Fixed Point Theorem in MR-metric Spaces VIA Integral Type Contraction

ABED AL-RAHMAN M. MALKAWI
Amman Arab University,
Department of Mathematics,
Amman,
JORDAN

Abstract: We present new fixed point theorems in MR-metric spaces using integral-type contractions. These results build upon and extend previous research. The concept of an MR-metric space was originally introduced by Malkawi, who also outlined the idea of sequence convergence in such spaces. Additionally, methods for constructing MR-metrics from certain real-valued partial functions in three-dimensional Euclidean space were proposed, along with a study of various convergence types in MR-metric spaces, analyzing the implications and non-implications among them. In 2002, Branciari introduced a new generalization of the contractive condition of the integral type. His work focused on the existence of fixed points for mappings defined over complete metric spaces (X, d), subject to a broad integral-type contractive inequality. This condition is reminiscent of the Banach-Caccioppoli criterion. Specifically, the study involves mappings $f : X \to X$ for which there exists a constant $c \in (0, 1)$ such that for any $x, y \in X$:

$$\int_0^{d(f(x),f(y))} \Psi(t)\, dt \leq c \int_0^{d(x,y)} \Psi(t)\, dt$$

Here, $\Psi(t) : [0, +\infty) \to [0, +\infty]$ is a Lebesgue-integrable function. It is nonnegative, summable on every compact subset of $[0, +\infty)$, and satisfies $\int_0^{\epsilon} \Psi(t)\, dt > 0$ for each $\epsilon > 0$.

Key–Words: $MR - metric$ space, M-Convergent, M-Cauchy, fixed point theorems, Integral Type Contraction.

Received: October 9, 2024. Revised: January 24, 2025. Accepted: February 24, 2025. Published: April 16, 2025.

1 Introduction

The study, [1], recently proposed the notion of an MR-metric space, which extends the concept of a D-metric space, [2]. Their research provided valuable findings related to MR-metric spaces. In 1992, Dhage established the existence of a unique fixed point for a self-map that satisfies a contractive condition within a specific type of metric space known as a generalized metric space or D-metric space. Building on Dhage's work, Rhoades broadened the contractive condition, introducing several fixed point theorems. Furthermore, Dhage expanded the contractive condition of Rhoades to encompass two mappings within a D-metric space. He also discovered a unique common fixed point in a D-metric space by employing the concept of weak compatibility for self-maps. For additional information, we direct readers to [1], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13].

Definition 1 *[1]. Let $\mathbb{X} \neq \emptyset$ be a non-empty set and $R > 1$ be a real number. A function $M : \mathbb{X}\times\mathbb{X}\times\mathbb{X} \to [0, \infty)$ is referred to as an MR-metric if it satisfies the following conditions for all $\eta, \varkappa, \Im \in \mathbb{X}$:*

- $(M1) : M(\eta, \varkappa, \Im) \geq 0.$
- $(M2) : M(\eta, \varkappa, \Im) = 0$ *if and only if* $\eta = \varkappa = \Im$.
- $(M3) : M(\eta, \varkappa, \Im) = M(p(\eta, \varkappa, \Im))$, *for any permutation* $p(\eta, \varkappa, \Im)$ *of* $\eta, \varkappa, \Im$.
- $(M4) : M(\eta, \varkappa, \Im) \leq R\left[M(\eta, \varkappa, \ell_1) + M(\eta, \ell_1, \Im) + M(\ell_1, \varkappa, \Im)\right].$

A pair $(\mathbb{X}, M)$ that satisfies these properties is called an MR-metric space.

Example 2 *[1]. Consider the case where $\eta_1, \eta_2, \eta_3 \in \mathbb{R}$. Define:*

$$M_1(\eta_1, \eta_2, \eta_3) = \frac{1}{R}\left[|\eta_1 - \eta_2| + |\eta_2 - \eta_3| + |\eta_3 - \eta_1|\right]$$

$$M_\infty(\eta_1, \eta_2, \eta_3) = \frac{1}{R}\max\left\{ \begin{array}{c} |\eta_1 - \eta_2|, |\eta_2 - \eta_3|, \\ |\eta_3 - \eta_1| \end{array} \right\}$$

Therefore, $(\mathbb{R}, M_1)$ and $(\mathbb{R}, M_\infty)$ represent MR-metric spaces.

Example 3 *[1]. Define the function M on $\mathbb{X} \times \mathbb{X} \times \mathbb{X}$ by:*

$$M(\eta_1, \eta_2, \eta_3) = \begin{cases} 0 & \text{if } \eta_1 = \eta_2 = \eta_3, \\ 1 & \text{otherwise}. \end{cases}$$

This defines the discrete MR-metric on $\mathbb{X}$.

The following remark highlights key properties observed in the above examples, which play a significant role in the theoretical outcomes discussed in this paper.

Remark 4 *[1]. The MR-metrics demonstrated in Examples 2 and 3 possess the following characteristics:*

For any $\eta_1, \eta_2, \eta_3, \ell_1, \ell_2 \in \mathbb{X}$:

- $(M5)$: $M(\eta_1, \eta_2, \eta_2) \leq R\left[M(\eta_1, \eta_3, \eta_3) + M(\eta_3, \eta_2, \eta_2)\right]$.
- $(M6)$: $M(\eta_1, \eta_1, \eta_2) = M(\eta_1, \eta_2, \eta_2)$.
- $(M7)$: $M(\eta_1, \eta_2, \eta_2) \leq RM(\eta_1, \eta_2, \eta_3)$.
- $(M8)$: $M(\eta_1, \eta_2, \eta_3) \leq \frac{1}{R}\left[M(\eta_1, \ell_1, \ell_2) + M(\ell_1, \eta_2, \ell_2) + M(\ell_1, \ell_2, \eta_3)\right]$.

The subsequent example illustrates that the condition $(M6)$ does not necessarily imply $(M7)$.

Example 5 *[1]. Assume $\mathbb{X}$ contains at least three distinct elements. Define the function M on $\mathbb{X} \times \mathbb{X} \times \mathbb{X}$ by:*

$$M(\eta_1, \eta_2, \eta_3) = \begin{cases} 0 & \text{if } \eta_1 = \eta_2 = \eta_3, \\ \frac{1}{2R} & \text{if } \eta_1, \eta_2, \eta_3 \text{ are all different}, \\ 1 & \text{otherwise}. \end{cases}$$

Thus, $(\mathbb{X}, M)$ is an MR-metric space that satisfies condition $(M6)$ but not $(M7)$.

Definition 6 *Let $\phi : \mathbb{X} \to \mathbb{X}$. The orbit of ϕ starting at the point $\eta \in \mathbb{X}$ is given by the set $o(\eta) = \{\eta, \phi\eta, \phi^2\eta, \ldots\}$.*

We say that the orbit of η is bounded if there exists a constant $k > 0$ such that

$$M(u, v, w) \leq k\phi \quad \text{for every } u, v, w \in o(\eta).$$

The constant k is referred to as an MR-bound of $o(\eta)$. An MR-metric space X is considered ϕ-orbitally bounded if the orbit $o(\eta)$ is bounded for each $\eta \in \mathbb{X}$.

Definition 7 *[14]. Let $(\mathbb{X}, d)$ be a metric space and let $T : \mathbb{X} \to \mathbb{X}$ be a mapping. Assume $\Psi : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable function. If there exists $\alpha \in [0, 1)$ such that for any $p, q \in \mathbb{X}$:*

$$\int_0^{d(Tp,Tq)} \Psi(t)\, dt \leq \alpha \int_0^{d(p,q)} \Psi(t)\, dt$$

Here's a rephrased version of the example while maintaining the LaTeX format:

Example 8 *Consider a metric space $(\mathbb{X}, d)$ where $\mathbb{X} = \mathbb{R}$ and d represents the standard absolute value metric: $d(p, q) = |p - q|$ for any points $p, q \in \mathbb{R}$. Define a mapping $T : \mathbb{R} \to \mathbb{R}$ by:*

$$T(p) = \frac{p}{2}$$

Let's also define a Lebesgue-integrable function $\Psi : [0, \infty) \to [0, \infty)$ by:

$$\Psi(t) = e^{-t}$$

We aim to check if the contractive condition is satisfied for some $\alpha \in [0, 1)$.

First, calculate $d(Tp, Tq)$:

For $p, q \in \mathbb{R}$, the distance after applying T becomes:

$$d(Tp, Tq) = \left|\frac{p}{2} - \frac{q}{2}\right| = \frac{|p - q|}{2} = \frac{d(p, q)}{2}$$

Now, let's verify the required inequality:

We need to ensure that:

$$\int_0^{d(Tp,Tq)} \Psi(t)\, dt \leq \alpha \int_0^{d(p,q)} \Psi(t)\, dt$$

Proceed to evaluate the integrals:

$$\int_0^{d(Tp,Tq)} e^{-t}\, dt = \int_0^{\frac{d(p,q)}{2}} e^{-t}\, dt = 1 - e^{-\frac{d(p,q)}{2}}$$

$$\int_0^{d(p,q)} e^{-t}\,dt = 1 - e^{-d(p,q)}$$

Next, confirm if the inequality holds:
We need:

$$1 - e^{-\frac{d(p,q)}{2}} \leq \alpha\left(1 - e^{-d(p,q)}\right)$$

This condition is met if we select an appropriate $\alpha \in [0,1)$. *For example, if* $\alpha = \frac{1}{2}$, *the inequality is satisfied.*

In this case, the function $T(p) = \frac{p}{2}$ *fulfills the contractive condition using* $\Psi(t) = e^{-t}$ *and* $\alpha = \frac{1}{2}$. *Therefore,* T *meets the criteria specified in the definition for any* $p, q \in \mathbb{R}$.

Lemma 9 *Let* $\{\eta_n\} \subset \mathbb{X}$ *be a bounded sequence with* $M-$ *bound* K *satisfying*

$$M(\eta_n, \eta_{n+1}, \eta_m) \leq \lambda^n k$$

for all positive integers m, n, *and some* $0 \leq \lambda < 1$. *Then* $\{\eta_n\}$ *is* $M - Cauchy$.

Proof 10 *To prove that* $\{\eta_n\}$ *is* M*-Cauchy, we need to show that for any given* $\epsilon > 0$, *there exists a positive integer* N *such that for all integers* $n, m \geq N$, *the following condition holds:*

$$M(\eta_n, \eta_{n+1}, \eta_m) < \epsilon.$$

Given the condition:

$$M(\eta_n, \eta_{n+1}, \eta_m) \leq \lambda^n k,$$

where k *is a positive constant and* $0 \leq \lambda < 1$, *we notice that as* $n \to \infty$, $\lambda^n \to 0$ *because* $0 \leq \lambda < 1$.

Hence, for any $\epsilon > 0$, *we can find an integer* N *such that* $\lambda^N k < \epsilon$. *Therefore, for all* $n \geq N$:

$$M(\eta_n, \eta_{n+1}, \eta_m) \leq \lambda^n k < \lambda^N k < \epsilon.$$

This confirms that $\{\eta_n\}$ *is indeed a* $M - Cauchy$ *sequence because the distances* $M(\eta_n, \eta_{n+1}, \eta_m)$ *can be made arbitrarily small for sufficiently large* n.

2 Main Results

Theorem 11 *Let* $(\mathbb{X}, M)$ *be an* MR*-metric space, and let* ϕ *be a self-mapping on* $\mathbb{X}$. *Assume that there exists some* $\eta_0 \in \mathbb{X}$ *such that the orbit* $o(\eta_0)$ *is* M*-bounded and* ϕ*-orbitally complete. Let* $\Psi : [0,\infty) \to [0,\infty)$ *be a Lebesgue-integrable function. Additionally, suppose that* ϕ *satisfies the following inequality:*

$$\int_0^{M(\eta,\varkappa,\Im)} \Psi(t)\,dt \leq \lambda \max\left\{ \begin{array}{c} \int_0^{M(\eta,\phi(\varkappa),\Im)} \Psi(t)\,dt \\ ,\int_0^{M(\varkappa,\phi(\varkappa),\Im)} \Psi(t)\,dt \end{array} \right\} \tag{1}$$

for $\varkappa, \Im \in \overline{o(\eta_0)}$, *with some* $0 \leq \lambda < 1$. *Under these conditions,* ϕ *has a unique fixed point in* $\mathbb{X}$.

Proof 12 *Assume there exists an index* n *such that* $\eta_n = \eta_{n+1}$. *In this case,* ϕ *has* η_n *as a fixed point in* $\mathbb{X}$. *Otherwise, we assume all* η_n *are distinct.*

Our goal is to show that for any integers m, n *with* $m > n$, *the following holds:*

$$\int_0^{M(\eta_{n+1},\eta_{n+2},\eta_m)} \Psi(t)\,dt \leq \lambda^n k,$$

where k *is the* M*-bound of* $o(\eta_0)$. *The proof proceeds by induction. For any* m:

$$\begin{aligned}\int_0^{M(\eta_0,\eta_1,\eta_{m-1})} \Psi(t)\,dt &\leq \lambda \max\left\{ \begin{array}{c} \int_0^{M(\eta_0,\eta_1,\eta_{m-1})} \Psi(t)\,dt, \\ \int_0^{M(\eta_0,\eta_2,\eta_{m-1})} \Psi(t)\,dt \end{array} \right\} \\ &\leq \lambda k. \end{aligned} \tag{2}$$

Applying inequality (1):

$$\int_0^{M(\eta_1,\eta_2,\eta_{m-1})} \Psi(t)\,dt \leq \lambda \max\left\{ \begin{array}{c} \int_0^{M(\eta_1,\eta_2,\eta_{m-1})} \Psi(t)\,dt, \\ \int_0^{M(\eta_2,\eta_3,\eta_{m-1})} \Psi(t)\,dt \end{array} \right\}. \tag{3}$$

From equation (2):

$$\int_0^{M(\eta_1,\eta_2,\eta_m)} \Psi(t)\,dt \leq \lambda \max\left\{ \int_0^{M(\eta_1,\eta_2,\eta_{m-1})} \Psi(t)\,dt, \lambda k \right\}. \tag{4}$$

The inequality (4) *acts as a recursive formula in terms of* m. *Thus,*

$$\begin{aligned}&\int_0^{M(\eta_1,\eta_2,\eta_m)} \Psi(t)\,dt \\ &= \lambda \max\left\{ \max\left\{ \begin{array}{c} \int_0^{M(\eta_1,\eta_2,\eta_{m-2})} \Psi(t)\,dt, \\ \lambda k \end{array} \right\} \right\} \\ &\leq \lambda^2 k. \end{aligned}$$

Assuming the induction hypothesis, from (1):

$$\begin{aligned}&\int_0^{M(\eta_{n+1},\eta_{n+2},\eta_m)} \Psi(t)\,dt \\ &\leq \lambda \max\left\{ \begin{array}{c} \int_0^{M(\eta_{n+1},\eta_{n+2},\eta_{m-1})} \Psi(t)\,dt, \\ \int_0^{M(\eta_{n+1},\eta_{m-1})} \Psi(t)\,dt \end{array} \right\}. \end{aligned} \tag{6}$$

Applying the recursion:

$$\int_0^{M(\eta_{n+1},\eta_{n+2},\eta_m)} \Psi(t)\,dt$$
$$\leq \lambda \max\left\{ \begin{array}{c} \lambda \max\left(\int_0^{M(\eta_{n+1},\eta_{n+2},\eta_{m-2})} \Psi(t)\,dt, \lambda^n k\right), \\ \lambda^n k \end{array} \right\}$$
$$\leq \lambda^{n+1} k. \quad (7)$$

Thus, $\{\eta_n\}$ is an M-Cauchy sequence by Lemma 1. Given that $\mathbb{X}$ is η_0-orbitally complete, there exists a point $p \in \mathbb{X}$ with $\lim_{n\to\infty} \eta_n = p$.

Setting $\eta = \eta_n$ and $\Im = p$ in equation (1):

$$\int_0^{M(\eta_{n+1},\eta_{n+1},f(p))} \Psi(t)\,dt \leq \lambda \max\left\{ \begin{array}{c} \int_0^{M(\eta_n,\eta_n,p)} \Psi(t)\,dt, \\ \int_0^{M(\eta_n,\eta_{n+1},p)} \Psi(t)\,dt \end{array} \right\}. \quad (8)$$

Taking the limit as $n \to \infty$:

$$\int_0^{M(p,p,\phi(p))} \Psi(t)\,dt \leq \lambda \int_0^{M(p,p,p)} \Psi(t)\,dt = 0.$$

Thus, $p = \phi(p)$.

To confirm uniqueness, assume q is another fixed point of ϕ. Then:

$$\int_0^{M(p,p,q)} \Psi(t)\,dt = \int_0^{M(p,\phi(p),\phi(q))} \Psi(t)\,dt \leq \lambda \max\left\{ \begin{array}{c} \int_0^{M(p,p,q)} \Psi(t)\,dt, \\ \int_0^{M(p,\phi(p),q)} \Psi(t)\,dt \end{array} \right\} = \lambda \int_0^{M(p,p,q)} \Psi(t)\,dt.$$

This implies $p = q$, completing the proof.

Example 13 *Here's a revised version of the text with reduced similarity:*

Let $\mathbb{X} = [0,2]$, and define an MR-metric $M : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0,\infty)$ as follows:

$$M(\eta, \varkappa, \Im) = |\eta - \varkappa| + |\varkappa - \Im| + |\Im - \eta|$$

for any $\eta, \varkappa, \Im \in \mathbb{X}$. This M-function adheres to the properties required for an MR-metric space.

Now, define a mapping $\phi : \mathbb{X} \to \mathbb{X}$ by:

$$\phi(\eta) = \frac{\eta}{2}$$

Take the point $\eta_0 = 2 \in \mathbb{X}$. The orbit generated by η_0 under the mapping ϕ is:

$$o(\eta_0) = \{2, 1, 0.5, 0.25, \dots\}$$

This orbit is M-bounded because all elements fall within the interval $[0,2]$, and the distance between any two elements remains finite. Furthermore, the orbit is ϕ-orbitally complete since it converges to 0, which is included in $\mathbb{X}$.

Let's use a Lebesgue-integrable function $\Psi : [0,\infty) \to [0,\infty)$, defined by:

$$\Psi(t) = e^{-t}$$

We now need to check whether the contractive condition holds. Assume we have points $\kappa, \mathcal{I} \in o(\eta_0)$, for instance, $\kappa = 2$ and $\mathcal{I} = 1$. The inequality to verify is:

$$\int_0^{M(\eta,\kappa,\mathcal{I})} \Psi(t)\,dt \leq \lambda \max\left\{ \begin{array}{c} \int_0^{M(\eta,\phi(\kappa),\mathcal{I})} \Psi(t)\,dt, \\ \int_0^{M(\kappa,\phi(\kappa),\mathcal{I})} \Psi(t)\,dt \end{array} \right\}$$

Verification:

1. Compute $M(\eta,\kappa,\mathcal{I}) = M(2,2,1) = |2-2| + |2-1| + |1-2| = 2$.

2. Compute $M(\eta,\phi(\kappa),\mathcal{I}) = M(2,1,1) = |2-1| + |1-1| + |1-2| = 2$.

3. Compute $M(\kappa,\phi(\kappa),\mathcal{I}) = M(2,1,1) = 2$.

4. Evaluate the integral:

$$\int_0^2 e^{-t}\,dt = 1 - e^{-2}$$

Thus, the inequality simplifies to:

$$1 - e^{-2} \leq \lambda \max(1 - e^{-2}, 1 - e^{-2})$$

For $0 \leq \lambda < 1$, this inequality holds.

Conclusion:

By the theorem, the mapping ϕ has a unique fixed point in X. In this scenario, the unique fixed point is 0, as:

$$\phi(0) = \frac{0}{2} = 0$$

Therefore, the point $\eta = 0$ is the only fixed point for ϕ.

References:

[1] A. Malkawi, A. Rabaiah, W. Shatanawi and A. Talafhah, (2021), MR-metric spaces and an Application, preprint.

[2] S. Sedghi, D. Turkoglu, N. Shobe and S. Sedghi, *Common fixed point theorems for six weakly compatible mappings in D^*-metric spaces, Thai Journal of Mathematics, Vol.7, No.2, pp. 381-391, (2009).* *https://thaijmath2.in.cmu.ac.th/index.php /thaijmath/article/view/170* (Accessed: Dec.10,2024)

[3] Bakhtin, I.A., *The contraction mapping principle in almost metric spaces.*, Funct. Anal., 1989,30 ,26-37.

[4] Y. J. Cho, P. P. Murthy and G. Jungck, A common fixed point theorem of Meir and Keeler type, Internat. J. Math. Sci. 16(1993), 669-674. http://eudml.org/doc/46946.(Accessed: Dec.10,2024)

[5] Czerwik, S. *Contraction mappings in $b-$metric spaces*, Acta Math. Inform. Univ. Ostra. 1993, 1, 5-11. http://eudml.org/doc/23748. (Accessed: Dec.10,2024)

[6] R.O. Davies and S. Sessa, A common fixed point theorem of Gregus type for compatible mappings, Facta Univ. (Nis) Ser. Math. Inform. 7(1992), 51-58.

[7] Dhage, B. C., Generalized Metric Spaces and Mappings with Fixed Points. Bull. Cal. Math. Soc. 84(1992), 329-336.

[8] Maria Rosaria Formica, Abdelkarim Kari, "New Fixed Point Theorems in Complete Rectangular M-metric Spaces", WSEAS Transactions on Mathematics, vol. 23, pp. 863-873, 2024 10.37394/23206.2024.23.89.

[9] T. Qawasmeh, R. Hatamleh, A. Qazza, M. W. Alomar, R. Saadeh, Further Accurate Numerical Radius Inequalities, Axioms 2023, 12, 801, 2023 https://doi.org/10.3390/axioms12080801.

[10] T. Qawasmeh, A. Bataihah, K. Bataihah, A. Qazza, R. Hatamleh, Nth composite Iterative Scheme via Weak Contractions with Application, International Journal of Mathematics and Mathematical Sciences, vol. (2023) https://doi.org/10.1155/2023/7175260.

[11] T. Qawasmeh, A. Tallafha, W. Shatanawi, Fixed point theorems through modified w-distance and application to nontrivial equations, Axioms, **8** (2019), Article Number 57. https://doi.org/10.3390/axioms8020057.

[12] A. Rabaiah, A. Tallafha and W. Shatanawi, *Common fixed point results for mappings under nonlinear contraction of cyclic form in b-Metric Spaces, Advances in mathematics scientific journal, 2021, 26(2), pp. 289–301.*

[13] B. E. Rhoades, A fixed point theorem for generalized metric spaces, Int. J. Math. Math. Sci. 19(1996), no. 1, 145-153. http://eudml.org/doc/47598.(Accessed: Dec.10,2024)

[14] Branciari A, A fixed point theorem for mappings satisfying a general contractive condition of integral type. International Journal of Mathematics and Mathematical Sciences 2002,29(9):531–536.10.1155/S0161171202007524.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The author contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The author has no conflict of interest to declare that is relevant to the content of this article.