Study and Analysis of Taylor's Series using The Modified Conformable Operator

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Abstract: In this paper, we explore an essential concept in mathematics: Taylor's series. To examine its behavior in fractional calculus, we utilize the modified conformable operator, which provides significant results by satisfying most properties of the classical derivative and establishing a strong connection between the usual and fractional derivatives. We then present theoretical results for the fractional Taylor series using the proposed operator. Finally, we demonstrate the effectiveness of this approach through numerical examples and simulations in Mathematica, which validate the obtained results.

Key-Words: Modified conformable operator, Taylor's series, Polynomials, Series expansions.

Received: October 16, 2024. Revised: February 2, 2025. Accepted: March 5, 2025. Published: April 24, 2025.

1 Introduction

Fractional derivatives have been an area of mathematical interest since the early development of calculus. In 1695, L'Hôpital posed a fundamental question about the interpretation of the derivative $\frac{d^n f}{dt^n}$ when $n = \frac{1}{2}$. Since then, researchers have made continuous efforts to define and develop fractional derivatives. The Riemann-Liouville fractional derivative, [1], Caputo fractional derivative, [1], Atana–Baleanu fractional derivative, [2], and Caputo–Fabrizio fractional derivative, [3], are some of the several formulations that have surfaced. The fact that many fractional derivatives do not completely meet classical features like the product rule and the chain rule presents a significant issue. A recent study, [4], covers this gap by proposing a new local fractional derivative based on the local fractional derivative – the conformable fractional derivative defined as follows:

Definition 1 *[4].*
The conformable fractional derivative of order α for a function $f : [0, \infty) \to \mathbb{R}$ is provided by

$$f^{(\alpha)}(x) = \lim_{\epsilon \to 0} \frac{f(x + \epsilon x^{1-\alpha}) - f(x)}{\epsilon},$$

for all $x > 0$ and $\alpha \in (0, 1)$, provided the limit exists.

We define

$$f^{(\alpha)}(0) = \lim_{x\to 0^+} f^{\alpha}(x),$$

if f is α-differentiable on some interval $(0,a)$ where $a>0$ and the limit $\lim_{x\to 0^+} f^{(\alpha)}(x)$ occurs.

This derivative, which stems from a simple limit definition, still retains many significant features of the traditional integer-order derivative. Numerous works have produced important theoretical and practical results, [5], [6], [7], [8], [9], [10]. for the conformable derivative that has been built and generalized upon. While this work has attracted much attention, in [11], authors explained the conformable derivative using fractional chords in terms of geometry who placed it under the spotlight. Since then, this derivative has been subject to increased interest, leading to the solution of numerous mathematical and physical problems. Additional information and applications are available in [12], [13], [14], [15], [16], [17].

Within this context, the focus of our work is on a modified conformable operator recently proposed in [18], which has been thoroughly examined regarding its enhanced characteristics and potential applications.

One of the most important tools of mathematical analysis is the Taylor series, which allows the representation of a given function as the sum of an infinite number of the functions' derivatives at a specific point. This remarkable technique, which enables balancing and analyzing functions in a wide range of areas, was first suggested in 1715, [19]. The Taylor series is recognized as the Maclaurin series, [20], but was referred to as Mclaurin series because of the 18th century use of this technique when evaluating at zero by Colin Maclaurin.

Our aim in this study is to investigate the behavior of Taylor series through fractional calculus using the modified conformable operator, [18]. This operator is quite helpful as it provides a broad generalization to the fractional case while retaining most of the properties of the classical derivative. In addition, we use this operator to obtain results for fractional Taylor series which we verified through Mathematica computational simulations and numerical.

This paper's remainder is organized as follows: The theoretical underpinnings and mathematical formulation of the modified conformable operator are presented in Section 2. Section 3 discusses the derivation and properties of the fractional Taylor series using this operator. Section 4 presents numerical examples and graphical simulations to illustrate the effectiveness of our approach. Finally, in Section 5, we conclude with key findings and possible future research directions.

2 Fundamentals of the Modified Conformable Operator

In this section, we present key definitions and properties of the modified conformable operator, denoted as D^{α}, of order α, where $0 < \alpha \leq 1$. Notably, this operator reduces to the identity operator for $\alpha = 0$ and to the classical differential operator for $\alpha = 1$.

Definition 2 *(Modified Conformable Differential Operator), [18].*
Let $0<\alpha\leq 1$. A differential operator D^{α} is termed a modified conformable operator if and only if it satisfies the following conditions:

$$D^0 f(x) = f(x),\ D^1 f(x) = \frac{d}{dx} f = f^{'}(x),\ \forall x \in \mathbb{R}. \tag{1}$$

Definition 3 *(A Class of Modified Conformable Derivative), [18].*
Let $0<\alpha\leq 1$, and let the functions $k_0, k_1 : [0,1]\times\mathbb{R}\to[0,\infty)$ be continuous and satisfy the conditions:

$$\begin{cases} \lim\limits_{\alpha\to 0^+} k_1(\alpha,x) = 1, & \lim\limits_{\alpha\to 0^+} k_0(\alpha,x) = 0, \\ \forall x\in\mathbb{R}, & \\ \lim\limits_{\alpha\to 1^-} k_1(\alpha,x) = 0, & \lim\limits_{\alpha\to 1^-} k_0(\alpha,x) = 1, \\ \forall x\in\mathbb{R}, & \\ k_1(\alpha,x)\neq 0,\ \alpha\in[0,1), & k_0(\alpha,x)\neq 0,\ \alpha\in(0,1], \\ \forall x\in\mathbb{R}. & \end{cases} \tag{2}$$

Then the following differential operator D^{α} defined by

$$D^{\alpha} f(x) = k_1(\alpha,x) f(x) + k_0(\alpha,x) f^{'}(x) \tag{3}$$

is termed a modified conformable operator, provided that $f(x)$ is differentiable.

Example 4 *Some classes of the modified conformable operator.*
(1) Let $k_1(\alpha,x) = (1-\alpha)x^{\alpha}$ and $k_0(\alpha,x) = \alpha x^{1-\alpha}$ for any $x\in(0,\infty)$ we find

$$\begin{aligned} D^{\alpha} f(x) &= k_1(\alpha,x) f(x) + k_0(\alpha,x) f^{'}(x) \\ &= (1-\alpha)x^{\alpha} f(x) + \alpha x^{1-\alpha} f^{'}(x). \end{aligned}$$

Based on the obtained operator, we get:

$$D^0 f(x) = f(x) \quad and \quad D^0 f(x) = f^{'}(x)$$

It means D^{α} satisfy condition (1), and one can easily prove that D^{α} satisfy condition (2), then we say that D^{α} is a class of modified conformable derivative.

(2) *Let* $k_1(\alpha,x) = \cos\left(\frac{\alpha\pi}{2}\right)x^{\alpha}$ *and* $k_0(\alpha,x) = \sin\left(\frac{\alpha\pi}{2}\right)x^{1-\alpha}$ *for any* $x \in (0,\infty)$ *we get*

$$\begin{aligned} D^{\alpha}f(x) &= k_1(\alpha,x)f(x) + k_0(\alpha,x)f^{'}(x) \\ &= \cos\left(\frac{\alpha\pi}{2}\right)x^{\alpha}f(x) + \sin\left(\frac{\alpha\pi}{2}\right)x^{1-\alpha}f^{'}(x). \end{aligned}$$

Similarly, the resulting operator satisfy conditions (1) *and* (2)*, then is a class of modified conformable derivative.*

However, an important limitation is that in general: $D^{\alpha}D^{\beta} \neq D^{\beta}D^{\alpha}$.

Definition 5 *(Partial Conformable Derivatives), [18].*
Let $0 < \alpha \leq 1$*, and let the functions* $k_0, k_1 : [0,1]\times\mathbb{R} \to [0,\infty)$ *be continuous and satisfy* (2)*. Given a function* $f(x,s) : \mathbb{R}^2 \to \mathbb{R}$ *such that* $\frac{d}{dx}f(x,s)$ *exists for each fixed* $s \in \mathbb{R}$*, the partial modified conformable differential operator* D_x^{α} *is defined as:*

$$D_x^{\alpha}f(x,s) = k_1(\alpha,x)f(x,s) + k_0(\alpha,x)\frac{\partial}{\partial x}f(x,s) \tag{4}$$

Definition 6 *(Modified Conformable Exponential Function), [18].*
Let $0 < \alpha \leq 1$*,* $s,x \in \mathbb{R}$ *with* $s \leq x$*, and let the functions* $m : [s,x] \to \mathbb{R}$ *be continuous.* $k_0, k_1 : [0,1]\times\mathbb{R} \to [0,\infty)$ *be continuous and satisfy* (2) *with* m/k_0 *and* k_1/k_0 *Riemann integrable on* $[s,x]$*. Then the modified conformable exponential function with respect to* D^{α} *is defined to be*

$$e_m(x,s) = e^{\int_s^x \frac{m(\lambda)-k_1(\alpha,\lambda)}{k_0(\alpha,\lambda)}d\lambda},\ e_0(x,s) = e^{\int_x^s \frac{k_1(\alpha,\lambda)}{k_0(\alpha,\lambda)}d\lambda}. \tag{5}$$

Based on (3) and (5), the following fundamental properties hold:

Lemma 7 *(Basic Derivatives), [18].*
Let the modified conformable differential operator D^{α} *be given as* (3)*, where* $0 < \alpha \leq 1$*. Let the function* $m : [s,x] \to \mathbb{R}$ *be continuous and the functions* $k_0, k_1 : [0,1]\times\mathbb{R} \to [0,\infty)$ *be continuous and satisfy* (2) *with* m/k_0 *and* k_1/k_0 *Riemann integrable on* $[s,x]$*. Assume the functions f and g are differentiable as needed. Then:*
(i) *Linearity:* $D^{\alpha}[af+bg] = aD^{\alpha}[f] + bD^{\alpha}[g]$*, for all* $a,b \in \mathbb{R}$*.*
(ii) $D^{\alpha}c = ck_1(\alpha,x)$*, for all constants* $c \in \mathbb{R},\ x \in \mathbb{R}$*.*
(iii) *Product Rule:* $D^{\alpha}[fg] = fD^{\alpha}[g] + gD^{\alpha}[f] - fgk_1(\alpha,x)$*, for all* $x \in \mathbb{R}$*.*
(iv) *Quotient Rule:* $D^{\alpha}\left[\frac{f}{g}\right] = \frac{gD^{\alpha}[f]-fD^{\alpha}[g]}{g^2} + \frac{f}{g}k_1(\alpha,x)$*, for all* $x \in \mathbb{R}$ *and* $g \neq 0$*.*
(v) *For* $\alpha \in (0,1]$ *and fixed* $s \in \mathbb{R}$*, the exponential function satisfies*

$$D_x^{\alpha}[e_m(x,s)] = m(x)e_m(x,s). \tag{6}$$

(vi) *For* $\alpha \in (0,1]$ *and for the exponential function* e_0 *given in* (5)*, we have*

$$D^{\alpha}\left[\int_a^x \frac{f(s)e_0(x,s)}{k_0(\alpha,s)}ds\right] = f(x). \tag{7}$$

Proof 8 *Back to [18].*

Definition 9 *(Modified Conformable Integral), [18].*
Let $0 < \alpha \leq 1$ *and* $x_0 \in \mathbb{R}$*. In light of* (5) *and Lemma* 7 (v) *and* (vi)*, define the antiderivative via*

$$\int D^{\alpha}f(x)d_{\alpha}x = f(x) + ce_0(x,x_0), \quad c \in \mathbb{R}.$$

In the same way define the integral of f over the closed interval $[a,b]$ *as follows :*

$$\begin{aligned} \int_a^b f(s)e_0(x,s)d_{\alpha}s &= \int_a^b \frac{f(s)e_0(x,s)}{k_0(\alpha,s)}ds, \\ d_{\alpha}s = \frac{ds}{k_0(\alpha,s)}. \end{aligned} \tag{8}$$

Therefore, we can write:

$$e_0(x,s) = e^{\int_x^s \frac{k_1(\alpha,\lambda)}{k_0(\alpha,\lambda)}d\lambda} = e^{\int_x^s k_1(\alpha,\lambda)d_{\alpha}\lambda}.$$

The next lemma outlines some fundamental properties of the modified conformable integral.

Lemma 10 *(Basic Integrals), [18].*
Let the conformable differential operator D^{α} *be given as in* (3) *and the integral be given as* (8) *with* $0 < \alpha \leq 1$*. Let the functions* $k_0, k_1 : [0,1]\times\mathbb{R} \to [0,\infty)$ *be continuous and satisfy* (2) *and let f and g be differentiable as needed. Then:*
(i) The derivative of the definite integral of f is given by

$$D^{\alpha}\left[\int_a^x f(s)e_0(x,s)d_{\alpha}s\right] = f(x).$$

(ii) The definite integral of the derivative of f is given by

$$\begin{aligned} \int_a^x D^{\alpha}\left[f(s)e_0(x,s)d_{\alpha}s\right] &= f(s)e_0(x,s)\mid_{s=a}^{x} \\ &= f(x) - f(a)e_0(x,a). \end{aligned}$$

(iii) An integration by parts formula is given as follow

$$\int_a^b f(x)D^{\alpha}[g(x)]e_0(b,x)d_{\alpha}x = f(x)g(x)e_0(b,x)\mid_{x=a}^{b}$$

$$-\int_a^b g(x)(D^\alpha[f(x)] - k_1(\alpha,x)f(x))e_0(b,x)d_\alpha x.$$

(iv) A version of the Leibniz rule for the differentiation of an integral is given by

$$D^\alpha\left[\int_a^x f(x,s)e_0(x,s)d_\alpha s\right] =$$

$$\int_a^x \left(D_x^\alpha[f(x,s)] - k_1(\alpha,x)f(x,s)\right)e_0(x,s)d_\alpha s + f(x,x).$$

If $e_0(x,s)$ is absent then by (4) *we have*

$$D^\alpha\left[\int_a^x f(x,s)d_\alpha s\right] = \int_a^x D_x^\alpha f(x,s)d_\alpha s + f(x,x).$$

Proof 11 *Founded in [18].*

3 Theoretical Results

In this section, we introduce functions that serve a role similar to polynomials in the Taylor series expansion. Our analysis is based on the modified conformable operator, which enables a comparative study between the standard derivative (when $\alpha = 1$) and its fractional counterpart.

Definition 12
Let the functions $\kappa_0, \kappa_1 : [0,1]\times\mathbb{R} \to [0,\infty)$ be continuous and satisfy the conditions given in (2) *are hold. When $\alpha = 1$ and $n \in \mathbb{N}_0$, the polynomials are given by:*

$$b_n(t,s) = \frac{1}{n!}(t-s)^n.$$

To generalize this concept in our framework, we define the functions $b_n : \mathbb{R}^2\times\mathbb{R} \to \mathbb{R}, n \in \mathbb{N}_0$, as follows:

$$b_0(t,s) = 1, \quad \forall t,s \in \mathbb{R} \tag{9}$$

and

$$b_n(t,s) = \int_s^t b_{n-1}(\lambda,s)d_\alpha\lambda, \quad n\in\mathbb{N}, \quad \forall t,s\in\mathbb{R}. \tag{10}$$

Lemma 13 *(Key Lemma)*
The following key relationship holds:

$$D_t^\alpha b_n(t,s) = b_{n-1}(t,s) + \kappa_1(\alpha,t)b_n(t,s) \tag{11}$$

Proof 14 *This result follows directly from Lemma* 7 (ii) *and Lemma* 10 (iv).

To establish and prove a Taylor-type expansion using the modified conformable derivative, we first introduce the following auxiliary result.

Lemma 15 *(Auxiliary Lemma)*
Let $n \in \mathbb{N}$, and assume that f is n-times differentiable. Suppose that the functions $m_k, 0 \le$ k $\le$ n-1, are differentiable at some $t\in\mathbb{R}$ and satisfy the recurrence relation:

$$D^\alpha m_{k+1}(t) = m_k(t) + \kappa_1(\alpha,t)m_{k+1}(t), \quad \forall\, 0\le k\le n-2. \tag{12}$$

Then, the following identity holds:

$$D^\alpha\left[\sum_{k=0}^{n-1}(-1)^k m_k\left(D^\alpha\right)^k f(t)\right] =$$

$$(-1)^{n-1}m_{k-1}\left(D^\alpha\right)^n f(t) + \left(D^\alpha m_0 - \kappa_1(\alpha,t)m_0\right)f(t).$$

Proof 16 *By applying Lemma 7 (i) and (iii) along* *with equation* (4)*, we obtain:*

$$
\begin{aligned}
&D^{\alpha}\left[\sum_{k=0}^{n-1}(-1)^{k}m_{k}\left(D^{\alpha}\right)^{k}f(t)\right] \\
&=\sum_{k=0}^{n-1}(-1)^{k}D^{\alpha}\left[m_{k}\left(D^{\alpha}\right)^{k}f(t)\right] \\
&=\sum_{k=0}^{n-1}(-1)^{k}\Big(m_{k}\left(D^{\alpha}\right)^{k+1}f(t) \\
&\quad+\left(D^{\alpha}m_{k}-\kappa_{1}(\alpha,t)m_{k}\right)\left(D^{\alpha}\right)^{k}f(t)\Big) \\
&=\sum_{k=0}^{n-1}(-1)^{k}m_{k}\left(D^{\alpha}\right)^{k+1}f(t) \\
&\quad+\sum_{k=0}^{n-1}(-1)^{k}\left(D^{\alpha}m_{k}-\kappa_{1}(\alpha,t)m_{k}\right)\left(D^{\alpha}\right)^{k}f(t) \\
&=\sum_{k=0}^{n-2}(-1)^{k}\Big(m_{k}\left(D^{\alpha}\right)^{k+1}f(t) \\
&+(-1)^{n-1}m_{n-1}\left(D^{\alpha}\right)^{n}f(t) \\
&\quad+\sum_{k=1}^{n-1}(-1)^{k}\left(D^{\alpha}m_{k}-\kappa_{1}(\alpha,t)m_{k}\right)\left(D^{\alpha}\right)^{k}f(t)\Big) \\
&+\left(D^{\alpha}m_{0}-\kappa_{1}(\alpha,t)m_{0}\right)f(t) \\
&=\sum_{k=0}^{n-2}(-1)^{k}\Big(m_{k}\left(D^{\alpha}\right)^{k+1}f(t) \\
&+(-1)^{n-1}m_{n-1}\left(D^{\alpha}\right)^{n}f(t) \\
&-\sum_{k=0}^{n-2}(-1)^{k}\left(m_{k}+\kappa_{1}(\alpha,t)m_{k+1}-\kappa_{1}(\alpha,t)m_{k+1}\right) \\
&\quad\times\left(D^{\alpha}\right)^{k+1}f(t)\Big)+\left(D^{\alpha}m_{0}-\kappa_{1}(\alpha,t)m_{0}\right)f(t) \\
&=\sum_{k=0}^{n-2}(-1)^{k}m_{k}\left(D^{\alpha}\right)^{k+1}f(t)+(-1)^{n-1}m_{n-1}\left(D^{\alpha}\right)^{n}f(t) \\
&\quad-\sum_{k=0}^{n-2}(-1)^{k}m_{k}\left(D^{\alpha}\right)^{k+1}f(t) \\
&+\left(D^{\alpha}m_{0}-\kappa_{1}(\alpha,t)m_{0}\right)f(t) \\
&=(-1)^{n-1}m_{n-1}\left(D^{\alpha}\right)^{n}f(t)+\left(D^{\alpha}m_{0}-\kappa_{1}(\alpha,t)m_{0}\right)f(t)
\end{aligned}
$$

holds at t.
The desired result follows by applying the differentiation rule and rearranging terms, thereby completing the proof of Lemma 15.

The following theorem, representing a key component of the main result, establishes Taylor's formula for the modified conformable derivative.

Theorem 17 *(Taylor's Formula for the Modified Conformable Derivative)*
Let $n \in \mathbb{N}$*, and suppose that* f *is* n*-times differentiable on* $[t_0, \infty)$*. For* $t, s \in [t_0, \infty)$*, define the functions* b_k *recursively as:*

$$b_0(t,s) = 1 \quad \forall t, s \in \mathbb{R},$$

and

$$b_{k+1}(t,s) = \int_s^t b_k(\lambda, s) d_\alpha \lambda, \quad \text{for } k \in \mathbb{N}_0.$$

Then, the following Taylor-type expansion holds:

$$
\begin{aligned}
f(t) &= e_0(t,s)\sum_{k=0}^{n-1}(-1)^k b_k(s,t)\left(D^\alpha\right)^k f(s) \\
&\quad+\int_s^t b_{n-1}(\lambda,t)\left(D^\alpha\right)^n f(\lambda)e_0(t,\lambda)d_\alpha\lambda,
\end{aligned}
$$

for all $t \in [t_0, \infty)$*.*

Proof 18 *By utilizing equation* (3) *and Lemma* 15*, we obtain*

$$D^{\alpha}\left[\sum_{k=0}^{n-1}(-1)^{k}b_{k}(\lambda,t)\left(D^{\alpha}\right)^{k}f(\lambda)\right] =$$

$$(-1)^{n-1}b_{k-1}(\lambda,t)\left(D^{\alpha}\right)^{n}f(\lambda), \quad \forall\lambda \in [t_0, \infty).$$

Integrating both sides from s *to* t *and using Lemma* 7 (ii)*, we obtain:*

$$
\begin{aligned}
&\int_s^t D^{\alpha}\left[\sum_{k=0}^{n-1}(-1)^{k}b_{k}(\lambda,t)\left(D^{\alpha}\right)^{k}f(\lambda)\right]e_0(t,\lambda)d_\alpha\lambda \\
&=\int_s^t(-1)^{n-1}b_{k-1}(\lambda,t)\left(D^{\alpha}\right)^{n}f(\lambda)e_0(t,\lambda)d_\alpha\lambda \\
&\left.\sum_{k=0}^{n-1}(-1)^{k}b_{k}(\lambda,t)\left(D^{\alpha}\right)^{k}f(\lambda)e_0(t,\lambda)\right|_{\lambda=s}^{t} \\
&=\int_s^t(-1)^{n-1}b_{k-1}(\lambda,t)\left(D^{\alpha}\right)^{n}f(\lambda)e_0(t,\lambda)d_\alpha\lambda \\
&f(t)-\sum_{k=0}^{n-1}(-1)^{k}b_{k}(s,t)\left(D^{\alpha}\right)^{k}f(s)e_0(t,s) \\
&=\int_s^t(-1)^{n-1}b_{k-1}(\lambda,t)\left(D^{\alpha}\right)^{n}f(\lambda)e_0(t,\lambda)d_\alpha\lambda.
\end{aligned}
$$

4 Numerical Methods

This section presents an application of the obtained results, focusing on two aspects: the analytical solution and the computational analysis. To clarify our method and approach, we utilize specific classes of the modified conformable operator, which depend on the functions κ_0 and κ_1.

4.1 Analytical Solution

In this part, we derive the analytical solution for selected numerical examples to compute the function $b_n(t,s)$ and the modified conformable exponential function $e_m(t,s)$.

Example 19 *For $\alpha \in (0,1]$ and $u_0 \in (0,\infty)$, let κ_1 satisfy* (2)*. Now, consider the function:*

$$\kappa_0(\alpha,t) = \alpha\,(u_0 t)^{1-\alpha}, \quad t \in [0,\infty).$$

Using (8)*, we obtain:*

$$d_\alpha\lambda = \frac{1}{\kappa_0(\alpha,\lambda)}d\lambda = \frac{\lambda^{\alpha-1}}{\alpha u_0^{1-\alpha}}d\lambda.$$

Starting with the initial condition $b_0(t,s)=1$, we compute $b_1(t,s)$ as follows:

$$b_1(t,s) = \int_s^t b_0(\lambda,s)d_\alpha\lambda = \int_s^t \frac{\lambda^{\alpha-1}}{\alpha u_0^{1-\alpha}}d\lambda.$$

Evaluating the integral gives:

$$b_1(t,s) = \frac{t^\alpha - s^\alpha}{\alpha^2 u_0^{1-\alpha}}.$$

Next, for $b_2(t,s)$, we have:

$$b_2(t,s) = \int_s^t b_1(\lambda,s)d_\alpha\lambda = \int_s^t \frac{\lambda^\alpha - s^\alpha}{\alpha^2 u_0^{1-\alpha}}d_\alpha\lambda.$$

Solving the integral yields:

$$b_2(t,,s) = \frac{1}{2!}\left(\frac{t^\alpha - s^\alpha}{\alpha^2 u_0^{1-\alpha}}\right)^2.$$

By continuing this process, we derive the general expression:

$$b_n(t,s) = \frac{1}{\text{n!}}\left(\frac{t^\alpha - s^\alpha}{\alpha^2 u_0^{1-\alpha}}\right)^n.$$

Notably, when $\alpha = 1$, this formula simplifies to:

$$b_n(t,s) = \frac{(t-s)^n}{\text{n!}},$$

which aligns with the classical case, as expected.

Example 20 *Consider the functions*
$k_0(\alpha,t) = t^{1-\alpha}\cos^2\left(\frac{\pi}{2}(1-\alpha)t^\alpha\right)$,
$k_1(\alpha,t) = \sin^2\left(\frac{\pi}{2}(1-\alpha)t^\alpha\right)$.
It is straightforward to verify that these functions satisfy equation (2)*. Since $b_0(t,s)=1$, we obtain*

$$b_1(t,s) = \int_s^t b_0(\lambda,s)d_\alpha\lambda = \int_s^t \frac{\lambda^{\alpha-1}}{\cos^2\left(\frac{\pi}{2}(1-\alpha)\lambda^\alpha\right)}d\lambda.$$

Using the substitution $u=\lambda^\alpha$, with $du = \alpha\lambda^{\alpha-1}d\lambda$, we rewrite the integral as

$$b_1(t,s) = \frac{1}{\alpha}\int_{s^\alpha}^{t^\alpha} \frac{1}{\cos^2\left(\frac{\pi}{2}(1-\alpha)u\right)}du.$$

Evaluating the integral, we obtain

$$b_1(t,s) = \frac{\tan\left(\frac{\pi}{2}(1-\alpha)t^\alpha\right) - \tan\left(\frac{\pi}{2}(1-\alpha)s^\alpha\right)}{\alpha(1-\alpha)\frac{\pi}{2}}.$$

To examine the limiting behavior as $\alpha \to 1$, we define $\chi = (1-\alpha)\frac{\pi}{2}$ and take the limit $\chi \to 0^+$ (which corresponds to $\alpha \to 1$). Using the Taylor series expansion, we obtain

$$b_1(t,s) = \frac{\tan(\chi t^\alpha) - \tan(\chi s^\alpha)}{\alpha\chi} = \frac{\chi t^\alpha - \chi s^\alpha}{\alpha\chi} + O(\chi^2).$$

Thus, in the limit,

$$\lim_{\chi\to 0^+\ \alpha\to 1} b_1(t,s) = t - s.$$

For $b_2(t,s)$, we compute

$$\begin{aligned} b_2(t,s) &= \int_s^t b_1(\lambda,s)d_\alpha\lambda \\ &= \frac{1}{\alpha\chi}\int_s^t \frac{\tan(\chi\lambda^\alpha) - \tan(\chi s^\alpha)\,\lambda^{\alpha-1}}{\cos^2\chi\lambda^\alpha}d\lambda. \end{aligned}$$

Using the same substitution $u=\lambda^\alpha$, we obtain

$$b_2(t,s) = \frac{1}{2\alpha^2\chi}\int_{s^\alpha}^{t^\alpha} \frac{\tan(\chi u) - \tan(\chi s^\alpha)}{\cos^2\chi u}du.$$

Evaluating the integral, we get

$$\begin{aligned} b_2(t,s) &= \frac{1}{2\alpha^2\chi^2}\left(\sec^2\chi t^\alpha - \sec^2\chi s^\alpha\right. \\ &- \left. 2\tan\chi t^\alpha \tan\chi s^\alpha + 2(\tan\chi s^\alpha)^2\right) \end{aligned}$$

Taking the limit as $\chi\to 0^+, \alpha\to 1$, we obtain

$$\lim_{\chi\to 0^+\ \alpha\to 1} b_2(t,s) = \frac{1}{2}(t-s)^2.$$

Similarly, we can analyze the modified conformable exponential function,

$$e_m(t,s) = e^{\int_s^t \frac{m-k_1(\alpha,\lambda)}{k_0(\alpha,\lambda)} d\lambda}$$

The integral evaluates to

$$\int_s^t \frac{m-k_1(\alpha,\lambda)}{k_0(\alpha,\lambda)} d\lambda = \frac{(m-1)\tan\left(\frac{\pi}{2}(1-\alpha)t^\alpha\right)}{\alpha(1-\alpha)\frac{\pi}{2}} + \frac{t^\alpha}{\alpha} + C.$$

where

$$C = -\frac{(m-1)\tan\left(\frac{\pi}{2}(1-\alpha)s^\alpha\right)}{\alpha(1-\alpha)\frac{\pi}{2}} - \frac{s^\alpha}{\alpha}$$

Thus,

$$e_m(t,s) = Ae^{\frac{(m-1)\tan\left(\frac{\pi}{2}(1-\alpha)t^\alpha\right)}{\alpha(1-\alpha)\frac{\pi}{2}} + \frac{t^\alpha}{\alpha}}$$

where $A = e^C$.
For the special case $m = 1$, *we obtain*

$$e_1(t,s) = Ae^{\frac{t^\alpha}{\alpha}}.$$

Taking the limit as $\chi \to 0^+, \alpha \to 1$, *we recover the standard exponential function,*

$$\lim_{\chi\to 0^+ \ \alpha\to 1} e_m(t,s) = Ae^{mt}.$$

This concludes the example.

4.2 Computational Analysis

To analyze the behavior of the modified conformable Taylor series, we utilize Mathematica to examine the impact of the operator classes, which depend on the functions $(\kappa_0, \ \kappa_1)$, the fractional order α, the standard case $\alpha = 1$ and the coefficient n. The following example illustrates this study.

Example 21 *For* $\alpha \in (0,1]$, *let* κ_1 *satisfy* (2), *and define*

$$\kappa_0(\alpha,t) = \alpha u_0^{\alpha-1} \quad , \quad u_0 \in (0,\infty).$$

Applying (8), *we obtain*

$$d_\alpha\lambda = \frac{1}{\kappa_0(\alpha,\lambda)} d\lambda = \frac{1}{\alpha u_0^{\alpha-1}} d\lambda.$$

Since $b_0(t,s) = 1$, *we compute*

$$\begin{aligned} b_1(t,s) &= \int_s^t b_0(\lambda,s) d_\alpha\lambda \\ &= \int_s^t \frac{1}{\alpha u_0^{\alpha-1}} d\lambda \\ &= \frac{t-s}{\alpha u_0^{\alpha-1}}. \end{aligned}$$

For the next term, we get

$$\begin{aligned} b_2(t,s) &= \int_s^t b_1(\lambda,s) d_\alpha\lambda \\ &= \int_s^t \frac{\lambda-s}{\left(\alpha u_0^{\alpha-1}\right)^2} d\lambda \\ &= \frac{(t-s)^2}{2!\left(\alpha u_0^{\alpha-1}\right)^2}. \end{aligned}$$

Extending this process, we derive the general formula:

$$b_n(t,s) = \frac{(t-s)^n}{\text{n!}\left(\alpha u_0^{\alpha-1}\right)^n}.$$

As expected, in the special case $\alpha = 1$ *we recover the standard result*

$$b_n(t,s) = \frac{(t-s)^n}{\text{n!}}.$$

The following figures illustrate the comparative study, showcasing the behavior of the modified conformable Taylor series in contrast to the classical and fractional case. These visual representations highlight the impact of different fractional orders α, *the choice of operator functions* κ_0 *and* κ_1, *and the variation in coefficients* n.

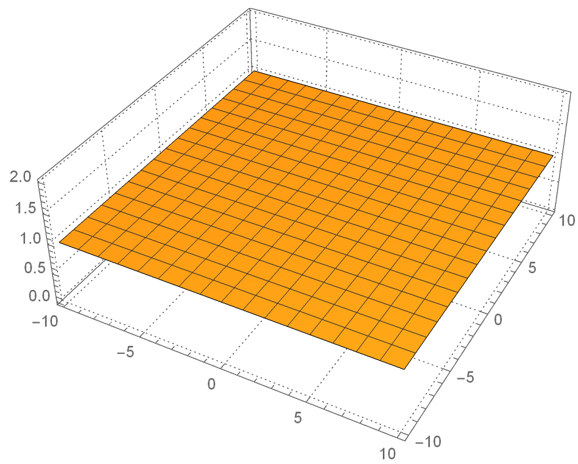

Figure 1: The function $b_0(t,s)$ in all cases.

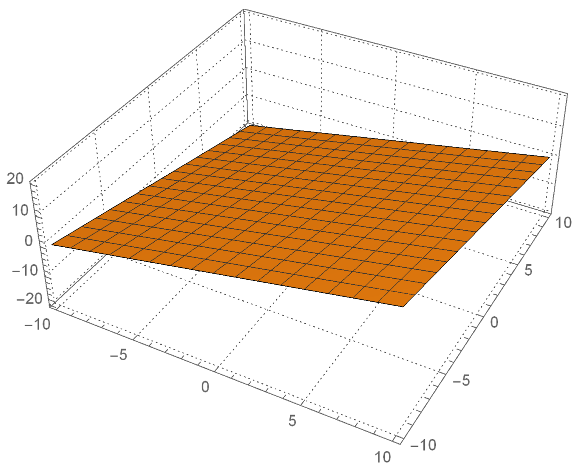

Figure 2: The function $b_n(t,s)$ with $n = 1$, $u = 3$, $\alpha = \frac{1}{2}$.

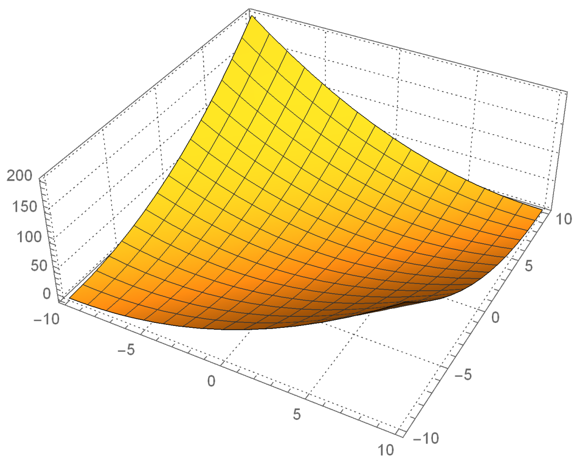

Figure 3: The function $b_n(t,s)$ with $n = 2$, $u = 3$, $\alpha = 1$.

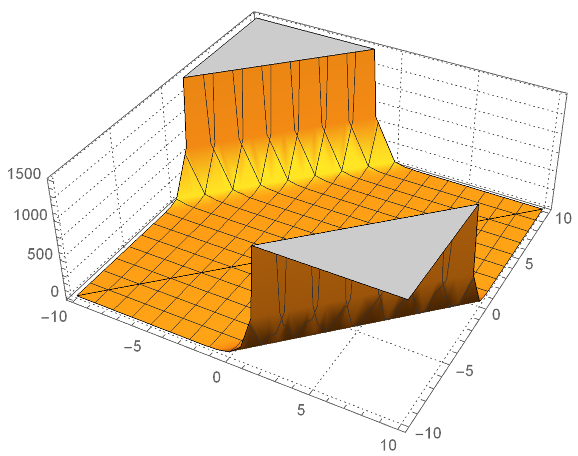

Figure 4: The function $b_n(t,s)$ with $n = 20$, $u = 3$, $\alpha = 1$.

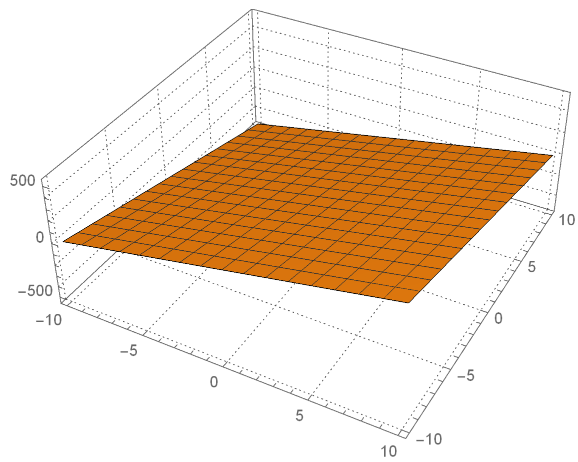

Figure 5: The function $b_n(t,s)$ with $n = 1$, $u = 3$, $\alpha = \frac{1}{10}$.

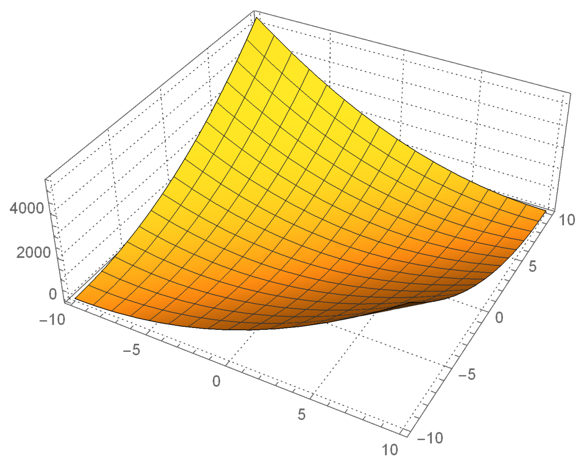

Figure 6: The function $b_n(t,s)$ with $n = 2$, $u = 3$, $\alpha = \frac{1}{10}$.

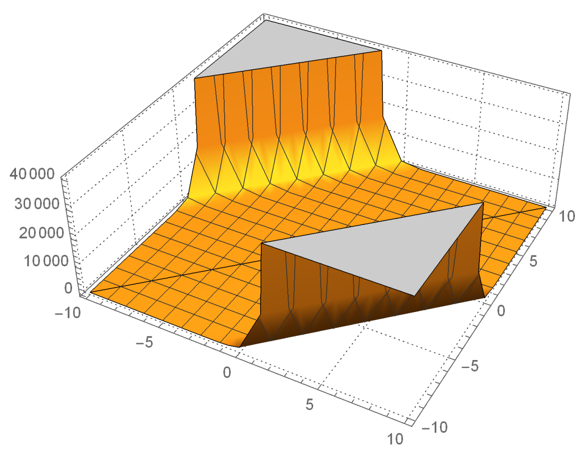

Figure 7: The function $b_n(t,s)$ with $n = 20$, $u = 3$, $\alpha = \frac{1}{10}$.

For $\alpha \in (0,1]$, *assume that* κ_1 *satisfy* (2), *and* *define*

$$\kappa_0(\alpha, t) = \sin\left(\frac{\alpha\pi}{2}\right) t^{1-\alpha}.$$

Under this assumption, we obtain the following results:

$$b_0(t, s) = 1,$$

$$b_1(t, s) = \frac{1}{\sin\left(\frac{\alpha\pi}{2}\right)} \frac{(t^\alpha - s^\alpha)}{\alpha}.$$

$$b_2(t, s) = \left(\frac{1}{\sin^2\left(\frac{\alpha\pi}{2}\right)}\right)\left(\frac{(t^\alpha - s^\alpha)^2}{2!\alpha^2}\right).$$

$$b_n(t, s) = \left(\frac{1}{\sin^n\left(\frac{\alpha\pi}{2}\right)}\right)\left(\frac{(t^\alpha - s^\alpha)^n}{n!\alpha^n}\right).$$

Discussion:

Based on the computational analysis of the previous example and considering that the functions $b_n(t, s)$ form part of the Taylor series, we conclude the following:

- *Figure 1 represent the function $b_0(t, s)$ which take the same behaviour under all conditions and in all cases.*
- *Different classes of the modified conformable differential operator yield the same results in the standard case ($\alpha = 1$) but produce distinct results in fractional cases ($\alpha \neq 1$).*
- *The choice of the modified conformable differential operator significantly influences the Taylor series expansion.*
- *Figure 2, Figure 3, and Figure 4 demonstrate that the coefficient n impacts the Taylor series.*
- *Figure 3, Figure 4, Figure 5, Figure 6 and Figure 7 indicate that the fractional order α affects the Taylor series behavior.*
- *A smaller n provides a more stable approximation, whereas a larger n leads to greater variations.*
- *The fractional order α has less effect on the function $b_n(t, s)$ than the number of terms (n).*

5 Conclusion and Future Work

5.1 Conclusion

This research represents the first attempt where traditional results are advanced toward a more complex framework based on fractions by implementing a Taylor-type expansion with the conformable derivative. Our investigation also produced notable theoretical contributions with a Taylor-series representation under the modified conformable operator and accompanying essential lemmas which were incorporated into the expanded framework. Moreover, I provided numerical examples that together with $\alpha = 1$ demonstrate the formulations consonant with traditional scenarios confirm the value of this result. The derived formulas illustrate the impact of the modified conformable operator on functions resembling polynomials, thereby enabling more flexible approaches to studying fractional calculus.

5.2 Future Work

The application of the modified conformable Taylor series to differential equations and dynamical systems is reserved for later studies focusing on modeling anomalous diffusion processes and memory-dependent processes. Moreover, developing efficient numerical methods for approximating solutions within this framework will be the primary focus of our future investigations. These results will also be applied in greater dimensions, and the stability and convergence properties of the proposed extension will be studied. Another very interesting problem which is useful to study the place of modified conformable operator in fractional calculus is: how to relate this operator with other fractional derivatives.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

The authors used Grammarly for the editorial corrections and maintained the wording of the text while preparing this document. The authors are solely accountable for the content of the publication after this service has been employed, followed by a review and edit process.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflicts of Interest

Regarding the subject matter of this paper, the authors have no conflicts of interest to disclose.