A New Construction of Rectifying Direction Curves for Quaternionic Space Q

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Abstract: Our article focuses on the study of quaternions topic introduced by Hamilton. Quaternions are a generalization of complex numbers and have multiple applications in mathematical physics. Another application of quaternions is robotics because what generalizes the imaginary axis is the family i, j, k modeling Euler angles and rotations in space. The first part of the article we recall the different definitions of how the algebra of quaternions is well constructed. The main results are given in the third part and concern: spatial quaternionics rectifying-direction (sqRD) curves and and spatial quaternionic rectifying-donor (sqRDnr) curves. We study a new tip of unit speed associated curves in $\mathbb{E}^3$, which is also used in robotic systems and kinematics, like a spatial quaternionic rectifying-direction curve and spatial quaternionic rectifying-donor curve. Then, we achieve qualification for the curves. Moreover, we present applications of spatial quaternionic rectifying-direction to some specific curves like helix, slant helix, Salkowski and anti-Salkowski curves or rectifying curves. In addition, we establish different theorems which generalize the results obtained on the quaternionic curves in Q. Then, we give some examples are finally discussed. Consequently, Our paper is centered around theoretical analysis in geometry rather than experimental investigations.

Key-Words: Associated curve, spatial quaternionic rectifying direction curve, spatial quaternionic rectifying donor curve, robotic systems.

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1 Introduction

Quaternions, one of the important building blocks of mathematical physics, were developed by Hamilton in 1843 with the motivation to expand complex numbers into three-dimensional space. Concentrating his studies on the triad number system defined as a+bi+cj, which has two complex and one real component, Hamilton aimed to represent a point in three-dimensional space with this number system. In this way, Hamilton tried to establish an algebraic system, but when he could not find any results in his studies, he abandoned this triad system and added a third imaginary component to this system. Therefore, he discovered real quaternions, which are very well known and frequently used in the scientific world today. After Hamilton, the other scientist studying on quaternions was [1]. The author used quaternions as a tool in his physical studies, and Maxwell, influenced by Tait, used quaternions in his work published in 1873 on electromagnetism, [2]. Although quaternions and vectors are different mathematical quantities, it was stated that vectors arise from different interpretations of quaternions, [3]. The vector algebra used today was found to be more useful by Gibbs and Heaviside as quaternions as expressed in Maxwell's electromagnetic theory and they laid the foundations of vector algebra used today, [4]. Quaternions are very unique in their structure and because of this structure they are very useful in the representation of rotational and translational motions. The study given by Chou on the derivation of the angular displacement, angular velocity, angular acceleration and momentum quantities associated with rotational motion is particularly interesting, [5]. In that studies by [5], [6], it was stated that when Euler parameters are considered as unit quaternions, quaternions have significant advantages in determining the orientation of a coordinate system compared to classical methods such as directional cosines. For this reason, rotational motion with quaternions has been studied by many researchers, [7], [8], [9]. In recent years, the studies on the representation of robotic systems have brought a different perspective to quaternions as if they were rediscovered, [10], [11]. On the other hand, some studies have shown that quaternions play an important role in control and simulation studies, [12], [13], [14]. In addition, quaternions can easily be used to represent physical quantities frequently used in classical mechanics. Quaternions also play an important role in relativistic mechanics, [15]. One of the application areas of quaternions is quantum mechanics. Since quaternions contain complex components, they can be used to represent wave functions and operators that are widely used in quantum mechanics, [16]. The studies of [17], [18], are remarkable in expressing the Schrödinger and Dirac equations. In these studies, the studies on quantum mechanics were examined both classically and relativistically. On the other hand, the curve theory is one of the important study areas in geometry. The curve theory studies date back to Newton in the 17th century. The Frenet-Serret formulas, which are one of the expressions characterizing curves, the most basic, and even used today, were found by Frenet (1847) and Serret (1851) independently from each other. After the Frenet frame of any curve was defined, a lot of special curve types such as plane, helix, slant helix have been defined, [19], [20], [21], [22], [23]. Some special curves can be also defined according to which Frenet plane where the position vector of this curve lies, that is, if the position vector of the space curve lies in the osculating plane, this curve is called the osculating curve, if the position vector lies in the normal plane, this curve is called the normal curve and if the position vector lies in the rectifying plane, this curve is called a rectifying curve, [24]. Moreover, a new curve pair defined by [25]. Then, many studies were carried out on the direction curves, [26], [27], [28], [29], [30].

Scientists have shown great interest in quaternions and have done many studies on quaternionic curves by combining quaternions with the theory of curves. First of all, the Serret-Frenet formulas of any curve in 3-dimensional real Euclidean space in $\mathbb{R}^3$ were produced by spatial quaternionic curves by [31]. With help of these formulas, the Serret-Frenet formulas of one-variable quaternion-valued functions (quaternionic curves) in $\mathbb{R}^4$ were obtained, [31]. The quaternionic rectifying curve in $\mathbb{E}^4$ and $\mathbb{E}^4_2$ were studied by [32], [33]. In addition to that, the quaternionic direction curves were given by [29]. Then a lot of papers were studied on the quaternionic curve and quaternionic direction curves, [29], [32], [33], [34], [35], [36], [37], [38], [39], [40], [41].

2 Preliminaries

The Serret-Frenet formulas of spatial (pure) quaternionic and quaternionic curves expressed by [31]. The study, [31], contributed greatly to the development of quaternions. Now, we give briefly these notions.

A real quaternion is given by $\mathfrak{q} = a_1 i + a_2 j + a_3 k + a_4$, where

$i)$ $i^2 = j^2 = k^2 = ijk = -1$

$ii)$ $ij = k = -ji,\ jk = i = -kj\ ki = j = -ik.$

in quaternionic space Q. It is worth noting here that quaternionic space Q is isomorphic to four dimensional real vector space $\mathbb{R}^4$. Now, we assume two quaternion $\mathfrak{q}_1 = S_{q_1} + V_{q_1} = a_1 i + a_2 j + a_3 k + a_4$ and $\mathfrak{q}_2 = S_{q_2} + V_{q_2} = b_1 i + b_2 j + b_3 k + b_4$ where $S_{q_1} = a_4, V_{q_1} = a_1 i + a_2 j + a_3 k$, $S_{q_2} = b_4, V_{q_2} = b_1 i + b_2 j + b_3 k$. Therefore, the quaternionic multiplication of these quaternions is

$$\mathfrak{q}_1 \times \mathfrak{q}_2 = S_{q_1} S_{q_2} - \langle V_{q_1}, V_{q_2} \rangle + S_{q_1} V_{q_2} + S_{q_2} V_{q_1} + V_{q_1} \wedge V_{q_2}.$$

which the vectorial and scalar product in $\mathbb{E}^3$ are considered $\wedge$ and $\langle , \rangle$, respectively. In addition, we assume that the quaternionic conjugate of $\mathfrak{q}$ is $\gamma\mathfrak{q}$. In that case, for the quaternions $\mathfrak{q}_1$ and $\mathfrak{q}_2$, the $\mathfrak{h}$-inner product is

$$\mathfrak{h}(\mathfrak{q}_1, \mathfrak{q}_2) = \frac{1}{2}(\mathfrak{q}_1 \times \gamma\mathfrak{q}_2 + \mathfrak{q}_2 \times \gamma\mathfrak{q}_1).$$

Now, we can give the definition of norm in quaternionic space Q with the aid of the $\mathfrak{h}$-inner product. Therefore, the norm of the quaternion q is expressed

$$\| q\|^2 = |h(q,q)| = q \times \gamma q = \gamma q \times q.$$

Moreover, the quaternion $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are expressed as $\mathfrak{h}$- orthogonal quaternions if the equation $\mathfrak{h}(\mathfrak{q}_1, \mathfrak{q}_2) = 0$ is hold.

Now, we expressed the definitions of the spatial (pure) quaternionic and quaternionic curves in quaternionic space Q from [31]. In that case, we take $I = [0,1] \subseteq \mathbb{R}$ therefore, the spatial quaternionic curve (ρ) is defined that

$$\begin{aligned} \rho : I \subset \mathbb{R} &\longrightarrow Q, \\ \varrho &\longrightarrow \rho(\varrho) = \rho_1(\varrho) i + \rho_2(\varrho) j + \rho_3(\varrho) k \end{aligned}$$

where $\varrho \in I$ is arc-length parameter of the spatial quaternionic curve (ρ) with non-zero curvatures. Therefore, if we take that the curve (ρ) is the spatial quaternionic curve with arc-length parameter. The following theorem for Serret-Frenet formulas can be given.

Theorem 1. *Suppose that the Frenet frame and curvatures of the spatial quaternionic curve (ρ) are $\{\mathfrak{t}, \mathfrak{n}, \mathfrak{b}\}$ and $\{\varkappa, \tau\}$, respectively. Therefore, the Frenet formulas of this curve (ρ) can be expressed that*

$$\mathfrak{t}' = \varkappa\mathfrak{n},\ \mathfrak{n}' = -\varkappa\mathfrak{t} + \tau\mathfrak{b},\ \mathfrak{b}' = -\tau\mathfrak{n} \qquad (1)$$

which $\{\varkappa, \tau\}$ are the curvature and torsion of curve (ρ), respectively and $\mathfrak{h}(\mathfrak{t}, \mathfrak{t}) = \mathfrak{h}(\mathfrak{n}, \mathfrak{n}) = \mathfrak{h}(\mathfrak{b}, \mathfrak{b}) = 1$, $\mathfrak{h}(\mathfrak{t}, \mathfrak{n}) = \mathfrak{h}(\mathfrak{t}, \mathfrak{b}) = \mathfrak{h}(\mathfrak{n}, \mathfrak{b}) = 0$, [31].

Now, we define the quaternionic curve, similarly. Let the unit interval in $\mathbb{R}$ be $I = [0,1] \subseteq \mathbb{R}$. Therefore, the real quaternionic curve is

$$\begin{aligned} \phi : I \subset \mathbb{R} \longrightarrow & Q, \\ \varrho \longrightarrow & \phi(\varrho) = \phi_0(\varrho) + \phi_1(\varrho)i + \phi_2(\varrho)j + \phi_3(\varrho)k \end{aligned}$$

where $\varrho \in I = [0,1]$ is the arc-length parameter along the smooth quaternionic curve (ϕ) with non-zero curvatures. Therefore, the following theorem about quaternionic Serret-Frenet formulas can be given.

Theorem 2. *Consider that the quaternionic curve in Q is (ϕ) with arch-length parameter $\varrho \in I$ and non-zero curvatures $\{\mathcal{K}, \varkappa, (\tau - \mathcal{K})\}$ and Frenet frame of this quaternionic curve (ϕ) is $\{\top, \eta, \beta_1, \beta_2\}$. Therefore, the Serret-Frenet formulas of (ϕ) are*

$$\begin{aligned} \top' &= \mathcal{K}\eta, \\ \eta' &= -\mathcal{K}\top + \varkappa\beta_1, \\ \beta_1' &= -\varkappa\eta + (\tau - \mathcal{K})\beta_2, \\ \beta_2' &= -(\tau - \mathcal{K})\beta_1 \end{aligned} \tag{2}$$

where $\mathcal{K} = \| \top'(\varrho) \|$, $\eta = \mathfrak{t} \times \top$, $\beta_1 = \mathfrak{n} \times \top$, $\beta_2 = \mathfrak{b} \times \top$ and there are some connections with the curvatures of (ρ) and (ϕ) such that the torsion $(\varkappa)$ of the (ϕ) is the principal curvature of the spatial quaternionic curve (ρ) and we assume that the torsion of the spatial quaternionic curve (ρ) is r, the principal curvatures of the quaternionic curve (ϕ) is $\mathcal{K}$ therefore, the bi-torsion of the quaternionic curve (ϕ) is $(\tau - \mathcal{K})$, [31].

Definition 1. *Suppose that the tangent vector field $\mathfrak{t}$ of the spatial quaternionic curve (ρ) intersects a constant vector with a constant angle for all points of (ρ) therefore, the spatial quaternionic curve (ρ) is called helix, [40].*

Theorem 3. *Consider that the curve (ρ) is with non-zero curvatures $\{\varkappa, \tau\}$ and (ρ) is helix. Therefore, the necessary and sufficient condition is*

$$\frac{\tau}{\varkappa} = constant \tag{3}$$

at every point, $\forall \varrho \in I$, [40].

Definition 2. *Let a spatial quaternionic curve in Q be (ρ). In that case, the normal vector field $\mathfrak{n}$ intersects a constant vector with a constant angle at every point of (ρ) therefore, the spatial quaternionic curve (ρ) is called slant helix, [41].*

Definition 3. *Assume that the spatial quaternionic curve ρ with non-zero curvatures $\{\varkappa, \tau\}$ is slant helix. Therefore, the necessary and sufficient condition is*

$$\frac{\varkappa^2}{(\tau^2 + \varkappa^2)^{\frac{3}{2}}} \left(\frac{\tau}{\varkappa}\right)' = constant \tag{4}$$

at every point of (ρ), [41].

Definition 4. *Let $\rho(\sigma) : I \rightarrow Q$ be spatial quaternionic curve and $\varkappa$ and τ be the curvatures of $\rho(\sigma)$. Therefore, $\rho(\sigma)$ is defined Salkowski curve such that $\varkappa$ is constant and torsion τ is non-constant. If $\varkappa$ is non-constant and τ is constant, $\rho(\sigma)$ is defined anti-Salkowski curve, [42], [43].*

Moreover, if the spatial quaternionic frame and curvatures of (ρ) are given by $\{\mathfrak{t}, \mathfrak{n}, \mathfrak{b}\}$ and $\{\varkappa, \tau\}$, respectively, then the vector $\tilde{D}(\sigma) = \frac{\tau}{\varkappa}(\sigma)\mathfrak{t}(\sigma) + \mathfrak{b}(\sigma)$ is defined as modified Darboux vector of (ρ), [40].

3 Main Theorems and Proofs

While we primarily focus on the geometric structure underlying quaternions used in physics and presents specific applications of spatial quaternionic curves, we also mentions the significance of quaternions in various fields such as robotics, control, simulation studies, classical mechanics, relativistic mechanics, and quantum mechanics. Given the extensive applications and ongoing research interest in quaternions, the our study suggests potential avenues for further exploration. For example, we discuss the representation of rotational motion using quaternions and highlights the advantages we offer over classical methods. Therefore, this study could inspire future studies to delve deeper into the application of quaternions in rotational dynamics and its implications for various fields such as robotics and mechanical engineering. Furthermore, our study touch upon the role of quaternions in quantum mechanics, indicating a potential area for future investigation. Since quaternions can represent wave functions and operators in quantum mechanics, readers may explore how quaternionic formulations can provide new insights or computational advantages in quantum mechanical simulations and analysis. Additionally, we mention the study of quaternionic curves in conjunction with the theory of curves, suggesting that there may be further advancements in understanding the geometric properties and applications of quaternionic curves. Future research could focus on developing new methods for analyzing and utilizing quaternionic curves in geometry, robotics, or other relevant fields. Therefore, in this section, firstly, we define spatial quaternionic rectifying-direction $(sqRD)$ curves and spatial quaternionic rectifying-donor $(sqRDnr)$ curves. Then, we obtain some characterizations of these curves with theorems. In addition, we give some corollaries about spatial quaternionic frames of these curves. After that, we express some applications of $(sqRD)$-curves such as whether these curves can be general and slant helices or plane curves, and spatial quaternionic rectifying-direction osculating curves $(sqRDO)$, spatial quaternionic rectifying-direction normal curves $(sqRDN)$, spatial quaternionic rectifying-direction rectifying $(sqRDR)$ curves. Consequently, we generalize the theorems and corollaries mentioned above for quaternionic curve in Q.

3.1 On the $(sqRD)$-curves

Definition 5. *Suppose that any spatial quaternionic curve is (ρ) and any unit vector field ξ is on the rectifying plane of (ρ) in $\mathbb{E}^3$. Therefore, this vector field ξ is defined as*

$$\xi(\sigma) = \mu(\sigma)\mathfrak{t}(\sigma) + \varpi(\sigma)\mathfrak{b}(\sigma) \tag{5}$$

where $\mu(\sigma) \neq 0, \varpi(\sigma) \neq 0$, $\mu(\sigma)^2 + \varpi(\sigma)^2 = 1$ and ξ' and $\mathfrak{n}$ are linearly dependent. In addition, if we assume the integral curve $\lambda : I \to \mathbb{E}^3$, parameterized with arc-length parameter σ, of the vector field ξ, then the integral curve (λ) is called $(sqRD)$-curve of (ρ) and (ρ) is also called $(sqRDnr)$-curve in $\mathbb{E}^3$.

Theorem 4. *Consider that the curve $\rho(\sigma)$ is spatial quaternionic curve, the unit vector field $\xi(\sigma) = \mu(\sigma)\mathfrak{t}(\sigma)+\varpi(\sigma)\mathfrak{b}(\sigma)$ is on the rectifying plane of (ρ). Therefore, the spatial quaternionic integral curve (λ) of ξ is a $(sqRD)$-curve of (ρ), the necessary and sufficient condition is that μ and ϖ are constants in equation (5).*

Proof. Let the vector field ξ be on the rectifying plane of curve (ρ) and (λ) be an integral curve with unit speed of ξ. Moreover, we suppose that (λ) is $(sqRD)$-curve of (ρ). Therefore, we can give $\xi(\sigma) = \mu(\sigma)\mathfrak{t}(\sigma)+\varpi(\sigma)\mathfrak{b}(\sigma)$ where $\mu^2(\sigma)+\varpi^2(\sigma) = 1$ from the equation (5). If we take derivative of the equation (5) with respect to σ, we get

$$\xi'(\sigma) = \mu'\mathfrak{t} + (\mu\varkappa - \varpi\tau)\mathfrak{n} + w'\mathfrak{b}. \tag{6}$$

one knows that ξ' and $\mathfrak{n}$ are linearly dependent, therefore we have

$$\begin{cases} \mu' = 0, \\ \mu\varkappa - \varpi\tau \neq 0, \\ \varpi' = 0. \end{cases} \tag{7}$$

and, consequently

$$\mu = \text{constant}, \quad \varpi = \text{constant}.$$

□

Now, we suppose that the angle between the vector fields ξ and $\mathfrak{t}$ is α, therefore we can write

$$\xi(\sigma) = \cos\alpha\mathfrak{t}(\sigma) + \sin\alpha\mathfrak{b}(\sigma), \tag{8}$$

and give the following corollary.

Corollary 1. *The angle α between the vector field ξ and tangent vector field $\mathfrak{t}$ of $(sqRDnr)$-curve (ρ) is constant.*

Theorem 5. *Consider that ξ is the vector field on the rectifying plane of the spatial quaternionic curve $\rho(\sigma) : I \to \mathbb{E}^3$ and the integral curve (λ) with unit speed of ξ is $(sqRD)$-curve of $(sqRDnr)$-curve (ρ). Therefore, the curve pair (λ, ρ) is a spatial quaternionic Bertrand curve pair.*

Proof. The unit speed curve (λ) is considered $(sqRD)$-curve of (ρ), $(sqRDnr)$-curve. In that case, since (λ) is spatial quaternionic integral curve of ξ, we know that $\lambda' = \xi$. Now, we assume that the spatial quaternionic frame of (λ) is $\{\bar{\mathfrak{t}}, \bar{\mathfrak{n}}, \bar{\mathfrak{b}}\}$. Therefore, we get $\lambda'' = \bar{\varkappa}\bar{\mathfrak{n}} = \xi'$ since the spatial quaternionic curve (λ) is unit speed curve where $\bar{\varkappa}$ is the curvature of (λ). On the other hand, ξ' and $\mathfrak{n}$ are linearly dependent, therefore $\mathfrak{n}$ is linearly dependent with $\bar{\mathfrak{n}}$. Consequently, the curves (λ, ρ) is spatial quaternionic Bertrand curves. □

Theorem 6. *Consider that the vector field ξ is on the rectifying plane of (ρ) and the quaternionic curvatures of the integral curve (λ) of ξ are $\{\bar{\varkappa}, \bar{\tau}\}$. Therefore, if the curve (λ) is $(sqRD)$-curve of $(sqRDnr)$-curve (ρ), then*

$$\begin{aligned} \bar{\varkappa} &= \cos\alpha\varkappa - \sin\alpha\tau, \\ \bar{\tau} &= \sin\alpha\varkappa + \cos\alpha\tau. \end{aligned} \tag{9}$$

where $\{\varkappa, \tau\}$ are quaternionic curvatures of (ρ) and the angle α is between ξ and $\mathfrak{t}$.

Proof. Let the integral curve (λ) be $(sqRD)$-curve of (ρ) with the vector field ξ and $\{\varkappa, \tau\}$, $\{\bar{\varkappa}, \bar{\tau}\}$ be Frenet curvatures of the spatial quaternionic curves (ρ, λ), respectively. In that case, if we use equation (8), Corollary (1) and Theorem (5), we get

$$\bar{\varkappa}\bar{\mathfrak{n}} = (\cos\alpha\varkappa - \sin\alpha\tau)\mathfrak{n},$$

and we can take

$$\bar{\varkappa} = \cos\alpha\varkappa - \sin\alpha\tau. \tag{10}$$

where $\bar{\mathfrak{n}} = \mathfrak{n}$ from Theorem (5).

Now, we can write

$$\bar{\mathfrak{b}} = \bar{\mathfrak{t}} \times \bar{\mathfrak{n}} = -\sin\alpha\mathfrak{t} + \cos\alpha\mathfrak{b}.$$

and

$$\bar{\mathfrak{b}}' = -(\cos\alpha\tau + \sin\alpha\varkappa)\mathfrak{n}. \tag{11}$$

On the other hand, from the definition of the unit speed spatial quaternionic curve (λ) we know $\bar{\tau} = -\langle\bar{\mathfrak{b}}', \bar{\mathfrak{n}}\rangle = -\langle\bar{\mathfrak{b}}', \mathfrak{n}\rangle$ consequently, with the equation (11) we get

$$\bar{\tau} = \cos\alpha\tau + \sin\alpha\varkappa. \tag{12}$$

□

In addition that, we can write

$$\varkappa = \cos\alpha\bar{\varkappa} + \sin\alpha\bar{\tau}, \ \ \tau = -\sin\alpha\bar{\varkappa} + \cos\alpha\bar{\tau}.$$

Corollary 2. *Let* (λ) *be* $(sqRD)$*-curve of* $(sqRDnr)$*-curve* (ρ)*. Therefore, the correlation between spatial quaternionic frame vectors of these curves can be given as*

$$\xi = \bar{\mathfrak{t}} = \cos\alpha\mathfrak{t} + \sin\alpha\mathfrak{b}, \quad \bar{\mathfrak{n}} = \mathfrak{n}, \quad \bar{\mathfrak{b}} = -\sin\alpha\mathfrak{t} + \cos\alpha\mathfrak{b},$$

or

$$\mathfrak{t} = \cos\alpha\bar{\mathfrak{t}} - \sin\alpha\bar{\mathfrak{b}}, \quad \mathfrak{n} = \bar{\mathfrak{n}}, \quad \mathfrak{b} = \sin\alpha\bar{\mathfrak{t}} + \cos\alpha\bar{\mathfrak{b}}. \tag{13}$$

3.2 Applications of $(sqRD)$-curves

Now, we give some applications for $(sqRD)$-curves such as general helices, slant helices, plane curves and we obtain $(sqRDO)$, $(sqRDN)$, and $(sqRDR)$-curves.

3.2.1 Applications to General Helices and Slant Helices

Now, we assume the second equation in the equation (7) and we see that the function $\frac{\tau}{\varkappa}(\sigma)$ is not a constant, provided that τ is non-zero. Then the following theorem can be written.

Theorem 7. *The* $(sqRDnr)$*-curve* (ρ) *of the curve* (λ) *is not a general helix.*

Now, we investigate the existence of condition to be a general helix for $(sqRD)$-curve (λ). If we consider $(sqRD)$-curve (λ) is a general helix, considering Theorem (6), we have

$$\frac{\bar{\tau}}{\bar{\varkappa}}(\sigma) = \frac{\sin\alpha\varkappa + \cos\alpha\tau}{\cos\alpha\varkappa - \sin\alpha\tau} = c = constant. \tag{14}$$

After that, considering the equation (14) and Corollary (1), one can see that the function

$$\frac{\tau}{\varkappa} = \frac{c - \tan\alpha}{1 + c\tan\alpha}.$$

Therefore, the curve (ρ) is a general helix where α is constant. In this case, there appears to be a contradiction with Theorem (7) and the following theorem can be written.

Theorem 8. *Let* (λ) *be a* $(sqRD)$*-curve of the spatial quaternionic curve* $\rho : I \to \mathbb{E}^3$*. In this case the following items are equivalent:*

i) Any $(sqRDnr)$*-curve* (ρ) *can not be a general helix.*

ii) Any $(sqRD)$*-curve* (λ) *of* (ρ) *can not be a general helix.*

Moreover, considering Theorem (6), we can give a similar theorem to Theorem (8) for slant helices.

Theorem 9. *Let* $\rho : I \to \mathbb{E}^3$ *be* $(sqRDnr)$*-curve and* (λ) *be* $(sqRD)$*-curve of* (ρ)*. In this case the* $(sqRDnr)$*-curve* (ρ) *is slant helix therefore, the necessary and sufficient condition is that* $(sqRD)$*-curve* (λ) *of* (ρ) *is a slant helix.*

Example 1. *Let the spatial quaternionic Salkowski curve be* (ρ) *and this curve be parametrized as follows.*

$$\rho(\sigma) = \frac{1}{\sqrt{1+y^2}} \begin{pmatrix} -\frac{1-z}{4(1+2z)}\sin((1+2z)\sigma) \\ -\frac{1+z}{4(1-2z)}\sin((1-2z)\sigma) - \frac{1}{2}\sin\sigma, \\ \frac{1-z}{4(1+2z)}\cos((1+2z)\sigma) \\ +\frac{1+z}{4(1-2z)}\cos((1-2z)\sigma) + \frac{1}{2}\cos\sigma, \\ \frac{1}{4z}\cos(2z\sigma) \end{pmatrix}$$

where $z = \frac{y}{\sqrt{1+y^2}}$*,* $y \neq \pm\frac{1}{\sqrt{3}}, 0$ *are constants. This curve is also a slant helix and we have*

$$\mathfrak{t}(\sigma) = -\begin{pmatrix} \cos\sigma\cos(z\sigma) + z\sin\sigma\sin(z\sigma), \\ \cos(z\sigma)\sin\sigma - z\cos\sigma\sin(z\sigma), \\ \frac{z}{y}\sin(z\sigma) \end{pmatrix},$$

$$\mathfrak{n}(\sigma) = z\left(\frac{\sin\sigma}{y}, -\frac{\cos\sigma}{y}, -1\right),$$

$$\mathfrak{b}(\sigma) = \begin{pmatrix} z\sin\sigma\cos(z\sigma) - \cos\sigma\sin(z\sigma), \\ -z\cos(z\sigma)\cos\sigma - \sin\sigma\sin(z\sigma), \\ \frac{z}{y}\cos(z\sigma) \end{pmatrix},$$

$$\varkappa(\sigma) = 1, \ \ \tau(\sigma) = \tan(z\sigma).$$

Then by choosing $\alpha = \frac{\pi}{4}$*, from* (8) *we get* $\xi(\sigma) = (x_1(\sigma), x_2(\sigma), x_3(\sigma))$ *where*

$$x_1(\sigma) = \frac{\sqrt{2}}{2}\begin{pmatrix} -\cos\sigma\cos(z\sigma) - z\sin\sigma\sin(z\sigma) \\ +z\sin\sigma\cos(z\sigma) - \cos\sigma\sin(z\sigma) \end{pmatrix},$$

$$x_2(\sigma) = \frac{\sqrt{2}}{2}\begin{pmatrix} -\cos(z\sigma)\sin\sigma + z\cos\sigma\sin(z\sigma) \\ -z\cos(z\sigma)\cos\sigma - \sin\sigma\sin(z\sigma) \end{pmatrix},$$

$$x_3(\sigma) = \frac{\sqrt{2}}{2}\frac{z}{y}\left(-\sin(z\sigma) + \cos(z\sigma)\right).$$

Now, we obtain $(sqRD)$*-curve* (λ) *as follows*

$$\lambda(\sigma) = \int_0^\sigma \lambda'(\sigma)d\sigma = \int_0^\sigma \xi(\sigma)d\sigma = (\lambda_1(\sigma), \lambda_2(\sigma), \lambda_3(\sigma))$$

where

$$\lambda_1(\sigma) = \int_0^\sigma \begin{bmatrix} \frac{\sqrt{2}}{2}(-\cos\sigma\cos(z\sigma) - z\sin\sigma\sin(z\sigma) \\ +z\sin\sigma\cos(z\sigma) - \cos\sigma\sin(z\sigma)) \end{bmatrix} d\sigma,$$

$$\lambda_2(\sigma) = \int_0^\sigma \begin{bmatrix} \frac{\sqrt{2}}{2}(-\cos(z\sigma)\sin\sigma + z\cos\sigma\sin(z\sigma) \\ -z\cos(z\sigma)\cos\sigma - \sin\sigma\sin(z\sigma)) \end{bmatrix} d\sigma,$$

$$\lambda_3(\sigma) = \int_0^\sigma \frac{\sqrt{2}}{2}\frac{z}{y}(-\sin(z\sigma) + \cos(z\sigma))d\sigma$$

(Fig. 1 and Fig. 2) and from Theorem (9)*,* (λ) *is also a slant helix.*

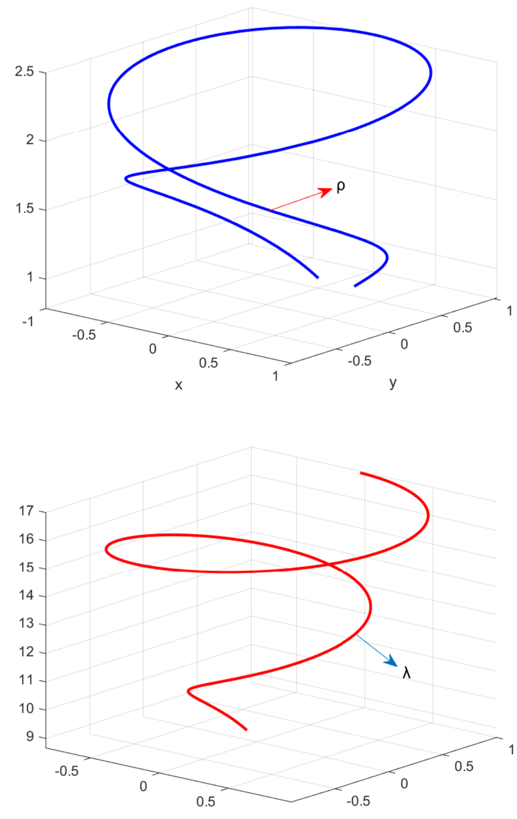

Figure 1: The Slant helix $(sqRDnr)$-curve (ρ) for $y = 1/10$ (left) and $(sqRD)$-curve (λ) of (ρ) (right).

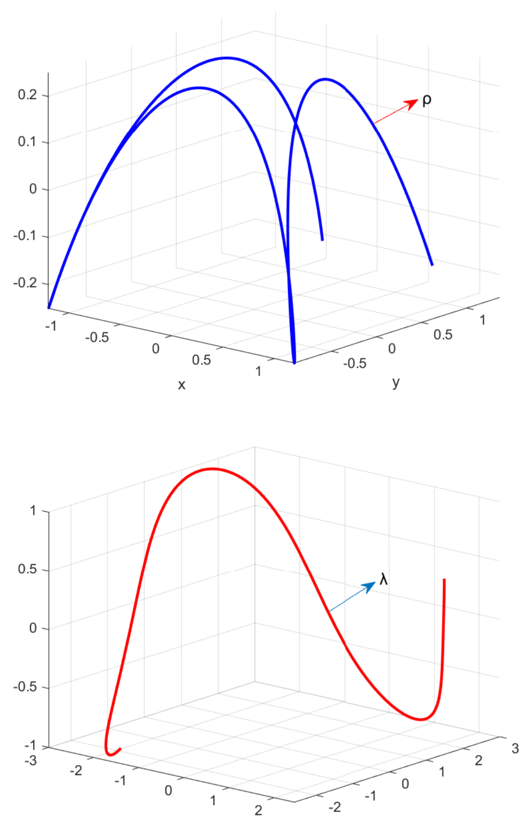

Figure 2: The Slant helix $(sqRDnr)$-curve (ρ) for $y = 1$ (left) and $(sqRD)$-curve (λ) of (ρ) (right).

3.2.2 Applications to Plane Curves

Now, we suppose that the $(sqRDnr)$-curve (ρ) is plane curve, especially. In this case, we know that the torsion of (ρ) is hold $\tau = 0$. Therefore, considering the equation (5) we have

$$\bar{\varkappa} = \cos\alpha\varkappa, \quad \bar{\tau} = \sin\alpha\varkappa \tag{15}$$

and $\frac{\bar{\tau}}{\bar{\varkappa}} = constant$ where $\{\bar{\varkappa}, \bar{\tau}\}$ are the spatial quaternionic curvatures of $(sqRD)$-curve (λ) and (λ) is general helix. In addition to that, if we consider that the $(sqRD)$-curve (λ) is general helix ($\frac{\bar{\tau}}{\bar{\varkappa}} = constant$), then the $(sqRDnr)$-curve (ρ) is general helix ($\frac{\tau}{\varkappa} = constant$). This situation contradicts with Theorem (7). Consequently, if (λ) is general helix, so that $\frac{\tau}{\varkappa} = constant$ the necessary and sufficient condition is the situation $\tau = 0$. Therefore we give the following proposition.

Proposition 1. *Assume that the $(sqRD)$-curve (λ) is spatial quaternionic rectifying direction curve of $(sqRDnr)$-curve (ρ). In this case, the $(sqRDnr)$-curve (ρ) is plane curve therefore, the necessary and sufficient condition is that the $(sqRD)$-curve (λ) is general helix.*

3.3 The $(sqRDO)$-curves

In this chapter, we describe spatial quaternionic rectifying-direction osculating $(sqRDO)$-curves in $\mathbb{E}^3$.

Definition 6. *Suppose that $\{\mathfrak{t}, \mathfrak{n}, \mathfrak{b}\}$ is the Frenet frame of $(sqRDnr)$-curve (ρ) and (λ) is $(sqRD)$-curve of (ρ). The curve (λ) is called $(sqRDO)$-curve of (ρ), if the position vector of (λ) for $\forall\sigma \in I$ lying on the osculating plane of its $(sqRDnrO)$-curve (ρ).*

Considering the definition of $(sqRDO)$-curve one can write the following equation

$$\lambda(\sigma) = u(\sigma)\mathfrak{t}(\sigma) + v(\sigma)\mathfrak{n}(\sigma) \tag{16}$$

where $u(\sigma)$, $v(\sigma)$ are non-zero differentiable functions dependent on parameter σ. Differentiating (16) and substituting the first equality of (13) in obtained equation, it follows

$$\cos\alpha\mathfrak{t} + \sin\alpha\mathfrak{b} = (u' - v\varkappa)\mathfrak{t} + (v' + u\varkappa)\mathfrak{n} + v\tau\mathfrak{b}.$$

and

$$\begin{cases} u' - v\varkappa = \cos\alpha, \\ v' + u\varkappa = 0, \\ v\tau = \sin\alpha, \end{cases}$$

finally,

$$u(\sigma) = \sin\alpha \frac{\tau'}{\varkappa\tau^2}, \quad v(\sigma) = \frac{\sin\alpha}{\tau}. \tag{17}$$

Substituting (17) in (16) we have

$$\lambda(\sigma) = \frac{\sin\alpha}{\tau}\left(\frac{\tau'}{\varkappa\tau}\mathfrak{t}(\sigma) + \mathfrak{n}(\sigma)\right).$$

Then the following theorem can be written.

Theorem 10. *Consider that (λ) is the $(sqRDO)$-curve of $(sqRDnrO)$-curve (ρ) in $\mathbb{E}^3$. In this case the following items are equivalent:*

i) (λ) is a (sqRDO)-curve of (ρ).

ii) The parametric representation of (λ) is given by

$$\lambda(\sigma) = \frac{\sin\alpha}{\tau}\left(\frac{\tau'}{\varkappa\tau}\mathfrak{t}(\sigma) + \mathfrak{n}(\sigma)\right)$$

where α is the constant angle between the curves (ρ) and (λ).

Theorem 11. *Suppose that (λ) is the (sqRDO)-curve of (ρ). In this case the following items are equivalent:*

i) The $(sqRDnrO)$-curve (ρ) is an anti-Salkowski curve in $\mathbb{E}^3$.

ii) The position vector of (sqRDO)-curve (λ) is linearly dependent with $\mathfrak{n}(\sigma)$ of (ρ).

Example 2. *Let the $(sqRDnrO)$-curve (ρ) be obtained as*

$$\rho(\sigma) = \left(-\tfrac{3}{4}\left(\tfrac{\cos 3\sigma}{9} + \cos\sigma\right), -\tfrac{3}{4}\left(\tfrac{\sin 3\sigma}{9} + \sin\sigma\right), \tfrac{\sqrt{3}}{2}\cos\sigma \right)$$

(Fig. 3). The required spatial quaternionic frame vectors and curvatures of $\rho(\sigma)$ are calculated by

$$\mathfrak{t}(\sigma) = \left(\tfrac{1}{4}\sin 3\sigma + \tfrac{3}{4}\sin\sigma, \ -\tfrac{1}{4}\cos 3\sigma - \tfrac{3}{4}\cos\sigma, \ -\tfrac{\sqrt{3}}{2}\sin\sigma \right),$$

$$\mathfrak{n}(\sigma) = \left(\tfrac{\sqrt{3}}{4}\tfrac{\cos(3\sigma)}{\cos\sigma} + \tfrac{\sqrt{3}}{4}, \ \tfrac{\sqrt{3}}{4}\tfrac{\sin(3\sigma)}{\cos\sigma} + \tfrac{\sqrt{3}}{4}\tan(\sigma), -\tfrac{1}{2}) \right)$$

and

$$\varkappa = \sqrt{3}\cos\sigma, \quad \tau = \sqrt{3}\sin\sigma,$$

respectively. Then, from Theorem (10), (sqRDO)-curve (λ) is obtained as follows,

$$\lambda(\sigma) = \left(\tfrac{\sin 3\sigma}{12\sin\sigma} + \tfrac{\cos 3\sigma}{4\cos\sigma} + \tfrac{1}{2}, -\tfrac{\cos 3\sigma}{12\sin\sigma} - \tfrac{\cos\sigma}{4\sin\sigma} + \tfrac{\sin\sigma}{4\cos\sigma} + \tfrac{\sin 3\sigma}{4\cos\sigma}, -\tfrac{2\sqrt{3}}{3} \right)$$

which is drawn in Fig 3.

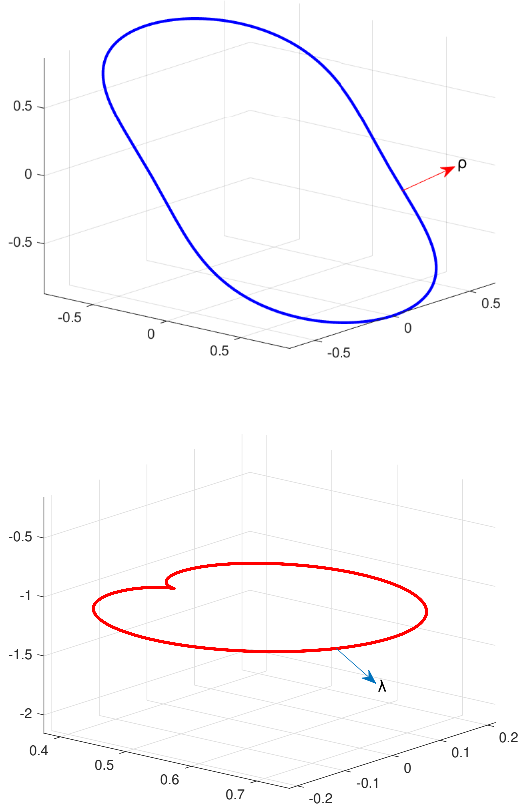

Figure 3: $(sqRDnrO)$-curve (ρ) (left) and $(sqRDO)$-curve (λ) (right).

3.4 The $(sqRDN)$-curves

Now, assume that (ρ) is $(sqRDnr)$-curve which have quaternionic frame $\{\mathfrak{t}, \mathfrak{n}, \mathfrak{b}\}$ and (λ) is $(sqRD)$-curve of (ρ). Therefore, (λ) is called spatial quaternionic rectifying-direction normal curve (or $(sqRDN)$-curve) of (ρ), if the position vector of (λ) always lies on the normal plane of its $(sqRDnrN)$-curve (ρ).

Therefore, one can write the parametric representation of (λ) as

$$\lambda(\sigma) = y(\sigma)\mathfrak{n}(\sigma) + z(\sigma)\mathfrak{b}(\sigma), \tag{18}$$

where $y(\sigma)$, $z(\sigma)$ are non-zero differentiable functions dependent on σ. Differentiating (18) and considering (13) in obtained equality, it follows

$$\cos\alpha\mathfrak{t} + \sin\alpha\mathfrak{b} = -y\varkappa\mathfrak{t} + (y' - z\tau)\mathfrak{n} + (z' + y\tau)\mathfrak{b}.$$

Therefore, we can find system of equations

$$\begin{cases} -y\varkappa = \cos\alpha, \\ y' - z\tau = 0, \\ z' + y\tau = \sin\alpha. \end{cases} \tag{19}$$

In this case, we get

$$y(\sigma) = -\frac{\cos\alpha}{\varkappa}, \quad z(\sigma) = \cos\alpha\frac{\varkappa'}{\varkappa^2\tau}. \tag{20}$$

Substituting (20) in (18), we get also

$$\lambda(\sigma) = -\frac{\cos\alpha}{\varkappa}\left(\mathfrak{n}(\sigma) - \frac{\varkappa'}{\varkappa\tau}\mathfrak{b}(\sigma)\right). \tag{21}$$

Therefore, we can give the following theorem.

Theorem 12. *Let (λ) be the $(sqRD)$-curve of the $(sqRDnr)$-curve (ρ). In this case, the following items are equivalent:*

i) (λ) is the $(sqRDN)$-curve of $(sqRDnrN)$-curve (ρ).

ii) The parametric representation of (λ) is

$$\lambda(\sigma) = -\frac{\cos\alpha}{\varkappa}\left(\mathfrak{n}(\sigma) - \frac{\varkappa'}{\varkappa\tau}\mathfrak{b}(\sigma)\right),$$

where the constant angle α between (ρ) and (λ).

Now, we assume that the $(sqRDnrN)$-curve (ρ) is Salowski curve in $\mathbb{E}^3$. Therefore, the curvature $(\varkappa)$ is constant and the $(sqRDN)$-curve (λ) is $\lambda(\sigma) = -\frac{\cos\alpha}{\varkappa}\mathfrak{n}(\sigma) = constant$. Consequently, the curve (λ) is non-regular curve and the following corollary can be given.

Corollary 3. *Consider that the curve (λ) is $(sqRDN)$-curve of the $(sqRDnrN)$-curve of (ρ). Therefore, (ρ) is not a Salkowski curve.*

3.5 The $(sqRDR)$-curves

Definition 7. *Let (ρ) be spatial quaternionic curve given by spatial quaternionic frame $\{\mathfrak{t}, \mathfrak{n}, \mathfrak{b}\}$ and (λ) be $(sqRD)$-curve of (ρ). The curve (λ) is called spatial quaternionic rectifying direction rectifying curve (or $(sqRDR)$-curve) of (ρ), if the position vector of (λ) every time lying on the rectifying plane of its $(sqRDnrR)$-curve (ρ).*

From this definition, the parametric representation of (λ) is

$$\lambda(\sigma) = \Upsilon(\sigma)\mathfrak{t}(\sigma) + \Phi(\sigma)\mathfrak{b}(\sigma), \tag{22}$$

where $\Upsilon(\sigma)$, $\Phi(\sigma)$ are non-zero differentiable functions dependent on the parameterx σ. Differentiating (22) and considering (13) in obtained equality, it follows

$$\cos\alpha\mathfrak{t} + \sin\alpha\mathfrak{b} = \Upsilon'\mathfrak{t} + (\Upsilon\varkappa - \Phi\tau)\mathfrak{n} + \Phi'\mathfrak{b}. \tag{23}$$

From (23) it follows,

$$\begin{cases} \Upsilon' = \cos\alpha, \\ \Upsilon\varkappa - \Phi\tau = 0, \\ \Phi' = \sin\alpha. \end{cases} \tag{24}$$

From the solution of system (24) we have

$$\Upsilon(\sigma) = (\cos\alpha)\sigma + c_1, \quad \Phi(\sigma) = (\sin\alpha)\sigma + c_2, \tag{25}$$

where $c_1,\ c_2$ are integration constant. From (25) and second equation of system (24) it follows

$$\frac{\varkappa}{\tau} = \frac{(\sin\alpha)\sigma + c_2}{(\cos\alpha)\sigma + c_1}. \tag{26}$$

Then if $c_1,\ c_2 = 0$, from (26) we get that $\frac{\varkappa}{\tau}$ is constant, i.e., ρ is a general helix. But this is a contradiction according to Theorem (7). Then, in (25) it should pointed out that the integration constants $c_1,\ c_2$ cannot be zero at the same time.

Now, substituting (25) in (22) and considering Corollary (2), we get;

$$\lambda(\sigma) = (\sigma + \gamma)\bar{\mathfrak{t}}(\sigma) + \zeta\bar{\mathfrak{b}}(\sigma), \tag{27}$$

where $\gamma = c_1\cos\alpha + c_2\sin\alpha,\ \zeta = c_2\cos\alpha - c_1\sin\alpha$ are non-zero constants. Differentiating (27) gives that

$$\frac{\bar{\tau}}{\bar{\varkappa}} = \frac{\sigma + \gamma}{\zeta}. \tag{28}$$

From (27) and (28) we can write

$$\lambda(\sigma) = \zeta\left(\frac{\bar{\tau}}{\bar{\varkappa}}\bar{\mathfrak{t}} + \bar{\mathfrak{b}}\right)(\sigma) = \zeta\bar{\tilde{D}}(\sigma),$$

where $\bar{\tilde{D}}(\sigma) = \frac{\bar{\tau}}{\bar{\varkappa}}\bar{\mathfrak{t}} + \bar{\mathfrak{b}}$ is the modified Darboux vector of (λ).Then the following theorem can be written as;

Theorem 13. *Suppose that (λ) is the $(sqRD)$-curve of the $(sqRDnr)$-curve (ρ). If (λ) is a $(sqRDR)$-curve in $\mathbb{E}^3$, then one can write,*

i) (λ) is a spatial quaternionic rectifying curve in $\mathbb{E}^3$ whose curvatures satisfy $\frac{\bar{\tau}}{\bar{\varkappa}} = \frac{\sigma+\gamma}{\zeta}$ where $\gamma,\ \zeta$ are non-zero constants.

ii) The position vector of $(sqRDR)$-curve (λ) is linearly dependent with the modified Darboux vector $\bar{\tilde{D}}$ of (λ).

The Theorem (13) provide a way for the $(sqRDR)$-curve using the $(sqRDnrR)$-curve as follows;

Corollary 4. *Suppose that (λ) is the $(sqRDR)$-curve of the $(sqRDnrR)$-curve (ρ). Therefore, the position vector of (λ) is given by*

$$\lambda(\sigma) = [(\cos\alpha)\sigma + c_1]\,\mathfrak{t}(\sigma) + [(\sin\alpha)\sigma + c_2]\,\mathfrak{b}(\sigma) \tag{29}$$

where α is the constant angle between the curves and c_1, c_2 are integration constants, non-zero.

Example 3. *Let the $(sqRDnrR)$-curve (ρ) be Salkowski curve. Then the parametrization of (ρ) is obtained as follows,*

$$\rho(\sigma)=\frac{9}{\sqrt{82}}\left(\begin{array}{c}\frac{\sqrt{82}-82}{8(41+\sqrt{82})}\sin\left(\left(1+\frac{\sqrt{82}}{41}\right)\sigma\right)\\+\frac{\sqrt{82}+82}{8(\sqrt{82}-41)}\sin\left(\left(1-\frac{\sqrt{82}}{41}\right)\sigma\right)-\frac{1}{2}\sin\sigma,\\\frac{82-\sqrt{82}}{8(41+\sqrt{82})}\cos\left(\left(1+\frac{\sqrt{82}}{41}\right)\sigma\right)\\-\frac{\sqrt{82}+82}{8(\sqrt{82}-41)}\cos\left(\left(1-\frac{\sqrt{82}}{41}\right)\sigma\right)+\frac{1}{2}\cos\sigma,\\\frac{9}{4}\cos\left(\frac{\sqrt{82}}{41}\sigma\right)\end{array}\right).$$

Then by choosing $\alpha=\pi/6$, $c_1=\sqrt{3}/2$, $c_2=1/2$ in (29)*, a $(sqRDR)$-curve of (ρ) is obtained as follows (Fig. 4)*

$$\lambda(\sigma)=\frac{\sigma+1}{2}\left(\sqrt{3}\mathfrak{t}(\sigma)+\mathfrak{b}(\sigma)\right),$$

where

$$\mathfrak{t}(\sigma)=\left(\begin{array}{c}-\cos\sigma\cos\left(\frac{\sqrt{82}}{82}\sigma\right)-\frac{\sqrt{82}}{82}\sin\sigma\sin\left(\frac{\sqrt{82}}{82}\sigma\right),\\-\sin\sigma\cos\left(\frac{\sqrt{82}}{82}\sigma\right)+\frac{\sqrt{82}}{82}\cos\sigma\sin\left(\frac{\sqrt{82}}{82}\sigma\right),\\-\frac{9\sqrt{82}}{82}\sin\left(\frac{\sqrt{82}}{82}\sigma\right)\end{array}\right),$$

$$\mathfrak{b}(\sigma)=\left(\begin{array}{c}\frac{\sqrt{82}}{82}\cos\left(\frac{\sqrt{82}}{82}\sigma\right)\sin\sigma-\cos\sigma\sin\left(\frac{\sqrt{82}}{82}\sigma\right),\\-\frac{\sqrt{82}}{82}\cos\left(\frac{\sqrt{82}}{82}\sigma\right)\cos\sigma-\sin\sigma\sin\left(\frac{\sqrt{82}}{82}\sigma\right),\\\frac{9\sqrt{82}}{82}\cos\left(\frac{\sqrt{82}}{82}\sigma\right)\end{array}\right).$$

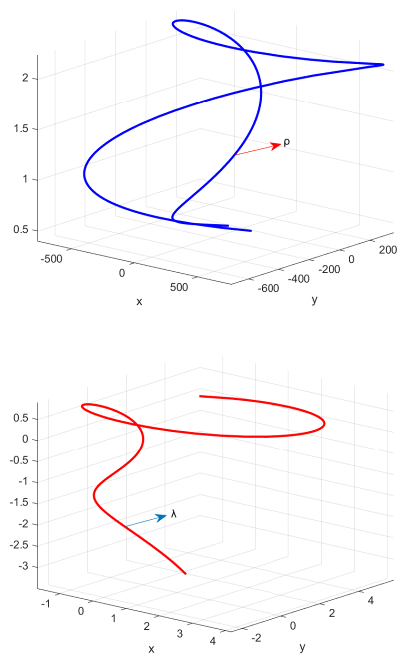

Figure 4: The $(sqRDnrR)$ Salkowski curve (ρ) (left) and the $(sqRDR)$-curve (λ) of (ρ) (right).

3.6 The (qRD)-curves

Now, we assume that a unit vector field χ of the quaternionic curve (ϕ) and a quaternionic integral curve (δ) of χ in Q. Then, we define the quaternionic rectifying direction curve ((qRD)-curve) and quaternionic rectifying-donor curve ($(qRDnr)$-curve). Consequently, we give definition and theorems about these curves.

Definition 8. *Assume that a vector field χ given as*

$$\chi(\varrho)=\psi_1(\varrho)\top(\varrho)+\psi_2(\varrho)\beta_1(\varrho)+\psi_3(\varrho)\beta_2(\varrho), \tag{30}$$

of the quaternionic curve $\phi:I\longrightarrow Q$ where ϱ is arbitrary parameter, $\psi_1(\varrho)\neq 0$, $\psi_2(\varrho)\neq 0$, $\psi_3(\varrho)\neq 0$, and $\chi'(\varrho)\notin Sp\{\top,\beta_1,\beta_2\}$. In that case, if the quaternionic integral curve of χ with arc-length parameter ϱ is (δ), then the curve (δ) is called (qRD)-curve of (ϕ) and the curve (ϕ) is called $(qRDnr)$-curve.

Theorem 14. *Assume that $\phi:I\longrightarrow Q$ is quaternionic curve and $\delta:J\longrightarrow Q$ is quaternionic integral curve of χ in equation* (30)*. In that case, (δ) is the (qRD)-curve of $(qRDnr)$-curve (ϕ), therefore the necessary and sufficient condition is that the equations*

$$\begin{cases}\psi_1(\varrho)=c=constant,\\\psi_2(\varrho)=\sin\left(\int(\tau-\mathcal{K})ds\right)\neq 0,\\\psi_3(\varrho)=\cos\left(\int(\tau-\mathcal{K})ds\right)\neq 0\end{cases}$$

are hold.

Proof. Assume that the quaternionic integral curve (δ) is (qRD)-curve of (ϕ). In that case, if we take derivative of the vector field $\chi(\varrho)$ in equation (30) with respect to ϱ, then we get

$$\begin{aligned}\chi'(\varrho)=&(\psi_1')\top+[\psi_1\mathcal{K}-\psi_2\varkappa]\eta+[\psi_2'-\psi_3(\tau-\mathcal{K})]\beta_1\\&+[\psi_3'+\psi_2(\tau-\mathcal{K})]\beta_2.\end{aligned}$$

We know that $\chi'(\varrho)\notin Sp\{\top,\beta_1,\beta_2\}$, and χ' is linear dependent with η, therefore we obtain

$$\begin{cases}\psi_1'=0,\\\psi_1\mathcal{K}-\psi_2\varkappa\neq 0,\\\psi_2'-\psi_3(\tau-\mathcal{K})=0,\\\psi_3'+\psi_2(\tau-\mathcal{K})=0.\end{cases}$$

and consequently, we get

$$\psi_1(\varrho)=c=constant,\quad \psi_2(\varrho)=\sin\left(\int(\tau-\mathcal{K})ds\right)\neq 0$$

$$\psi_3(\varrho) = \cos\left(\int(\tau - \mathcal{K})ds\right) \neq 0.$$

□

Theorem 15. *Assume that (δ) is the (qRD)-curve of the $(qRDnr)$-curve (ϕ) and the quaternionic curvatures of these curves (ϕ, δ) are $\{\mathcal{K}, \varkappa, (\tau - \mathcal{K})\}$ and $\{\bar{\mathcal{K}}, \bar{\varkappa}, (\bar{\tau} - \bar{\mathcal{K}})\}$, respectively. Therefore, the relationships between the quaternionic curvatures are given*

$$\bar{\mathcal{K}} = \sqrt{\varkappa^2 + \mathcal{K}^2},$$

$$\bar{\varkappa} = \frac{\sqrt{\varkappa^2(\tau-\mathcal{K})^2(\varkappa^2+\mathcal{K}^2) + (\varkappa'\mathcal{K} - \varkappa\mathcal{K}')^2}}{\varkappa^2+\mathcal{K}^2},$$

$$\bar{\tau} - \bar{\mathcal{K}} = \frac{\sqrt{\varkappa^2+\mathcal{K}^2}}{\varkappa^2(\tau-\mathcal{K})^2(\varkappa^2+\mathcal{K}^2) + (\varkappa'\mathcal{K} - \varkappa\mathcal{K}')^2} \cdot \begin{pmatrix} (\tau-\mathcal{K}).\left[2\varkappa'(\mathcal{K}'\varkappa - \mathcal{K}\varkappa')\right. \\ +\varkappa\mathcal{K}(\varkappa'' - \varkappa(\tau-\mathcal{K})^2) \\ -\mathcal{K}''\varkappa^2 + \\ \left.(\varkappa^2\mathcal{K}' - 2\varkappa\varkappa'\mathcal{K}')(\tau-\mathcal{K})'\right] \end{pmatrix}.$$

Proof. Let the quaternionic apparatus of the (qRD)-curve and $(qRDnr)$-curve be $\{\bar{\top}, \bar{\eta}, \bar{\beta}_1, \bar{\beta}_2, \bar{\mathcal{K}}, \bar{\varkappa}, (\bar{\tau} - \bar{\mathcal{K}})\}$ and $\{\top, \eta, \beta_1, \beta_2, \mathcal{K}, \varkappa, (\tau - \mathcal{K})\}$, respectively. In that case, we can write

$$\delta' = \bar{\top} = \eta \tag{31}$$

and with the aid of derivative with respect to (ρ), we have

$$\bar{\top}' = \eta' \Rightarrow \bar{\mathcal{K}}\bar{\eta} = -\mathcal{K}\top + \varkappa\beta_1.$$

Therefore, we obtain the principal quaternionic curvature of (δ) as

$$\bar{\mathcal{K}} = \sqrt{\varkappa^2 + \mathcal{K}^2}.$$

On the other hand, if we use the equation $\bar{\varkappa} = \frac{\|\top\times\eta\times\gamma'''\|}{\|\gamma''\|}$ and we obtain

$$\gamma''' = -\varkappa'\top + (-\mathcal{K}^2 - \varkappa^2)\eta + \varkappa'\beta_1 + \varkappa(\tau-\mathcal{K}))\beta_2$$

then we get the torsion of quaternionic (RD)-curve (δ) as

$$\bar{\varkappa} = \frac{\sqrt{\varkappa^2(\tau-\mathcal{K})^2(\varkappa^2+\mathcal{K}^2) + (\varkappa'\mathcal{K} - \varkappa\mathcal{K}')^2}}{\varkappa^2+\mathcal{K}^2}$$

Consequently, if we make necessary adjustments in $\bar{\tau} - \bar{\mathcal{K}} = \frac{\mathfrak{h}(\gamma^4, \beta_2)}{\|\top\times\eta\times\gamma'''\|}$, then we find the bi-torsion of (δ) as

$$\bar{\tau} - \bar{\mathcal{K}} = \frac{\sqrt{\varkappa^2+\mathcal{K}^2}}{\varkappa^2(\tau-\mathcal{K})^2(\varkappa^2+\mathcal{K}^2) + (\varkappa'\mathcal{K} - \varkappa\mathcal{K}')^2} \cdot \begin{pmatrix} (\tau-\mathcal{K}).\left[2\varkappa'(\mathcal{K}'\varkappa - \mathcal{K}\varkappa')\right. \\ +\varkappa\mathcal{K}(\varkappa'' - \varkappa(\tau-\mathcal{K})^2) \\ -\mathcal{K}''\varkappa^2 + \\ \left.(\varkappa^2\mathcal{K}' - 2\varkappa\varkappa'\mathcal{K}')(\tau-\mathcal{K})'\right] \end{pmatrix}.$$

□

4 Conclusion

In many fields of study such as mathematics, physics, robotics and digital technology, using quaternions (and therefore non-commutative structure) derived from complex numbers instead of complex numbers creates a new field of study, and therefore quaternions provide great convenience in this field of study. In 1843, Hamilton discovered quaternions while working on generalizing complex numbers to three dimensions. While Hamilton hoped that quaternions would be characterized by three numbers when he generalized the complex numbers, he found that four numbers were necessary. Hamilton was always interested in the geometric interpretation of quaternions and explored the role of quaternions in explaining three-dimensional rotations with pure quaternions. The main use of quaternions after Hamilton was expressed is the notation of quaternions instead of physical theories in Cartesian coordinates. The best-known example of this is Maxwell's Treatise on Electricity and Magnetism. In these years, there was talk of the value of using non-coordinate methods of modern differential geometry instead of using the old tensor methods in space-time physics. In addition, quaternions were also very popular in many other fields of physics. First of all, also Maxwell used quaternions in the results of the equations, which will be named after him. Electromagnetism has been studied many times by many scientists with dual, complex or real quaternions. In addition, the representation of electromagnetism in the matter environment was reconstructed using quaternion algebra. Derivatives of quaternions from high-dimensional algebras are used in order to gain a new perspective on a system involving the gravitational force. In addition, these algebras were used to unify both gravity and electromagnetism. Quaternionic structures are discussed in the fields of physics, acoustics, plasma and so on. The success of quaternions was not only limited to mathematics and physics, but also used as a very successful method in robotic studies. In this study, we have defined the new type associated curves of a space curve, which is parameterized by arc-length in $\mathbb{E}^3$. We have named these associated curves as $(sqRD)$-curve and $(sqRDnr)$-curve. We have examined relationships between some specific curves like helix, slant helix, Salkowski, anti-Salkowski curve, spatial quaternionic rectifying curves and these new type curves. We have showed that although $(sqRD)$-curve of spatial quaternionic curve are not general helix, they are slant helix. Furthermore, spatial quaternionic rectifying-direction curve of spatial quaternionic curve is an $(sqODR)$-curve of the same curve. In that case, the curve is an anti-Salkowski curve. The similar results have been obtained for $(sqNDR)$-curve and $(sqRDR)$-curve of the curve. These results are quite remarkable. Therefore, we think that this study will create a bridge in the fields of physics and geometry with the help of quaternions. In addition, we have continued theoretical and practical research.

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