On Coincidence and Common Fixed Points for Compatible Hybrid Mappings in Partial Metric Spaces Related to Partial Order

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Abstract: - In this paper, we prove the existence of coincidence points for hybrid mapping pairs and their common fixed points by utilizing the compatibility properties. Our new results extend the theorems on coincidence points and common fixed points of compatible hybrid mapping pairs in partial metric spaces under partial ordering. Additionally, we establish new theorems on coincidence and common fixed points in partially ordered sets. Some illustrative examples are also provided.

Key-Words: - Compatible mapping, hybrid mapping pairs, coincidence point, common fixed point, partial ordering, p-Pompeiu-Hausdorff metric spaces.

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1 Introduction

Mathematicians are still developing the discussion of fixed points. One direction of development often done is in the metric space used. The results of the development of fixed points in various metric spaces that have recently been carried out include [1], [2], [3]. Some extensions of fixed points to the coincidence case and common fixed points are used to analyze the stability of iterative processes, demonstrate the existence of solutions to equations, and other related purposes. Common fixed points topic stems from a conjecture made by [4] in 1954: for any $\vartheta\in[0,1]$ if the single-valued mapping $\hbar$ and l satisfy $\hbar\,\mathrm{l}(\vartheta)=\mathrm{l}\hbar\,(\vartheta)$ then the common fixed point of $\hbar$ and l in $[0,1]$ can be shown. Several researchers have addressed this conjecture, such as those in [5], [6], [7]. In 1967, [8] extended the discussion to the metric space (ω,d); he proved a common fixed point theorem for commute mappings in (ω,d). Since then, research on common fixed points has begun to develop rapidly.

Several researchers involved set-valued, single-valued, and even their compositions in its development. In 1977, [9] proposed the concept of commutativity for the composition of set-valued and single-valued mappings. Furthermore, they also showed common fixed points in topological vector spaces. In 1988, [10] proposed the term weakly commuting mappings and their common fixed point. In 1989, [11] proposed the compatibility concept for combining set-valued and single-valued mappings (hybrid mappings) and showed the common fixed point. They followed the results of [12] and [13]. 2017, [14] extended the results of [11] on partial metric spaces. In 1995, [15] extended the result of compatible set-valued and single-valued mapping to the t-weak-compatible mapping. In 2023, [16] extended the result of [15] to partial metric spaces.

On the other side, [17] pioneered extending the fixed point concept to partially ordered sets. In partially ordered sets, the exploration of fixed points expands, as seen from the results in references [18], [19], [20], [21], [22], [23], [24]. Recently, the common fixed points study of hybrid mapping has been extended to partially ordered sets. Furthermore, [23] proved the common fixed points theorem for single-valued mappings on partially ordered partial metric spaces. Also, [25] extended the results of [11] to partially ordered sets. Motivated by this work, we will establish the theorem on coincidence and common fixed points for compatible hybrid mappings on partially ordered sets. In this regard, our results expand the findings of [14] and [16] in partially ordered sets.

2 Preliminaries

Let ω be nonempty sets. Suppose (ω, d) is a metric space and $CB(\omega)$ represents the class of all $\Sigma \subseteq \omega$ and $\Sigma \neq \emptyset$ that are closed and bounded. Define

$$H(\Sigma, \Omega) = max\{d(\Sigma, \Omega), d(\Omega, \Sigma)\}, \quad (1)$$

for all $\Sigma, \Omega \in CB(\omega)$, where $d(\Sigma, \Omega) = sup\{d(\vartheta, \Omega): \vartheta \in \Sigma\}$, and $d(\vartheta, \Omega) = inf\{d(\vartheta, \varsigma): \varsigma \in \Omega\}$. Mapping H is a metric on $CB(\omega)$, which we call the Pompeiu-Hausdorff metric on $CB(\omega)$ [26]. The pairs $(CB(\omega), H)$ are called by Pompeiu-Hausdorff metric space. Additionally, the completeness of $(CB(\omega), H)$ depends on the completeness (ω, d) [12], [27], [28].

In 1992, [29] and [30] proposed partial metric as a broader version of the standard metric, wherein an object's distance from itself need not always be zero. Let $\omega \neq \emptyset$, a partial metric on ω is $p: \omega \times \omega \to [0, \infty)$ a mapping which satisfies the following conditions:

1. $p(\vartheta, \varsigma) = p(\varsigma, \vartheta)$,
2. If $p(\vartheta, \vartheta) = p(\vartheta, \varsigma) = p(\varsigma, \varsigma)$ then $\vartheta = \varsigma$,
3. $p(\vartheta, \vartheta) \leq p(\vartheta, \varsigma)$,
4. $p(\vartheta, \eta) + p(\varsigma, \varsigma) \leq p(\vartheta, \varsigma) + p(\varsigma, \eta)$,

for all $\vartheta, \varsigma, \eta \in \omega$. Furthermore, a pair (ω, p) is partial metric space. Several properties of sequence in this space we refer to [20], [21], [22], [24], [29], [30], [31] and reference therein.

Furthermore, [31] extend the Pompeiu-Hausdorff metric to a more general (partial) metric. They presented partial Pompeiu-Hausdorff metric space induced by partial metric p. Consider (ω, p) as a partial metric space, where $CB^p(\omega)$ represents the class of all subsets $\Sigma \subseteq \omega$ and $\Sigma \neq \emptyset$ that closed and bounded.

The mapping $H^p: CB^p(\omega) \times CB^p(\omega) \to [0, \infty)$ define

$$H^p(\Sigma, \Omega) = max\{p(\Sigma, \Omega), p(\Omega, \Sigma)\}, \quad (2)$$

for $\Sigma, \Omega \in CB^p(\omega)$, where $p(\Sigma, \Omega) = sup\,\{p(\vartheta, \varsigma): \vartheta \in \Sigma\}$, and also we have $p(\vartheta, \Omega) = inf\,\{p(\vartheta, \varsigma): \varsigma \in \Omega\}$. The mapping H^p is a partial Pompeiu-Hausdorff metric induced by p [19], [31], [32]. The pairs $(CB^p(\omega), H^p)$ are called partial Pompeiu-Hausdorff metric spaces or p-Pompeiu-Hausdorff metric spaces. Every Pompeiu-Hausdorff metric is a p-Pompeiu-Hausdorff metric, but the reverse is not necessarily true (see Example 2.6 and Remark 2.7 in [31]). Furthermore, $(CB^p(\omega), H^p)$ is complete whenever (ω, p) is complete. Furthermore, some properties regarding the p-Pompeiu-Hausdorff metric space are stated in Theorem 2.1, 2.2, 2.3, and Lemma 2.4.

Theorem 2.1. [31], [32] *Suppose (ω, p) is a partial metric space. For each $\Sigma, \Omega, \Upsilon \in CB^p(\omega)$, satisfy*

1. $p(\Sigma, \Sigma) = sup\{p(\vartheta, \vartheta): \vartheta \in \Sigma\}$,
2. $p(\Sigma, \Sigma) \leq p(\Sigma, \Omega)$,
3. $p(\Sigma, \Omega) = 0$ *implies that* $\Sigma = \Omega$,
4. $p(\Sigma, \Omega) \leq p(\Sigma, \Upsilon) + p(\Upsilon, \Omega) - inf\{p(\eta, \eta): \eta \in \Upsilon\}$.

Theorem 2.2. [31], [32], [33] *Suppose (ω, p) is a partial metric space and $\Sigma \subseteq \omega$ and $\Sigma \neq \emptyset$, then*

$$\vartheta \in \bar{\Sigma} \text{ iff } p(\vartheta, \Sigma) = p(\vartheta, \vartheta).$$

Theorem 2.3. [31], [32] *Suppose (ω, p) is a partial metric space. Every $\Sigma, \Omega, \Upsilon \in CB^p(\omega)$, satisfy*

1. $H^p(\Sigma, \Omega) \leq H^p(\Sigma, \Omega)$,
2. $H^p(\Sigma, \Omega) = H^p(\Omega, \Sigma)$,
3. $H^p(\Sigma, \Omega) \leq H^p(\Sigma, \Upsilon) + H^p(\Upsilon, \Omega) - inf\{p(\eta, \eta): \eta \in \Upsilon\}$,
4. *If* $H^p(\Sigma, \Omega) = 0$ *then* $\Sigma = \Omega$.

Lemma 2.4. *Suppose (ω, p) is a partial metric space. If $\Sigma, \Omega \in CB^p(\omega)$, $\vartheta \in \Sigma$, then there is $\varsigma \in \Omega$ such that*

$$p(\vartheta, \varsigma) \leq H^p(\Sigma, \Omega).$$

Furthermore, we give some definitions of partial ordering.

Definition 2.5. [34] *A binary relation $\preccurlyeq$ that is defined on ω is said to be a partial order relation when it satisfies:*

1. *$\vartheta \preccurlyeq \vartheta$ (reflexivity),*
2. *If $\vartheta \preccurlyeq \varsigma$ and $\varsigma \preccurlyeq \vartheta$ then $\vartheta = \varsigma$ (antisymmetry),*
3. *If $\vartheta \preccurlyeq \varsigma$ and $\varsigma \preccurlyeq \eta$ then $\vartheta \preccurlyeq \eta$ (transitivity),*

for every ϑ, ς and $\eta \in \omega$. A pair $(\omega, \preccurlyeq)$ is called a partially ordered set.

Definition 2.6. [34] *Suppose that $(\omega, \preccurlyeq)$ be a partially ordered set, $\vartheta, \varsigma \in \omega$. Elements ϑ and ς are called comparable elements of ω if either $\vartheta \preccurlyeq \varsigma$ or $\varsigma \preccurlyeq \vartheta$.*

Definition 2.7. [18] *Suppose that $(\omega, \preccurlyeq)$ is a partially ordered set. The mapping t is called by dominating mapping if $\vartheta \preccurlyeq t(\vartheta)$ for each $\vartheta \in \omega$.*

For example, $\omega = [0,1]$ with usual ordering $\leq$ and mapping $t: \omega \to \omega$ where $t(\vartheta) := \vartheta^{1/n}$ for any n positive integers. We can show that $\vartheta \leq t(\vartheta)$ for $\vartheta \in \omega$ or t is dominating maps.

Definition 2.8. *If pair $(\omega, \preccurlyeq)$ is a partially ordered set and pair (ω, p) is a partial metric space, then we call the triple $(\omega, p, \preccurlyeq)$ a partially ordered partial metric space.*

For example, suppose we have
$\omega = \{(0,0), (0, -1/4), (1/4, 0), (-1/4, 1/4)\} \subseteq \mathbb{R}^2$
with the order $\preccurlyeq$ defined as $(\vartheta_1, \vartheta_2)$ and $(\varsigma_1, \varsigma_2) \in \omega$ with $(\vartheta_1, \vartheta_2) \preccurlyeq (\varsigma_1, \varsigma_2)$ if and only if $\vartheta_1 \leq \varsigma_1$ and $\vartheta_2 \leq \varsigma_2$. We know that $(\omega, \preccurlyeq)$ is a partially ordered set. Let $p: \omega \times \omega \to \mathbb{R}^2$ is defined as

$$p(\vartheta, \varsigma) = p\big((\vartheta_1, \vartheta_2).(\varsigma_1, \varsigma_2)\big) := max\{|\vartheta_1 - \varsigma_1|, |\vartheta_2 - \varsigma_2|\},$$

for all $\vartheta, \varsigma \in \omega$ so that the partial metric spaces (ω, p). Thus, $(\omega, p, \preccurlyeq)$ is partially ordered partial metric spaces.

Definition 2.9. *If partial metric spaces (ω, p) in Definition 2.8 is complete, then the triple $(\omega, p, \preccurlyeq)$ is also complete.*

The partial metric space (ω, p) above is complete partial metric space since every Cauchy sequence is convergent in ω. Therefore, the partially ordered partial metric spaces $(\omega, p, \preccurlyeq)$ above are complete.

Next, definitions of common fixed points and coincidence points of hybrid mapping pairs are given in the following definitions.

Definition 2.10. [14], [16] *Let given $T: \omega \to CB(\omega)$ and $t: \omega \to \omega$. A point $\eta \in \omega$ is called a common fixed point of mapping T and t if $\eta = t(\eta) \in T(\eta)$. A point $\vartheta \in \omega$ is said to be a coincidence point of hybrid mapping pairs T and t when $t(\vartheta) \in T(\vartheta)$.*

Furthermore, [11] introduced the new concept, the compatibility term for hybrid mapping pairs T and t (see Definition 2 in [11]) generalizes the compatibility concept for single-valued mappings t and s (see Definition 1 in [11]). Now, we propose the extension of the notion compatible mapping of hybrid mapping pairs in partial metric spaces.

Definition 2.11. *Suppose that we defined the mappings $T: \omega \to CB^p(\omega)$ and $t: \omega \to \omega$ in partial metric spaces (ω, p).*

1. *The mappings T and t are compatible iff $tT(\vartheta) \in CB^p(\omega)$ and*
$$H^p\big(Tt(\vartheta_n), tT(\vartheta_n)\big) \to 0,$$
with (ϑ_n) is sequence in ω where $t(\vartheta_n) \to \eta \in \Upsilon$ and $T(\vartheta_n) \to \Upsilon$ where $\Upsilon \in CB^p(\omega)$.
2. *The mappings T and t are t-weak compatible if $tT(\vartheta) \in CB^p(\omega)$ and satisfy*
$$lim_{n\to\infty} H^p\big(tT(\vartheta_n), Tt(\vartheta_n)\big) \leq lim_{n\to\infty} H^p\big(Tt(\vartheta_n), T(\vartheta_n)\big),$$
and
$$lim_{n\to\infty} p\big(tT(\vartheta_n), Tt(\vartheta_n)\big) \leq lim_{n\to\infty} H^p\big(Tt(\vartheta_n), T(\vartheta_n)\big),$$
whenever $(\vartheta_n) \subseteq \omega$ where $t(\vartheta_n) \to \eta \in \Upsilon$ and $T(\vartheta_n) \to \Upsilon$ where set $\Upsilon \in CB^p(\omega)$.

From Definition 2.11, the hybrid mapping pairs T and t are compatible iff $tT(\vartheta) \in CB^p(\omega)$ for all $\vartheta \in \omega$ and

$$H^p(Tt(\vartheta_n), tT(\vartheta_n)) = H^p(tT(\vartheta_n), tT(\vartheta_n)),$$

where (ϑ_n) is sequence in ω such that sequences $t(\vartheta_n) \to \eta \in \Upsilon$ and $T(\vartheta_n) \to \Upsilon$ where $\Upsilon \in CB^p(\omega)$ [14]. Moreover, the set of weakly compatible mappings on partial metric spaces contains the set of compatible mappings [14], [16].

The following lemma gives some compatibility properties for set-valued mappings T and single-valued mappings t.

Lemma 2.12. *Given partial metric spaces (ω, p) and $T: \omega \to CB^p(\omega)$ be set-valued mappings that are continuous on ω. If there exist single-valued mappings $t: \omega \to \omega$ is continuous on ω where $t(\eta) \in T(\eta)$ for some $\eta \in \omega$, then the mapping T and t are compatible.*

Proof. By using the continuity of t on ω, $T(\vartheta) \in CB^p(\omega)$ for all $\vartheta \in \omega$, then $tT(\vartheta) \in CB^p(\omega)$ for all $\vartheta \in \omega$. Let (ϑ_n) sequence on ω where $T(\vartheta_n) \to \Upsilon$, $t(\vartheta_n) \to \eta$, whenever $\Upsilon \in CB^p(\omega)$ and $\eta \in \Upsilon$. We can take $\eta \in \Upsilon$ where $t(\eta) \in T(\eta)$. By the continuity of mappings T and t implies

$$\begin{aligned} &lim_{n\to\infty} H^p\big(Tt(\vartheta_n), tT(\vartheta_n)\big) \\ &\leq lim_{n\to\infty} (H^p(Tt(\vartheta_n), t(\eta)) + H^p(t(\eta), t(\eta)) \\ &\quad + H^p(t(\eta), tT(\vartheta_n))) \\ &\leq lim_{n\to\infty} H^p\big(Tt(\vartheta_n), t(\eta)\big) \\ &\quad + lim_{n\to\infty} H^p\big(t(\eta), t(\eta)\big) \\ &\quad + lim_{n\to\infty} H^p(t(\eta), tT(\vartheta_n)) \\ &= 0 \end{aligned}$$

Therefore, by Definition 2.11 part (1), we obtain T and t are compatible.

Lemma 2.12 above extends Lemma 4 in [35] to partial metric spaces. Furthermore, Lemma 2.13 below is an extension of Lemma 1 (in [35]) or LEMMA (in [11]) in partial metric spaces.

Lemma 2.13. *Let $T: \omega \to CB^p(\omega)$ and $t: \omega \to \omega$ be compatible mappings. The condition $t(\eta) \in T(\eta)$ for some $\eta \in \omega$ implies*

$$tT(\eta) = Tt(\eta).$$

Proof. Suppose that $\vartheta_n = \eta$ for each n, then we obtain $t(\vartheta_n) = t(\eta) \to t(\eta)$ and $T(\vartheta_n) \to \Upsilon$, where $\Upsilon = t(\eta)$. Therefore, by $t(\eta) \in T(\eta)$ and using compatibility of T and t then,

$$H^p(tT(\eta), Tt(\eta)) = H^p(Tt(\vartheta_n), tT(\vartheta_n)) \to 0,$$

as $n \to \infty$. Hence, we obtain

$$H^p(tT(\eta), Tt(\eta)) = 0.$$

It implies $tT(\eta) = Tt(\eta)$.

In 2017, [14] proved the common fixed point theorem for hybrid mapping pairs that are compatible in partial metric spaces.

Theorem 2.14. *Suppose (ω, p) is a complete partial metric space. If the mappings $t: \omega \to \omega$ and $T: \omega \to CB^p(\omega)$ are compatible and continuous mappings that satisfy $T(\omega) \subseteq t(\omega)$, and*

$$H^p\big(T(\vartheta), T(\varsigma)\big) \le c \max\{ p\big(t(\vartheta), t(\varsigma)\big), p\big(t(\vartheta), T(\vartheta)\big), p\big(t(\varsigma), T(\varsigma)\big), 1/2\, (p(t(\vartheta), T(\varsigma)) + p(t(\varsigma), T(\vartheta)))\}$$

for some c with $0 \le c < 1$, it implies there exist $\eta \in \omega$ where $t(\eta) \in T(\eta)$.

In 2023, [16] recently established the following common fixed point theorem for t-weak compatible hybrid pair mappings in partial metric space.

Theorem 2.15. *Suppose (ω, p) is a complete partial metric space. If the mappings $t: \omega \to \omega$ and $T: \omega \to CB^p(\omega)$ be a t-weak compatible and continuous mappings that satisfy $T(\omega) \subseteq t(\omega)$, and*

$$H^p\big(T(\vartheta), T(\varsigma)\big) \le c \max\{ p\big(t(\vartheta), t(\varsigma)\big), p\big(t(\vartheta), T(\vartheta)\big), p\big(t(\varsigma), T(\varsigma)\big), 1/2\, (p(t(\vartheta), T(\varsigma)) + p(t(\varsigma), T(\vartheta)))\}$$

for some c with $0 \le c < 1$, it implies there exist $\eta \in \omega$ where $t(\eta) \in T(\eta)$.

In this paper, Theorem 2.14 and 2.15 are extended to partially ordered sets to get the coincidence and the common fixed point for compatible hybrid mapping pairs and t-weak compatible hybrid mapping pairs in partially ordered partial metric spaces.

3 Main Results

3.1 Compatible Hybrid Mapping Pairs

We extend Theorem 2.14 in the setting of partial order sets as in Theorem 3.1.

Theorem 3.1. *Suppose $(\omega, p, \preccurlyeq)$ is complete partially ordered partial metric space. Assume that the mapping $t: \omega \to \omega$ and $T: \omega \to CB^p(\omega)$ be a continuous mapping and satisfy*

$$H^p\big(T(\vartheta), T(\varsigma)\big) \le c \max\{ p\big(t(\vartheta), t(\varsigma)\big), p\big(t(\vartheta), T(\vartheta)\big), p\big(t(\varsigma), T(\varsigma)\big), 1/2\, (p(t(\vartheta), T(\varsigma)) + p(t(\varsigma), T(\vartheta)))$$

for some c with $0 \le c < 1$ and for all comparable element $\vartheta, \varsigma \in \omega$. If the specified conditions are satisfied

1. *$T(\omega) \subseteq t(\omega)$,*
2. *If $t(\varsigma) \in T(\vartheta)$ then $\vartheta \preccurlyeq \varsigma$,*
3. *T and t are compatible.*

then $t(\eta) \in T(\eta)$ for $\eta \in \omega$.

Proof. Take ϑ_0 be an arbitrary element of ω. We can choose $\vartheta_1 \in \omega$ such that $t(\vartheta_1) \in T(\vartheta_0)$, and this is possible since $T(\vartheta) \subseteq t(\omega)$. Therefore, by assumption 2, $\vartheta_0 \preccurlyeq \vartheta_1$. If $c = 0$, then

$$p\big(t(\vartheta_1), T(\vartheta_1)\big) \le H^p\big(T(\vartheta_0), T(\vartheta_1)\big) = 0.$$

Since $T(\vartheta_1) \in CB^p(\omega)$ then $t(\vartheta_1) \in T(\vartheta_1)$. It means the proof is done.

In the other case, assuming that $c \neq 0$, we have $\kappa := 1/\sqrt{c} > 1$. Since $t(\vartheta_1) \in T(\vartheta_0)$ then by Lemma 2.4, there exist a point $\varsigma_1 \in T(\vartheta_1)$ such that

$$p\big(\varsigma_1, t(\vartheta_1)\big) \le H^p\big(T(\vartheta_1), T(\vartheta_0)\big) \le \kappa\, H^p\big(T(\vartheta_1), T(\vartheta_0)\big).$$

So, for comparable element $\vartheta_0 \preccurlyeq \vartheta_1$ we have

$$p\big(\varsigma_1, t(\vartheta_1)\big) < \kappa\, H^p\big(T(\vartheta_1), T(\vartheta_0)\big).$$

Let consider that $T(\vartheta_1) \subseteq t(\omega)$, therefore we can choose $\vartheta_2 \in \omega$ with $\varsigma_1 = t(\vartheta_2) \in T(\vartheta_1)$. By assumption 2 we have $\vartheta_1 \preccurlyeq \vartheta_2$.

Futhermore, since $t(\vartheta_2) \in T(\vartheta_1)$ then by Lemma 2.4, we can find $\varsigma_2 \in T(\vartheta_2)$ such that

$$p\big(\varsigma_2, t(\vartheta_2)\big) \le H^p\big(T(\vartheta_2), T(\vartheta_1)\big).$$

Since $\kappa > 1$ then for comparable element $\vartheta_1 \preccurlyeq \vartheta_2$ we get

$$p\big(\varsigma_2, t(\vartheta_2)\big) \le \kappa H^p\big(T(\vartheta_2), T(\vartheta_1)\big).$$

By assumtion (1), there exist $\vartheta_3 \in \omega$ with $\varsigma_2 = t(\vartheta_3) \in T(\vartheta_2)$.

In general, for $\vartheta_{n-1} \in \omega$ we can find ϑ_n where $\varsigma \in t(\vartheta_{n+1}) \in T(\vartheta_n)$, and also

$$p\big(\varsigma, t(\vartheta_n)\big) \le \kappa\, H^p\big(T(\vartheta_n), T(\vartheta_{n-1})\big),$$

for all $n \ge 1$ and comparable element $\vartheta_{n-1} \preccurlyeq \vartheta_n$ for every n. Since ϑ_n comparable for all n, then we have

$$p\big(t(\vartheta_{n+1}), t(\vartheta_n)\big) < \kappa\, H^p\big(T(\vartheta_n), T(\vartheta_{n-1})\big)$$
$$\le \frac{1}{\sqrt{c}} . c \max\{p\big(t(\vartheta_n), t(\vartheta_{n-1})\big), p\big(t(\vartheta_n), T(\vartheta_n)\big), p\big(t(\vartheta_{n-1}), T(\vartheta_{n-1})\big), 1/2\, \big(p\big(t(\vartheta_n), T(\vartheta_{n-1})\big) + p\big(t(\vartheta_{n-1}), T(\vartheta_n)\big)\big)\}$$
$$\le \sqrt{c} \max\{p\big(t(\vartheta_n), t(\vartheta_{n-1})\big), p\big(t(\vartheta_n), T(\vartheta_n)\big),$$

$p(t(\vartheta_{n-1}), T(\vartheta_{n-1})), 1/2\ (p(t(\vartheta_n), T(\vartheta_{n-1}))$
$+p(t(\vartheta_{n-1}), T(\vartheta_n)))\}$
$\leq \sqrt{c}\max\{p(t(\vartheta_n), t(\vartheta_{n-1})), p(t(\vartheta_n), t(\vartheta_{n+1})),$
$p(t(\vartheta_{n-1}), t(\vartheta_n)), 1/2\ (p(t(\vartheta_n), t(\vartheta_n))$
$+p(t(\vartheta_{n-1}), t(\vartheta_{n+1})))\}$
$\leq \sqrt{c}\max\{p(t(\vartheta_n), t(\vartheta_{n-1})), p(t(\vartheta_n), t(\vartheta_{n+1})),$
$1/2\ (p(t(\vartheta_n), t(\vartheta_n)) + p(t(\vartheta_n), t(\vartheta_{n+1})))\}$
$\leq \sqrt{c}\max\{p(t(\vartheta_n), t(\vartheta_{n-1})), p(t(\vartheta_n), t(\vartheta_{n+1}))\}$
i.e., $p(t(\vartheta_{n+1}), t(\vartheta_n)) \leq \sqrt{c}\, p(t(\vartheta_n), t(\vartheta_{n-1}))$ for all $n \geq 2$. Therefore, by continuing this process

$$p(t(\vartheta_{n+1}), t(\vartheta_n)) \leq \left(\sqrt{c}\right)^{n-1} p(t(\vartheta_2), t(\vartheta_1)), \quad (3)$$

for every $n \in \mathbb{N}$. Since $\sqrt{c} < 1$, then $p(t(\vartheta_{n+1}), t(\vartheta_n)) \to 0$ as $n \to \infty$. It means $(t(\vartheta_n))$ is Cauchy sequences in ω. Since partial metric spaces ω is complete, it implies $t(\vartheta_n) \to \eta$, where $\eta \in \omega$. Furthermore, from inequalities (3) we get

$$H^p(T(\vartheta_{n-1}), T(\vartheta_n)) \leq c\, p(t(\vartheta_n), t(\vartheta_{n-1})).$$

Since $(t(\vartheta_n))$ is the Cauchy sequence, this must imply $(T(\vartheta_n))$ is also a Cauchy sequence in $(CB^p(\omega), H^p)$. Since $(CB^p(\omega), H^p)$ is complete, thus we have set $\Upsilon \in CB^p(\omega)$ such that $T(\vartheta_n) \to \Upsilon$ as $n \to \infty$. Furthermore,

$$\begin{aligned}(\eta, \Upsilon) &\leq p(\eta, t(\vartheta_n)) + p(t(\vartheta_n), \Upsilon) \\ &\quad - p(t(\vartheta_n), t(\vartheta_n)) \\ &\leq p(\eta, t(\vartheta_n)) + p(t(\vartheta_n), \Upsilon) \\ &\leq p(\eta, t(\vartheta_n)) + H^p(T(\vartheta_n), \Upsilon)\end{aligned}$$

Since $p(\eta, t(\vartheta_n)) \to 0$ and $H^p(T(\vartheta_n), \Upsilon) \to 0$ as $n \to \infty$, then $p(\eta, \Upsilon) = 0$. Since $\Upsilon \in CB^p(\omega)$ then $\eta \in \Upsilon$.

Since T and t are compatible, then

$$H^p(Tt(\vartheta_n), tT(\vartheta_n)) \to 0$$

as $n \to \infty$, then

$$\begin{aligned}&p(t(\eta), T(\eta)) \\ &\leq p(t(\eta), tt(\vartheta_n)) + p(tt(\vartheta_n), T(\eta)) \\ &\quad - p(tt(\vartheta_n), tt(\vartheta_n)) \\ &\leq p(t(\eta), tt(\vartheta_n)) + p(tt(\vartheta_n), T(\eta)) \\ &\leq p(t(\eta), tt(\vartheta_n)) + H^p(tT(\vartheta_n), T(\eta)) \\ &\leq p(t(\eta), tt(\vartheta_n)) + H^p(tT(\vartheta_n), Tt(\vartheta_n)) \\ &\quad + H^p(Tt(\vartheta_n), T(\eta))\end{aligned}$$

By compatibility of mappings T and t, and also $t(\vartheta_n) \to \eta$ as $n \to \infty$ therefore

$$p(t(\eta), T(\eta)) = 0.$$

Since $T(\eta) \in CB^p(\omega)$, it implies $t(\eta) \in T(\eta)$. This result completes the proof.

Example 3.2. Let we consider this conditions:
$\omega = \{(0,0), (0, -1/4), (1/4, 0), (-/4, 1/4)\} \subseteq \mathbb{R}^2$ with the order $\preccurlyeq$ defined as $(\vartheta_1, \vartheta_2)$ and $(\varsigma_1, \varsigma_2) \in \omega$ with $(\vartheta_1, \vartheta_2) \preccurlyeq (\varsigma_1, \varsigma_2)$ iff $\vartheta_1 \leq \varsigma_1$ and $\vartheta_2 \leq \varsigma_2$.

Let $p: \omega \times \omega \to \mathbb{R}^2$ be a partial metric on ω, where

$$\begin{aligned}p(\vartheta, \varsigma) &= p((\vartheta_1, \vartheta_2).(\varsigma_1, \varsigma_2)) \\ &:= max\{|\vartheta_1 - \varsigma_1|, |\vartheta_2 - \varsigma_2|\},\end{aligned}$$

for all $\vartheta, \varsigma \in \omega$ so that the partial metric spaces (ω, p) is complete. Let mappings $t: \omega \to \omega$ and $T: \omega \to CB^p(\omega)$ be defined as follows:

$$t(\vartheta) = \vartheta$$

and

$$\mathrm{T}(\vartheta) = \mathrm{T}((\vartheta_1, \vartheta_2)) = \begin{cases}\{(0,0)\} & \vartheta_1 \geq \vartheta_2 \\ \{(0,0), (0, -1/4)\} & \vartheta_1 < \vartheta_2\end{cases}$$

T and t are continuous $T(\omega) \subseteq t(\omega) = \omega$. Let $\vartheta_n = (1/n, 1/(n+1))$ is sequence in ω, thus we have $T(\vartheta_n) \to \{(0,0)\}$ and $t(\vartheta_n) \to (0,0)$ as $n \to \infty$. It implies

$$H^p(Tt(\vartheta_n), tT(\vartheta_n)) \to 0$$

as $n \to \infty$. This means that T and t are compatible mappings. We get three pairs of comparable elements from ω: $(0, -1/4)$ and $(0,0)$, $(0, -1/4)$ and $(1/4, 0)$ and $(0,0)$ and $(1/4, 0)$. Suppose that $c = 1/2$, that all assumptions in Theorem 3.1 are satisfied. We are reviewing the following cases:

Case 1.
If $\vartheta = (0, -1/4)$ and $\varsigma = (0,0)$ then
$H^p(T(\vartheta), T(\varsigma)) = 0$, and
$\max\{p(t(\vartheta), t(\varsigma)), p(t(\vartheta), T(\vartheta)), p(t(\varsigma), T(\varsigma)),$
$1/2\left(p(t(\vartheta), T(\varsigma)) + p(t(\varsigma), T(\vartheta))\right)\}$ is equal to
$\max\{1/4, 1/4, 0, 1/2(1/4 + 1/4)\} = 1/4$

Case 2.
If $\vartheta = (0,0)$ and $\varsigma = (1/4, 0)$ then
$H^p(T(\vartheta), T(\varsigma)) = 0$, and
$\max\{p(t(\vartheta), t(\varsigma)), p(t(\vartheta), T(\vartheta)), p(t(\varsigma), T(\varsigma)),$
$1/2(p(t(\vartheta), T(\varsigma)) + p(t(\varsigma), T(\vartheta)))\}$ is equal to
$max\{1/4, 0, 1/4, 1/2(1/4 + 1/4)\} = 1/4$

Case 3.
If $\vartheta = (0, -1/4)$ and $\varsigma = (1/4, 0)$ then
$H^p(T(\vartheta), T(\varsigma)) = 0$, and
$\max\{p(t(\vartheta), t(\varsigma)), p(t(\vartheta), T(\vartheta)), p(t(\varsigma), T(\varsigma)),$
$1/2(p(t(\vartheta), T(\varsigma)) + p(t(\varsigma), T(\vartheta)))\}$ is equal to
$max\{1/4, 1/4, 1/4, 1/2(1/4 + 1/4) = 1/4.$

In all the above cases, it is clearly shown that
$H^p(T(\vartheta), T(\varsigma))$
$\leq c\max\{p(t(\vartheta), t(\varsigma)), p(t(\vartheta), T(\vartheta)), p(t(\varsigma), T(\varsigma)),$
$1/2(p(t(\vartheta), T(\varsigma + p(t(\varsigma), T(\vartheta)))\},$
for all comparable element. Therefore, the conditions of Theorem 3.1 are satisfied, confirming that $\{(0,0)\}$ is the coincidence point for t and T.

By replacing assumption 3 on Theorem 3.1 with the condition: "$t(\eta) \in T(\eta)$ for some $\eta \in \omega$", we will have results that t and T are compatible by referring to Lemma 2.12. Furthermore, referring to Theorem 3.1 and Lemma 2.13, we obtain Corollary 3.3 below.

Corollary 3.3. *Let $(\omega, p, \preccurlyeq)$ be a complete partially ordered partial metric space. Suppose that mappings $t: \omega \to \omega$ and $T: \omega \to CB^p(\omega)$ are continuous mappings and satisfy*

$$H^p\big(T(\vartheta), T(\varsigma)\big) \le c \max\{p(t(\vartheta), t(\varsigma)), p(t(\vartheta), T(\vartheta)), p(t(\varsigma), T(\vartheta)), 1/2(p(t(\vartheta), T(\varsigma) + p(t(\varsigma), T(\vartheta)))\},$$

for some c with $0 \le c < 1$ and for all comparable element $\vartheta, \varsigma \in \omega$. If the specified conditions are satisfied

1. $T(\omega) \subseteq t(\omega)$,
2. If $t(\varsigma) \in T(\vartheta)$ then $\vartheta \preccurlyeq \varsigma$,
3. $tT(\vartheta) = Tt(\vartheta)$,

it implies there exist $\eta \in \omega$ where $t(\eta) \in T(\eta)$.

Corollary 3.4. *Let $(\omega, p, \preccurlyeq)$ be a complete partially ordered partial metric space. Assume that mappings $t: \omega \to \omega$ and $T: \omega \to CB^p(\omega)$ are continuous mappings and satisfy*

$$H^p\big(T(\vartheta), T(\varsigma)\big) \le c\, p(t(\vartheta), t(\varsigma)),$$

for some c with $0 \le c < 1$ and for all comparable element $\vartheta, \varsigma \in \omega$. If the following assumptions hold

1. $T(\omega) \subseteq t(\omega)$,
2. If $t(\varsigma) \in T(\vartheta)$ then $\vartheta \preccurlyeq \varsigma$,
3. $tT(\vartheta) = Tt(\vartheta)$,

it implies there exist $\eta \in \omega$ where $t(\eta) \in T(\eta)$.

3.2 Generalized Results for Compatible Hybrid Mapping Pairs

Following the notations of [10], we used ψ to represent the family of all real functions ψ of $[0, \infty)$ into $[0, \infty)$, with ψ is right-continuous, non-decreasing, and $\psi(\vartheta) < \vartheta$ for each $\vartheta > 0$. We can see one of ψ's properties on the Lemma 3.5 below.

Lemma 3.5. [7], [35] *Let real function $\psi: [0, \infty) \to [0, \infty)$ be non-decreasing and right-continuous on $[0, \infty)$. We have $\psi^n(\vartheta) \to 0$, as $n \to \infty$ iff*

$$\psi(\vartheta) < \vartheta,$$

for every $\vartheta > 0$.

By assuming that f dominates maps, then using Lemma 3.5, we have results as in Theorem 3.7 below. Theorem 3.6 is the generalization of Theorem 3.1.

Theorem 3.6. *Suppose $(\omega, p, \preccurlyeq)$ be a complete partially ordered partial metric space, and if mappings $t: \omega \to \omega$ and $T: \omega \to CB^p(\omega)$ are continuous mappings and*

$$H^p\big(T(\vartheta), T(\varsigma)\big) \le \psi(\max\{p(t(\vartheta), t(\varsigma)), p(t(\vartheta), T(\vartheta)), p(t(\varsigma), T(\vartheta)), 1/2(p(t(\vartheta), t(\varsigma)) + p(t(\varsigma), T(\vartheta))\}),$$

for all comparable elements $\vartheta, \varsigma \in \omega$, and ψ a real function that meets the conditions specified in Lemma 3.5. If the specified conditions are satisfied

1. *$T(\omega) \subseteq t(\omega)$,*
2. *If $t(\varsigma) \in T(\vartheta)$ then $\vartheta \preccurlyeq \varsigma$,*
3. *The hybrid pairs T and t are compatible,*
4. *t is dominating mapping, i.e. $\vartheta \preccurlyeq t(\vartheta)$ for each $\vartheta \in \omega$,*

then hybrid mapping pairs t and T have a common fixed point on ω.

Proof. Let $\vartheta_0 \in \omega$ be any arbitrary element. Since $T(\omega) \subseteq t(\omega)$, we can take $\vartheta_1 \in \omega$ where $\varsigma_1 = t(\vartheta_1) \in T(\vartheta_0)$ then we have $\vartheta_0 \preccurlyeq \vartheta_1$. Repeating this argument, we obtain: if $\vartheta_{n-1} \in \omega$ there is $\vartheta_n \in \omega$ where

$$y_n = t(\vartheta_n) \in T(\vartheta_{n-1}),$$

so we have $\vartheta_{n-1} \preccurlyeq \vartheta_n$ for each n. Furthermore,

$$\begin{aligned} p\big(t(\vartheta_n), t(\vartheta_{n+1})\big) &\le H^p\big(T(\vartheta_{n-1}), T(\vartheta_n)\big) \\ &\le \psi(\max\{p\big(t(\vartheta_{n-1}), t(\vartheta_n)\big), p(t(\vartheta_{n-1}), T(\vartheta_{n-1})), p(t(\vartheta_n), T(\vartheta_n)), \\ &\quad 1/2(p(t(\vartheta_{n-1}), T(\vartheta_n)) + p(t(\vartheta_n), T(\vartheta_{n-1})))\} \\ &\le \psi(\max\{p\big(t(\vartheta_{n-1}), t(\vartheta_n)\big), p(t(\vartheta_{n-1}), t(\vartheta_n)), p\big(t(\vartheta_n), t(\vartheta_{n+1})\big), 1/2(p(t(\vartheta_{n-1}), t(\vartheta_{n+1})) \\ &\quad + p(t(\vartheta_n), t(\vartheta_n)))\} \\ &\le \psi(\max\{p\big(t(\vartheta_{n-1}), t(\vartheta_n)\big), p(t(\vartheta_n), t(\vartheta_{n+1})), 1/2(p(t(\vartheta_{n-1}), t(\vartheta_n)) + p(t(\vartheta_n), t(\vartheta_{n+1})))\} \\ &\le \psi(max\{p(t(\vartheta_{n-1}), t(\vartheta_n)), p(t(\vartheta_n), t(\vartheta_{n+1})). \end{aligned}$$

Therefore, we obtain:

$$p\big(t(\vartheta_n), t(\vartheta_{n+1})\big) \le \psi\Big(p\big(t(\vartheta_{n-1}), t(\vartheta_n)\big)\Big). \quad (4)$$

By continuing this process then, we obtain

$$\begin{aligned} p\big(t(\vartheta_n), t(\vartheta_{n+1})\big) &\le \psi\Big(p\big(t(\vartheta_{n-1}), t(\vartheta_n)\big)\Big) \\ &\le \psi\psi\Big(p\big(t(\vartheta_{n-1}), t(\vartheta_{n-1})\big)\Big) \\ &= \psi^2\Big(p\big(t(\vartheta_{n-2}), t(\vartheta_{n-1})\big)\Big) \\ &\le \psi\psi^2\Big(p\big(t(\vartheta_{n-2}), t(\vartheta_{n-1})\big)\Big) \\ &= \psi^3\Big(p\big(t(\vartheta_{n-3}), t(\vartheta_{n-2})\big)\Big) \\ &\vdots \\ &\le \psi^n\Big(p\big(t(\vartheta_0), t(\vartheta_1)\big)\Big) \end{aligned}$$

Therefore

$$p\big(t(\vartheta_n), t(\vartheta_{n+1})\big) \le \psi^n\Big(p\big(t(\vartheta_0), t(\vartheta_1)\big)\Big) \quad (5)$$

Since $p(t(\vartheta_0), t(\vartheta_1)) > 0$, then

$$\psi^n\Big(p\big(t(\vartheta_0), t(\vartheta_1)\big)\Big) \to 0 \text{ as } n \to \infty$$

by Lemma 3.5. This condition implies $p\big(t(\vartheta_n), t(\vartheta_{n+1})\big) \to 0$ as $n \to \infty$. It means that sequence $\big(t(\vartheta_n)\big)$ is Cauchy in ω. As we know that ω is complete, this condition guarantees the existence of $\eta \in \omega$ such that $t(\vartheta_n) \to \eta$ as $n \to \infty$. Furthermore, from the inequality (4) and (5) also show that

$$H^p\big(T(\vartheta_{n-1}), T(\vartheta_n)\big) \le \psi\Big(p\big(t(\vartheta_{n-1}), t(\vartheta_n)\big)\Big) \le \psi^n\Big(p\big(t(\vartheta_0), t(\vartheta_1)\big)\Big).$$

Therefore, $\big(T(\vartheta_n)\big)$ is the Cauchy sequence in $(CB^p(\omega), H^p)$. Since $(CB^p(\omega), H^p)$ is complete, then there exist set $\Upsilon \in CB^p(\omega)$ where $T(\vartheta_n) \to \Upsilon$ as $n \to \infty$. Furthermore, we have

$$\begin{aligned} p(\eta, \Upsilon) &\le p\big(\eta, t(\vartheta_n)\big) + p(t(\vartheta_n), \Upsilon) \\ &\quad - p\big(t(\vartheta_n), t(\vartheta_n)\big) \\ &\le p\big(\eta, t(\vartheta_n)\big) + p(t(\vartheta_n), \Upsilon) \\ &\le p\big(\eta, t(\vartheta_n)\big) + H^p(T(\vartheta_n), \Upsilon). \end{aligned}$$

Taking $n \to \infty$ then $p(\eta, \Upsilon) = 0$. It implies $\eta \in \Upsilon$, since $\Upsilon \in CB^p(\omega)$. By assumption 3, T and t, the hybrid pair mappings are compatible then $H^p\big(Tt(\vartheta_n), tT(\vartheta_n)\big) \to 0$, as $n \to \infty$. Furthermore, we have $t^2(\vartheta_{n+1}) \in tT(\vartheta_n)$ and by assumption 4 we have $\vartheta_n \preccurlyeq t(\vartheta_n)$ thus ϑ_n and $t(\vartheta_{n+1})$ are comparable. Therefore,

$$\begin{aligned} &H^p\big(Tt(\vartheta_{n+1}), T(\vartheta_n)\big) \\ &\le \psi(\max\{p\big(t^2(\vartheta_{n+1}), t(\vartheta_n)\big), \\ &\quad p\big(t^2(\vartheta_{n+1}), Tt(\vartheta_{n+1})\big), p\big(t(\vartheta_n), T(\vartheta_n)\big), \\ &\quad 1/2(p\big(t^2(\vartheta_{n+1}), T(\vartheta_n)\big) \\ &\quad + p(t(\vartheta_n), Tt(\vartheta_{n+1})))\}) \\ &\le \psi(\max\{p(tT(\vartheta_n), t(\vartheta_n)), H^p(tT(\vartheta_n), Tt(\vartheta_{n+1})), \\ &\quad p(t(\vartheta_n), T(\vartheta_n)), 1/2(H^p\big(tT(\vartheta_n), T(\vartheta_n)\big) \\ &\quad + p(t(\vartheta_n), Tt(\vartheta_{n+1})))\}) \\ &\le \psi(\max\{H^p(tT(\vartheta_n), Tt(\vartheta_n)) \\ &\quad + H^p(Tt(\vartheta_n), T(\vartheta_n)) + p(T(\vartheta_n), t(\vartheta_n)), \\ &\quad H^p\Big(tT(\vartheta_n), Tt(\vartheta_n) + p\big(Tt(\vartheta_n), t(\vartheta_n)\big)\Big) \\ &\quad + p\big(t(\vartheta_n), Tt(\vartheta_{n+1})\big), p\big(t(\vartheta_n), T(\vartheta_n)\big), \\ &\quad 1/2(p(tT(\vartheta_n), T(\vartheta_n)) \\ &\quad + p(t(\vartheta_n), Tt(\vartheta_{n+1})))\}) \\ &\le \psi(\max\{H^p(tT(\vartheta_n), Tt(\vartheta_n)) \\ &\quad + H^p\big(Tt(\vartheta_n), T(\vartheta_n)\big) + p\big(T(\vartheta_n), t(\vartheta_n)\big), \\ &\quad H^p(tT(\vartheta_n), Tt(\vartheta_n) + p(Tt(\vartheta_n), t(\vartheta_n)) \\ &\quad + p(t(\vartheta_n), Tt(\vartheta_{n+1}))\}) \\ &\le \psi(\max\{H^p\big(Tt(\vartheta_n), tT(\vartheta_n)\big) \\ &\quad + H^p\big(Tt(\vartheta_n), T(\vartheta_n)\big) + p\big(T(\vartheta_n), t(\vartheta_n)\big), \\ &\quad H^p\big(Tt(\vartheta_n), tT(\vartheta_n)\big) + p\big(Tt(\vartheta_n), t(\vartheta_n)\big) \\ &\quad + p(t(\vartheta_n), Tt(\vartheta_{n+1}))\}) \\ &\le \psi(\max\{H^p(Tt(\vartheta_n), tT(\vartheta_n)) \\ &\quad + H^p(Tt(\vartheta_n), T(\vartheta_n)) + p\big(T(\vartheta_n), t(\vartheta_n)\big), \\ &\quad H^p\big(Tt(\vartheta_n), tT(\vartheta_n)\big) + H^p\big(Tt(\vartheta_n), T(\vartheta_{n-1})\big) \\ &\quad + H^p(T(\vartheta_{n-1}), Tt(\vartheta_{n+1}))\}). \end{aligned}$$

Since ψ is non-decreasing, right-continuous, and also T and t are compatible then by taking $n \to \infty$

$$\begin{aligned} &H^p(T(\eta)), \Upsilon) \\ &\le \psi(\max\{0 + H^p(T(\eta), \Upsilon) + p(\Upsilon, \eta), \\ &\quad 0 + H^p(T(\eta), \Upsilon) + H^p(\Upsilon, T(\eta))\}) \\ &\le \psi(\max\{0 + H^p(T(\eta), \Upsilon) + 0, \\ &\quad 0 + H^p(T(\eta), \Upsilon) + H^p(\Upsilon, T(\eta))\}) \\ &\le \psi(max\{H^p(T(\eta), \Upsilon), 2H^p(T(\eta), \Upsilon)\}) \\ &\le \psi\big(2H^p(T(\eta), \Upsilon)\big) \\ &\le 2\, \psi(H^p(T(\eta), \Upsilon)) \end{aligned}$$

Since $\psi(\vartheta) < \vartheta$ for every $\vartheta > 0$, then we have $H^p(T(\eta), \Upsilon) = 0$. Thus,

$$T(\eta) = \Upsilon. \tag{6}$$

Let's consider that

$$H^p\big(Tt(\vartheta_n), tT(\vartheta_n)\big) \to H^p(T(\eta), t(\Upsilon)),$$

as $n \to \infty$. On the other side, since T and t are compatible, then we have

$$H^p\big(Tt(\vartheta_n), tT(\vartheta_n)\big) \to 0, \tag{7}$$

as $n \to \infty$. So, we obtain $H^p(T(\eta), t(\Upsilon)) = 0$, i.e.,

$$T(\eta) = t(\Upsilon). \tag{8}$$

Therefore, since $\eta \in \Upsilon$, $t(\eta) \in t(\Upsilon)$ and also from equation (6) and (8), then

$$t(\eta) \in T(\eta) = t(\Upsilon) = \Upsilon.$$

We also have $tT(\vartheta_n) \to T(\Upsilon)$. Therefore, $H^p\big(Tt(\vartheta_n), T(\Upsilon)\big) \to 0$ as $n \to \infty$ by (7).

Furthermore, by assumption 4 we have $t(\vartheta_n)$ and ϑ_n are comparable then we obtain

$$\begin{aligned} &p\big(t^2(\vartheta_{n+1}), t(\vartheta_{n+1})\big) \\ &\le H^p(tT(\vartheta_n), T(\vartheta_n)) \\ &\le H^p\big(tT(\vartheta_n), Tt(\vartheta_n)\big) + H^p\big(Tt(\vartheta_n), T(\vartheta_n)\big) \\ &\quad - H^p\big(Tt(\vartheta_n), Tt(\vartheta_n)\big) \\ &\le H^p(tT(\vartheta_n), Tt(\vartheta_n)) + H^p(Tt(\vartheta_n), T(\vartheta_n)) \\ &\le H^p\big(tT(\vartheta_n), Tt(\vartheta_n)\big) \\ &\quad + \psi(\max\{p(t^2(\vartheta_n), t(\vartheta_n)), p\big(t^2(\vartheta_n), Tt(\vartheta_n)\big), \\ &\quad p(t(\vartheta_n), T(\vartheta_n)), 1/2(p\big(t^2(\vartheta_n), T(\vartheta_n)\big) \\ &\quad + p(t(\vartheta_n), Tt(\vartheta_n)))\}) \end{aligned}$$

Taking $n \to \infty$, thus

$$\begin{aligned} &p(t(\eta), \eta) \\ &\le 0 + \psi(\max\{p(t(\eta), \eta), p(t(\eta), T(\Upsilon)), p(\eta, \Upsilon), \\ &\quad 1/2(p(t(\eta), \Upsilon) + p(\eta, t(\eta)))\}) \\ &= 0 + \psi(max\{p(t(\eta), \eta), 0, 0, 1/2(0 + 0)\}) \\ &= \psi(p(t(\eta), \eta)). \end{aligned}$$

As a result of $p(t(\eta), \eta) = 0$, i.e., $t(\eta) = \eta$. It means the point η is the fixed point of t. This condition allows the results $\eta = t(\eta) \in T(\eta)$. This condition indicates that η serves as a common fixed point of hybrid mapping pairs t and T.

Example 3.7. Suppose that T, t and ω are as defined in Example 3.2. For $\psi(\vartheta) = 2\vartheta/3$, we can show that Theorem 3.6 is satisfied, and we obtain $\vartheta = (0,0)$ where $\vartheta = t(\vartheta) \in T(\vartheta)$.

3.3 Compatible t-Weak Hybrid Mapping Pairs

In this section, we extend Theorem 2.15 in the setting of partially ordered sets.

Theorem 3.8. *Let $(\omega, p, \preccurlyeq)$ be a complete partially ordered partial metric space. If $t: \omega \to \omega$ and $T: \omega \to CB^p(\omega)$ be a continuous mappings and satisfy*

$$H^p(T(\vartheta), T(\varsigma)) \le c \max\{p(t(\vartheta), t(\varsigma)), p(t(\vartheta), T(\vartheta)), p(t(\varsigma), T(\varsigma)), 1/2(p(t(\vartheta), T(\varsigma)) + p(t(\varsigma), T(\vartheta)))\},$$

for some c with $0 \le c < 1$ and for all comparable element $\vartheta, \varsigma \in \omega$. If the specified conditions are hold

1. *$T(\omega) \subseteq t(\omega)$,*
2. *If $t(\varsigma) \in T(\vartheta)$ then $\vartheta \preccurlyeq \varsigma$,*
3. *t and T are t-weak compatible,*
4. *If $\varsigma_n \in T(\vartheta_n)$ such that $\varsigma_n \to \varsigma = t(\vartheta)$ then $\vartheta_n \preccurlyeq \vartheta$ for all n,*
5. *t is dominating mappings,*

thus $t(\eta) \in \ (\eta)$ for $\eta \in \omega$.

Proof. Analogue with Theorem 3.1's proof, we only need to verify that η is coincidence point of hybrid mapping pairs t and T. Considering t and T as t-weak compatible mappings, then

$$lim_{n\to\infty} H^p(tT(\vartheta_n), Tt(\vartheta_n)) \le lim_{n\to\infty} H^p(Tt(\vartheta_n), T(\vartheta_n)),$$

and

$$lim_{n\to\infty} p(tT(\vartheta_n), Tt(\vartheta_n)) \le lim_{n\to\infty} H^p(Tt(\vartheta_n), T(\vartheta_n)).$$

Since t and T are continuous then

$$H^p(t(\Upsilon), T(\eta)) \le H^p(T(\eta), \Upsilon) \quad (9)$$

and $p(t(\Upsilon), \eta) \le H^p(T(\eta), \Upsilon)$. Since $\eta \in \Upsilon$ then $p(t(\eta), \eta) \le p(t(\Upsilon), \eta)$. Thus

$$p(t(\eta), \eta) \le H^p(T(\eta), \Upsilon).$$

Furthermore,

$$\begin{aligned} &p(t(\eta)), T(\eta)) \\ &\le p(t(\eta), tt(\vartheta_{n+1})) + p(tt(\vartheta_{n+1}), T(\eta)) - p(tt(\vartheta_{n+1}), tt(\vartheta_{n+1})) \\ &\le p(t(\eta), tt(\vartheta_{n+1})) + H^p(tT(\vartheta_n), T(\eta)) - p(tt(\vartheta_{n+1}), tt(\vartheta_{n+1})) \\ &\le p(t(\eta), tt(\vartheta_{n+1})) + H^p(tT(\vartheta_n), Tt(\vartheta_n)) + H^p(Tt(\vartheta_n), T(\eta)) - H^p(Tt(\vartheta_n), Tt(\vartheta_n)) \\ &\quad - p(tt(\vartheta_{n+1}), tt(\vartheta_{n+1})) \end{aligned}$$

Taking $n \to \infty$ then

$$p(t(\eta)), T(\eta)) \le H^p(t(\Upsilon), T(\eta)).$$

By (9) we obtain $p(t(\eta)), T(\eta)) \le p(T(\eta)), \Upsilon)$. Since $t(\vartheta_n) \to \eta = t(\vartheta)$ by assumption 4 we have $\vartheta_n \preccurlyeq \vartheta$. Also, by assumption 5, t is dominating mappings then $\vartheta \preccurlyeq t(\vartheta)$, then we obtain $\vartheta_n \preccurlyeq \eta$. Therefore

$$\begin{aligned} &H^p(T(\vartheta_n), T(\eta)) \\ &\le c \max\{p(t(\vartheta_n), t(\eta)), p(t(\vartheta_n), T(\vartheta_n)), p(t(\eta), T(\eta)), 1/2(p(t(\vartheta_n), T(\eta)) + p(t(\eta), T(\vartheta_n)))\} \\ &\le c \max\{p(t(\vartheta_n), t(\eta)), p(t(\vartheta_n), T(\vartheta_n)), p(t(\eta), T(\eta)), 1/2(p(t(\vartheta_n), T(\eta)) + p(t(\eta), t(\vartheta_n)) + p(t(\vartheta_n), T(\vartheta_n)))\} \end{aligned}$$

Taking $n \to \infty$ we have

$$\begin{aligned} &H^p(\Upsilon, T(\eta)) \\ &\le c \max\{p(\eta, t(\eta)), p(\eta, \Upsilon), p(t(\eta), T(\eta)), 1/2(p(\eta, T(\eta)) + p(t(\eta), \eta) + p(\eta, \Upsilon))\} \\ &\le c \max\{H^p(T(\eta), \Upsilon), p(\eta, \eta), H^p(t(\Upsilon), T(\eta)), 1/2(H^p(T(\eta), \Upsilon) + p(T(\eta), \Upsilon) + p(\eta, \eta))\} \end{aligned}$$

then we have

$$H^p(\Upsilon, T(\eta)) \le c\,(H^p(T(\eta), \Upsilon) + p(\eta, \eta)).$$

Since $0 \le c < 1$ and $p(\eta, \eta) \le H^p(T(\eta), \Upsilon)$ then $H^p(\Upsilon, T(\eta)) = 0$.

Furthermore, since $p(t(\eta), T(\eta)) \le H^p(\Upsilon, T(\eta))$ then $p(t(\eta), T(\eta)) = 0$. Let's consider that $T(\eta) \in CB^p(X)$ then $t(\eta) \in T(\eta)$. This completes the proof.

Since the sets of weakly compatible mappings on partial metric spaces include compatible mappings, Example 3.2 is also applicable to Theorem 3.8.

3.4 Generalized Results for t-Weak Compatible Mappings

In this section, we generalize Theorem 3.6 to apply to t-weak compatible mappings T and t as stated in Theorem 3.9.

Theorem 3.9. *Let $(\omega, p, \preccurlyeq)$ be a complete partially ordered partial metric space and assume that the mappings $t: \omega \to \omega$ and $T: \omega \to CB^p(\omega)$ are continuous and satisfy*

$$H^p(T(\vartheta), T(\varsigma)) \le \psi(\max\{p(t(\vartheta), t(\varsigma)), p(t(\vartheta), T(\vartheta)), p(t(\varsigma), T(\vartheta)), 1/2(p(t(\vartheta), t(\varsigma)) + p(t(\varsigma), T(\vartheta))\}),$$

for all comparable elements $\vartheta, \varsigma \in \omega$, where ψ a real function that satisfies Lemma 3.6's conditions. If the specified conditions are satisfied

1. *$T(\omega) \subseteq t(\omega)$,*
2. *If $t(\varsigma) \in T(\vartheta)$ then $\vartheta \preccurlyeq \varsigma$,*
3. *The pairs T and t are t-weak compatible,*
4. *If $\varsigma_n \in T(\vartheta_n)$ such that $\varsigma_n \to \varsigma = t(\vartheta)$ then $\vartheta_n \preccurlyeq \vartheta$ for all n,*
5. *t is dominating mapping, i.e. $\vartheta \preccurlyeq t(\vartheta)$ for each $\vartheta \in \omega$,*

it implies t and T, the hybrid mapping pairs have a common fixed point on ω.

Proof. The proof of Theorem 3.9 is on a similar line with Theorem 3.1's proof. We must verify that η is the coincidence point of hybrid mapping pairs T and t. When t and T are t-weak compatible hybrid mapping pairs, then

$$lim_{n\to\infty}H^p\big(tT(\vartheta_n),Tt(\vartheta_n)\big) \leq lim_{n\to\infty}H^p(Tt(\vartheta_n),T(\vartheta_n)),$$

and

$$lim_{n\to\infty}p\big(tT(\vartheta_n),Tt(\vartheta_n)\big) \leq lim_{n\to\infty}H^p(Tt(\vartheta_n),T(\vartheta_n)).$$

By continuity of T and t,

$$H^p\big(t(\Upsilon),T(\eta)\big) \leq H^p(T(\eta),\Upsilon), \qquad (10)$$

and $p(t(\Upsilon),\eta) \leq H^p(T(\eta),\Upsilon)$. Since $\eta \in \Upsilon$ then $p(t(\eta),\eta) \leq p(t(\Upsilon),\eta)$. Thus

$$p(t(\eta),\eta) \leq H^p(T(\eta),\Upsilon).$$

Therefore

$$\begin{aligned}
&p(t(\eta)),T(\eta))\\
&\leq p\big(t(\eta),tt(\vartheta_{n+1})\big)+p\big(tt(\vartheta_{n+1}),T(\eta)\big)\\
&\quad -p(tt(\vartheta_{n+1}),tt(\vartheta_{n+1}))\\
&\leq p\big(t(\eta),tt(\vartheta_{n+1})\big)+H^p\big(tT(\vartheta_n),T(\eta)\big)\\
&\quad -p(tt(\vartheta_{n+1}),tt(\vartheta_{n+1})).\\
&\leq p\big(t(\eta),tt(\vartheta_{n+1})\big)+H^p\big(tT(\vartheta_n),Tt(\vartheta_n)\big)\\
&\quad +H^p(Tt(\vartheta_n),T(\eta))-H^p(Tt(\vartheta_n),Tt(\vartheta_n))\\
&\quad -p(tt(\vartheta_{n+1}),tt(\vartheta_{n+1}))
\end{aligned}$$

Taking $n\to\infty$ on the both side, we obtain $p(t(\eta)),T(\eta)) \leq H^p(t(\Upsilon),T(\eta))$. By (3.7) we have

$$p(t(\eta)),T(\eta)) \leq p(T(\eta)),\Upsilon).$$

Since $t(\vartheta_n)\to\eta=t(\vartheta)$ by assumption 4 we will have $\vartheta_n \preccurlyeq \vartheta$. Also, by assumption 5, t is dominating mappings then $\vartheta \preccurlyeq t(\vartheta)$, then we obtain $\vartheta_n \preccurlyeq \eta$. Consequently

$$\begin{aligned}
&H^p\big(T(\vartheta_n),T(\eta)\big)\\
&\leq \psi(\max\{p(t(\vartheta_n),t(\eta)),p(t(\vartheta_n),T(\vartheta_n)),\\
&\quad p(t(\eta),T(\eta)),\\
&\quad 1/2(p(t(\vartheta_n),T(\eta))+p(t(\eta),T(\vartheta_n)))\})\\
&\leq \psi(\max\{p(t(\vartheta_n),t(\eta)),p(t(\vartheta_n),T(\vartheta_n)),\\
&\quad p(t(\eta),T(\eta)),1/2(p(t(\vartheta_n),T(\eta))\\
&\quad +p(t(\eta),t(\vartheta_n))+p(t(\vartheta_n),T(\vartheta_n)))\})
\end{aligned}$$

Taking $n\to\infty$ we have

$$\begin{aligned}
&H^p\big(\Upsilon,T(\eta)\big)\\
&\leq \psi(\max\{p(\eta,t(\eta)),p(\eta,\Upsilon),p(t(\eta),T(\eta)),\\
&\quad 1/2(p(\eta,T(\eta))+p(t(\eta),\eta)+p(\eta,\Upsilon))\})\\
&\leq \psi(\max\{H^p(T(\eta),\Upsilon),p(\eta,\eta),H^p(t(\Upsilon),T(\eta)),\\
&\quad 1/2(H^p(T(\eta),\Upsilon)+p(T(\eta),\Upsilon)+p(\eta,\eta))\})\\
&\leq \psi((H^p(T(\eta),\Upsilon)+p(\eta,\eta))).
\end{aligned}$$

Since $\psi(\vartheta)<\vartheta$ for $\vartheta>0$ and

$$p(\eta,\eta) \leq H^p(T(\eta),\Upsilon)$$

then $H^p(\Upsilon,T(\eta))=0$. Furthermore, since we have $p\big(t(\eta),T(\eta)\big) \leq H^p(\Upsilon,T(\eta))$ then

$$p(t(\eta),T(\eta))=0.$$

Consider $T(\eta)\in CB^p(\omega)$ then $t(\eta)\in T(\eta)$. Since $H^p(\Upsilon,T(\eta))=0$ then $T(\eta)=\Upsilon$. Since

$$H^p\big(t(\Upsilon),T(\eta)\big) \leq H^p(T(\eta),\Upsilon)=0,$$

thus $H^p(t(\Upsilon),T(\eta))=0$, it means $t(\Upsilon)=T(\eta)$. Also, since $\eta\in\Upsilon$ and $t(\eta)\in T(\eta)$ then

$$t(\eta)\in T(\eta)=T(\Upsilon)=\Upsilon.$$

Furthermore, since ϑ_n are comparable for every n and t is dominating mapping then, ϑ_n and $t(\vartheta_{n+1})$ are comparable. Thus

$$\begin{aligned}
&p\big(tt(\vartheta_{n+1}),t(\vartheta_{n+1})\big)\\
&\leq H^p\big(tT(\vartheta_n),T(\vartheta_n)\big)\\
&\leq H^p\big(tT(\vartheta_n),Tt(\vartheta_n)\big)+H^p\big(Tt(\vartheta_n),T(\vartheta_n)\big)\\
&\quad -H^p(Tt(\vartheta_n),Tt(\vartheta_n))\\
&\leq H^p(tT(\vartheta_n),Tt(\vartheta_n))+H^p(Tt(\vartheta_n),T(\vartheta_n))\\
&\leq H^p\big(tT(\vartheta_n),Tt(\vartheta_n)\big)\\
&\quad +\psi(\max\{p(tt(\vartheta_n),t(\vartheta_n)),p(tt(\vartheta_n),Tt(\vartheta_n),\\
&\quad p\Big(t\big(\vartheta_n,T(\vartheta_n)\big)\Big),\\
&\quad 1/2\Big(p\big(tt(\vartheta_n)\big),T(\vartheta_n)\Big)+p(t(\vartheta_n),Tt(\vartheta_n))\}).
\end{aligned}$$

Taking $n\to\infty$ on the both side

$$\begin{aligned}
&p(t(\eta),\eta)\\
&\leq H^p\big(t(\Upsilon),T(\eta)\big)\\
&\quad +\psi(\max\{p(t(\eta),\eta),p(t(\eta),T(\eta),p(\eta,\Upsilon),\\
&\quad 1/2(p(t(\eta)),\Upsilon)+p(\eta,T(\eta))\}).
\end{aligned}$$

Therefore, $p(t(\eta),\eta)=0$, this means $t(\eta)=\eta$. Consequently, $\eta=t(\eta)\in T(\eta)$.

Similarly, we have the fact that weak compatible mappings include that of the sets of compatible mappings, so Example 3.7 also holds for Theorem 3.9.

4 Conclusion

In this article, we prove several theorems, including coincidence point theorems and common fixed point theorems, for hybrid mappings: single-valued mappings and set-valued mappings. These theorems introduce a novel condition setting within a space involving partial orders. The compatibility condition of hybrid mappings is utilized to demonstrate the existence of coincidence points. Additionally, the membership relation between single-valued and set-valued mappings is an extra assumption to establish common fixed points. The same approach is applied by exploring the relationship between compatible and weakly compatible mappings to prove the existence of coincidence points and common fixed points for t-weak compatible mappings. Furthermore, we propose a more general contraction principle incorporating a non-decreasing, right-continuous ψ function, which is subsequently employed to prove common fixed point theorems for both compatible and t-weak compatible mappings.

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Conflict of Interest

The authors have no conflicts of interest to declare.