \tag{9}$$

where $c =$ constant.

Proof. $(\Rightarrow)$ Let ζ be a $\mathrm{W}_3\mathrm{W}_4$-curve with non-vanishing curvatures κ_n, κ_g^2, and τ_g^1 on $\mathcal{M}$. Then, by the definition, we can write

$$\zeta(s) = \mu_1(s)\mathrm{W}_3(s) + \mu_2(s)\mathrm{W}_4(s) \tag{10}$$

for some functions $\mu_i(s)$, $(i = 1,2)$. If we differentiate (10) according to s and use (1), we obtain:

$$\begin{cases} (\mu_1(s)\tau_g^1(s))' = 0, \\ (\mu_1(s)\kappa_g^2(s))' + \mu_2(s)\kappa_n(s) = 0, \\ \mu_2'(s) - \mu_1(s)\kappa_n(s)\kappa_g^2(s) - 1 = 0. \end{cases} \tag{11}$$

From the first equation of (11), we have:

$$\mu_1(s) = \frac{c}{\tau_g^1(s)},$$

where c is an integration constant. The second equation of (11) yields:

$$\mu_2(s) = -\frac{c}{\kappa_n(s)}\left(\frac{\kappa_g^2(s)}{\tau_g^1(s)}\right)'.$$

If we substitute these results into the third equation of (11), we get (9).

(⇐) Let us assume that the curvatures of ζ satisfy the equation (9). Let:

$$B(s) = \zeta(s) - \frac{c}{\tau_g^1(s)}\mathrm{W}_3(s) + \frac{c}{\kappa_n(s)}\left(\frac{\kappa_g^2(s)}{\tau_g^1(s)}\right)'\mathrm{W}_4(s).$$

If we differentiate $B(s)$ according to s, we obtain a zero vector. This means $B(s)$ is a constant vector. Thus, $\zeta(s)$ is congruent to a $\mathrm{W}_3\mathrm{W}_4$-curve.

Definition 3.3 If the position vector of a geodesic curve on an orientable hypersurface $\mathcal{M} \subset \mathbb{E}^4$ always lies in its $\mathrm{W}_2\mathrm{W}_3$-plane, it is called $\mathrm{W}_2\mathrm{W}_3$-curve.

Theorem 3.3 Let σ be a geodesic curve with arc-length parameter s lying on an orientable hypersurface $\mathcal{M}$ with nonvanishing curvatures κ_n, κ_g^2, and τ_g^1. Then, σ is congruent to a $\mathrm{W}_2\mathrm{W}_3$-curve if and only if the curvatures of σ satisfy the equation:

$$\begin{cases} c_1\left(\frac{\tau_g^1(s)}{\kappa_n(s)}\right)' + c_2\kappa_n(s) + 1 = 0, \\ c_2\left(\frac{\tau_g^1(s)}{\kappa_g^2(s)}\right)' - c_1\kappa_g^2(s) = 0, \end{cases} \tag{12}$$

where $c_1 = \text{constant}$ and $c_2 = \text{constant}$.

Proof. (⇒) Let σ be a $\mathrm{W}_2\mathrm{W}_3$-curve with non-vanishing curvatures κ_n, κ_g^2, and τ_g^1 on $\mathcal{M}$. Then, from the definition of the $\mathrm{W}_2\mathrm{W}_3$-curve, we can write

$$\sigma(s) = \nu_1(s)\mathrm{W}_2(s) + \nu_2(s)\mathrm{W}_3(s) \tag{13}$$

for some functions $\nu_i(s)$, $(i = 1,2)$. Taking the derivative of (13) according to s and using (1) yield:

$$\begin{cases} (\nu_1(s)\kappa_n(s))' = 0, \\ (\nu_2(s)\kappa_g^2(s))' = 0, \\ (\nu_1(s)\tau_g^1(s))' + \nu_2(s)\kappa_n(s)\kappa_g^2(s) + 1 = 0, \\ (\nu_2(s)\tau_g^1(s))' - \nu_1(s)\kappa_n(s)\kappa_g^2(s) = 0. \end{cases} \tag{14}$$

If we use the first equation of (14), we get:

$$\nu_1(s) = \frac{c_1}{\kappa_n(s)},$$

where c_1 is an integration constant. The second equation of (14) gives:

$$\nu_2(s) = \frac{c_2}{\kappa_g^2(s)},$$

where c_2 is an integration constant. Then, we obtain the equation (12), if we substitute these results into the third and fourth equations of (14).

(⇐) Let us assume that the curvatures of the curve σ satisfy the equation (12). If we consider the vector:

$$C(s) = \sigma(s) - \frac{c_1}{\kappa_n(s)}\mathrm{W}_2(s) - \frac{c_2}{\kappa_g^2(s)}\mathrm{W}_3(s),$$

and differentiate it according to s, we find zero vector. So, $C(s)$ is a constant vector. Then, $\sigma(s)$ is congruent to a $\mathrm{W}_2\mathrm{W}_3$-curve.

Example 3.1 Let us consider the hypersurface $\mathcal{M}$ with its parametric equation:

$$\begin{aligned} X(u,v,w) = (&(1+\frac{3v}{\sqrt{10}})\sin(\ell_1 u) - \frac{3\sqrt{21}}{14}w\cos(\ell_1 u), \\ &(1+\frac{3v}{\sqrt{10}})\cos(\ell_1 u) + \frac{3\sqrt{21}}{14}w\sin(\ell_1 u), \\ &(1-\frac{v}{\sqrt{10}})\sin(\ell_2 u) + \frac{\sqrt{7}}{14}w\cos(\ell_2 u), \\ &(1-\frac{v}{\sqrt{10}})\cos(\ell_2 u) - \frac{\sqrt{7}}{14}w\sin(\ell_2 u)), \end{aligned}$$

where $\ell_1 = \frac{1}{2}$, $\ell_2 = \frac{\sqrt{3}}{2}$. It is easy to verify that the curve

$$\begin{aligned} \delta(s) &= X(s,0,0) \\ &= (\sin(\ell_1 s), \cos(\ell_1 s), \sin(\ell_2 s), \cos(\ell_2 s)) \end{aligned}$$

is a geodesic on $\mathcal{M}$, where $s \in I \subset \mathbb{R}$. Then, we can construct the extended Darboux frame of the second kind along δ. If we use the formulas given in [12], we find the curvatures of δ as:

$$\kappa_n(s) = \frac{\sqrt{5}}{2\sqrt{2}}, \quad \kappa_g^2(s) = \frac{\sqrt{3}}{\sqrt{10}}, \quad \tau_g^1(s) = \frac{1}{2}\frac{\sqrt{3}}{\sqrt{10}}.$$

Thus, we can see that these curvatures do not satisfy the equation (6). It means δ is not a $\mathrm{W}_1\mathrm{W}_2$-curve in $\mathbb{E}^4$. However, the above curvatures satisfy (9) with $c = -\frac{\sqrt{2}}{\sqrt{5}}$. This means δ is a $\mathrm{W}_3\mathrm{W}_4$-curve in $\mathbb{E}^4$. If we substitute the results into (10), we have:

$$\delta(s) = -\frac{4}{\sqrt{3}}\mathrm{W}_3(s) + 0.\mathrm{W}_4(s),$$

where

$$\mathrm{W}_3(s) = \frac{\sqrt{3}}{\sqrt{10}}\left(\mathrm{N}(s) - \frac{1}{2}\mathrm{D}(s)\right).$$

Moreover, the curvatures of δ satisfy also the equation (12) with $c_1 = 0, c_2 = -\frac{2\sqrt{2}}{\sqrt{5}}$ which yields δ as a $\mathrm{W}_2\mathrm{W}_3$-curve in $\mathbb{E}^4$.

4 Ruled Hypersurfaces Obtained by the Vector Fields W_i

Let δ be a geodesic curve with arc-length parameter s lying on an orientable hypersurface $\mathcal{M}$ with nonvanishing curvatures κ_n, κ_g^2, and τ_g^1. Let us consider the vector fields W_i $(1 \leq i \leq 4)$ defined in (4). If we normalize the orthogonal sets $\{\mathrm{W}_1, \mathrm{W}_2\}$ and $\{\mathrm{W}_2, \mathrm{W}_3\}$, we find the orthonormal frames $\{\mathrm{W}_1, \overline{\mathrm{W}}_2\}$ and $\{\overline{\mathrm{W}}_2, \overline{\mathrm{W}}_3\}$, where

$$\overline{\mathrm{W}}_2(s) = \frac{\mathrm{W}_2(s)}{||\mathrm{W}_2(s)||} = \frac{1}{\sqrt{\kappa_n^2(s) + (\tau_g^1)^2(s)}}\left[\kappa_n(s)\mathrm{E}(s) - \tau_g^1(s)\mathrm{T}(s)\right]$$

and

$$\overline{\mathrm{W}}_3(s) = \frac{\mathrm{W}_3(s)}{||\mathrm{W}_3(s)||} = \frac{1}{\sqrt{(\kappa_g^2)^2(s) + (\tau_g^1)^2(s)}}\left[\kappa_g^2(s)\mathrm{N}(s) - \tau_g^1(s)\mathrm{D}(s)\right],$$

respectively. Differentiating these vector fields according to s give:

$$\overline{\mathrm{W}}_2{}'(s) = \rho_1(s)\left[\kappa_n(s)\mathrm{T}(s) + \tau_g^1(s)\mathrm{E}(s)\right] + \rho_2(s)\mathrm{D}(s)$$

and

$$\overline{\mathrm{W}}_3{}'(s) = -\rho_3(s)\mathrm{T}(s) + \rho_4(s)\left[\kappa_g^2(s)\mathrm{D}(s) + \tau_g^1(s)\mathrm{N}(s)\right],$$

where

$$\rho_1(s) = \left[\left(\frac{\kappa_n}{\tau_g^1}\right)' \frac{(\tau_g^1)^2}{\left(\kappa_n^2 + (\tau_g^1)^2\right)^{3/2}}\right](s),$$

$$\rho_2(s) = \frac{\kappa_n \kappa_g^2}{\sqrt{\kappa_n^2 + (\tau_g^1)^2}}(s),$$

$$\rho_3(s) = \frac{\kappa_n \kappa_g^2}{\sqrt{(\kappa_g^2)^2 + (\tau_g^1)^2}}(s),$$

$$\rho_4(s) = \left[\left(\frac{\kappa_g^2}{\tau_g^1}\right)' \frac{(\tau_g^1)^2}{\left((\kappa_g^2)^2 + (\tau_g^1)^2\right)^{3/2}}\right](s).$$

Now, let us consider the orthonormal frames $\{\mathrm{W}_1(s), \overline{\mathrm{W}}_2(s)\}$ and $\{\overline{\mathrm{W}}_2(s), \overline{\mathrm{W}}_3(s)\}$. We define the ruled hypersurfaces:

$$\varphi(s,u,v) = \delta(s) + u\mathrm{W}_1(s) + v\overline{\mathrm{W}}_2(s), \quad (15)$$

and

$$\psi(s,u,v) = \delta(s) + u\overline{\mathrm{W}}_2(s) + v\overline{\mathrm{W}}_3(s),$$

where $u, v \in \mathbb{R}, s \in I$, and call them as $\mathrm{W}_1\overline{\mathrm{W}}_2$-ruled hypersurface and $\overline{\mathrm{W}}_2\overline{\mathrm{W}}_3$-ruled hypersurface of δ, respectively.

Theorem 4.1 Let δ be a geodesic curve with arc-length parameter s lying on an orientable hypersurface $\mathcal{M}$. Then, (s_0, u_0, v_0) is a singular point of the $\mathrm{W}_1\overline{\mathrm{W}}_2$-ruled hypersurface of δ if and only if the equation

$$\kappa_n(s_0) - u_0\left(\kappa_g^2\tau_g^1\right)(s_0) + v_0\left[\rho_1\left((\kappa_n)^2 + (\tau_g^1)^2\right)\right](s_0) = 0. \quad (16)$$

holds.

Proof. If we calculate the partial derivatives of $\varphi(s,u,v)$, we find:

$$\varphi_s = [1 + v\rho_1(s)\kappa_n(s)]\mathrm{T}(s) + \left[v\rho_1(s)\tau_g^1(s) - u\kappa_g^2(s)\right]\mathrm{E}(s) + v\rho_2(s)\mathrm{D}(s),$$

$$\varphi_u = \mathrm{D}(s),$$

$$\varphi_v = \left(\frac{-\tau_g^1}{\sqrt{\kappa_n^2 + (\tau_g^1)^2}}\right)(s)\mathrm{T}(s) + \left(\frac{\kappa_n}{\sqrt{\kappa_n^2 + (\tau_g^1)^2}}\right)(s)\mathrm{E}(s).$$

So, we get :

$$\varphi_s \otimes \varphi_u \otimes \varphi_v = \frac{\kappa_n(s) - u\left(\kappa_g^2\tau_g^1\right)(s) + v\left[\rho_1\left(\kappa_n^2 + (\tau_g^1)^2\right)\right](s)}{\left(\sqrt{\kappa_n^2 + (\tau_g^1)^2}\right)(s)}\mathrm{N}(s). \quad (17)$$

It is known that (s_0, u_0, v_0) is a singular point of the $\mathrm{W}_1\overline{\mathrm{W}}_2$-ruled hypersurface of δ if and only if $(\varphi_s \otimes \varphi_u \otimes \varphi_v)(s_0, u_0, v_0) = 0$. Thus, the claim is clear from (17).

Corollary 4.1 If we write $u = v = 0$ in (15), we find the points of the curve δ. Then, $\delta(s)$ is a regular point of $\varphi(s,u,v)$ for all $s \in I$.

Proposition 4.1 Let δ be a geodesic curve with arc-length parameter s lying on an orientable hypersurface $\mathcal{M}$. The $W_1\overline{W}_2$-ruled hypersurface associated with δ is developable.

Proof. We get:

$$rank[\mathrm{T}, \mathrm{W}_1, \overline{\mathrm{W}}_2, \mathrm{W}_1', \overline{\mathrm{W}}_2{}'] = rank\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-\tau_g^1}{\sqrt{\kappa_n^2 + (\tau_g^1)^2}} & \frac{\kappa_n}{\sqrt{\kappa_n^2 + (\tau_g^1)^2}} & 0 & 0 \\ 0 & -\kappa_g^2 & 0 & 0 \\ \rho_1\kappa_n & \rho_1\tau_g^1 & \rho_2 & 0 \end{bmatrix} = 3.$$

Then, from (3), the $W_1\overline{W}_2$-ruled hypersurface associated with δ is developable.

Theorem 4.2 Let δ be a geodesic curve with arc-length parameter s lying on an orientable hypersurface $\mathcal{M}$. Then, (s_0, u_0, v_0) is a singular point of the $\overline{W}_2\overline{W}_3$-ruled hypersurface of δ if and only if the equations

$$\begin{cases} \kappa_n(s_0) + u_0\left[\rho_1\left(\kappa_n^2 + (\tau_g^1)^2\right)\right](s_0) \\ \qquad -v_0(\kappa_n\rho_3)(s_0) = 0, \\ v_0\left[\rho_4\left((\kappa_g^2)^2 + (\tau_g^1)^2\right)\right](s_0) + u_0\left(\kappa_g^2\rho_2\right)(s_0) = 0 \end{cases} \tag{18}$$

hold.

Proof. If we calculate the partial derivatives of $\psi(s, u, v)$, we obtain:

$$\psi_s = [1 + u\rho_1(s)\kappa_n(s) - v\rho_3(s)]\mathrm{T}(s) + [u\rho_1(s)\tau_g^1(s)]\mathrm{E}(s) + [u\rho_2(s) + v\rho_4(s)\kappa_g^2(s)]\mathrm{D}(s) + [v\rho_4(s)\tau_g^1(s)]\mathrm{N}(s),$$

$$\psi_u = \left(\frac{-\tau_g^1}{\sqrt{\kappa_n^2 + (\tau_g^1)^2}}\right)(s)\mathrm{T}(s) + \left(\frac{\kappa_n}{\sqrt{\kappa_n^2 + (\tau_g^1)^2}}\right)(s)\mathrm{E}(s),$$

$$\psi_v = \left(\frac{\kappa_g^2}{\sqrt{(\kappa_g^2)^2 + (\tau_g^1)^2}}\right)(s)\mathrm{N}(s) - \left(\frac{\tau_g^1}{\sqrt{(\kappa_g^2)^2 + (\tau_g^1)^2}}\right)(s)\mathrm{D}(s).$$

So, we have:

$$\psi_s \otimes \psi_u \otimes \psi_v = \Gamma_1\left(-\kappa_n(s)\mathrm{T}(s) - \tau_g^1(s)\mathrm{E}(s)\right) + \Gamma_2\left(\kappa_g^2(s)\mathrm{D}(s) + \tau_g^1(s)\mathrm{N}(s)\right), \tag{19}$$

where

$$\Gamma_1 = \frac{v\{\rho_4\left((\kappa_g^2)^2 + (\tau_g^1)^2\right)\}(s) + u\left(\kappa_g^2\rho_2\right)(s)}{\sqrt{\kappa_n^2 + (\tau_g^1)^2}\sqrt{(\kappa_g^1)^2 + (\tau_g^1)^2}}$$

and

$$\Gamma_2 = \frac{\kappa_n(s) + u\{\rho_1\left(\kappa_n^2 + (\tau_g^1)^2\right)\}(s) - v(\kappa_n\rho_3)(s)}{\sqrt{\kappa_n^2 + (\tau_g^1)^2}\sqrt{(\kappa_g^1)^2 + (\tau_g^1)^2}}.$$

Thus, the claim is clear from (19).

Corollary 4.2 $\delta(s)$, $\forall s \in I$, is a regular point of the $\overline{W}_2\overline{W}_3$-ruled hypersurface.

Proposition 4.2 Let δ be a geodesic curve with arc-length parameter s lying on an orientable hypersurface $\mathcal{M}$. The $\overline{W}_2\overline{W}_3$-ruled hypersurface associated with δ is non-developable.

Proof. We have:

$$rank[\mathrm{T}, \overline{W}_2, \overline{W}_3, \overline{W}_2{}', \overline{W}_3{}'] = 4.$$

Then, from (2), the $\overline{W}_2\overline{W}_3$-ruled hypersurface associated with δ is non-developable.

Example 4.1 Let us consider the unit speed geodesic curve δ given in Example 3.1. Since:

$$\overline{W}_2(s) = \frac{1}{2\sqrt{7}}\left(5\mathrm{E}(s) - \sqrt{3}\,\mathrm{T}(s)\right),$$

$$\overline{W}_3(s) = \frac{1}{\sqrt{5}}\left(2\mathrm{N}(s) - \mathrm{D}(s)\right),$$

we obtain the $\overline{W}_2\overline{W}_3$-ruled hypersurface of δ as:

$$\psi(s, u, v) = ((1 - \frac{v}{\sqrt{2}})\sin(\ell_1 s) - \frac{3\sqrt{21}}{14}u\cos(\ell_1 s), (1 - \frac{v}{\sqrt{2}})\cos(\ell_1 s) + \frac{3\sqrt{21}}{14}u\sin(\ell_1 s), (1 - \frac{v}{\sqrt{2}})\sin(\ell_2 s) + \frac{\sqrt{7}}{14}u\cos(\ell_2 s), (1 - \frac{v}{\sqrt{2}})\cos(\ell_2 s) - \frac{\sqrt{7}}{14}u\sin(\ell_2 s)).$$

If we consider the equation (18), we find the singular points of the $\overline{W}_2\overline{W}_3$-ruled hypersurface of δ as $(\eta, 0, \sqrt{2})$, $\forall \eta \in I$.

5 Conclusion

This paper focuses on a geodesic curve on an orientable hypersurface with nonvanishing curvatures of extended Darboux frame of the second kind in Euclidean 4-space $\mathbb{E}^4$. We define the linearly independent vector fields:

$$W_1 = \mathrm{D}, \quad W_2 = -\tau_g^1\mathrm{T} + \kappa_n\mathrm{E},$$
$$W_3 = \kappa_g^2\mathrm{N} - \tau_g^1\mathrm{D}, \quad W_4 = \mathrm{T}$$

along the curve. These vector fields enable us to rewrite the derivatives of the extended Darboux frame field vectors as ternary products of related vector fields. We also defined the W_1W_2-plane, W_3W_4-plane, and W_2W_3-plane along the curve and showed that the W_1W_2-plane and W_3W_4-plane play the role of Darboux vector. Furthermore, we introduced the W_1W_2-curve, W_3W_4-curve, and W_2W_3-curve and obtained their characterizations. Finally, two ruled hypersurfaces related to the newly defined vector fields have been constructed. Two examples have been given as applications of our results. Similar investigations can be researched for the extended Darboux frame of the first kind, or any other special frames along a curve in Euclidean 4-space or other spaces.

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