Computational and variational formulations of unilateral problems for structures made of composite materials (laminates, functionally graded materials)

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Abstract: In the present paper buckling problems of constructions with a single delamination are conducted. Structures are made of laminates and functionally graded materials (FGM). The first part of the work is devoted to the formulation of contact problems with the aid of various functional inequalities. Then computational models are discussed. Finally two particular problems dealing with buckling of spherical shells and compressed rectangular plates. The results demonstrate that the unsymmetric configurations of FGM structures leads to the reduction of buckling loads for structures with delamination.

Keywords: Contact (unilateral) problems, functionally graded materials, variational inequalities, buckling, delaminations.

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Introduction

MULTI-layered composite structures (laminates and those made of functionally graded materials or nanostructures) may be subject to various forms of local (matrix, fiber cracking, fiber separation or delamination) or global (buckling or free vibration) failure. In the case of the simultaneous occurrence of global and local forms of damage, the mathematical and numerical description of the deformation and final failure of the analyzed structures is drastically changed and becomes difficult – see the experimental results shown in Fig. 1.

The efficient modeling of unilaterlal (contact) problems is still a challenge in non-linear implicit structural analysis. The broader discussion and studies of possible contact problems in mechanics is presented in Ref. [1], [2]. In this area the variety of problems can be formulated and solved:

- three dimensional (3D) static and dynamic analysis [3], [4]
- two dimensional (2D) static and dynamic problems connected with the analysis of beam/plated or shell problems [5]

However, it should be pointed out that the correct and accurate solution of the above problems requires a different than classical approach due to existence unilateral boundary conditions. The mathematical formulation of such problems is carried out with the use of the variational inequalities – see Panagiotopoulos [6], Muc [7].

The importance and complexity of numerical approach is underlined in different papers [8], [9], [10], [11], [12], [13] where various numerical methods have been studied characterizng the application of dual methods, nonlinear programming methods, asymptotic methods, the Ritz method.

In the literature the current investigations are carried out for different material properties of structures, i. e. isotropic, laminates [14], [15], [16], [17], [18], [19]. It should be mentioned that Lazarev, Kovtunenko [20] considered the 2D Signorini-Fichera problem for composites bodies with a rigid inclusion..

In 1981 Chai et al. [21] considered the problem of single delamination with local buckling. The one dimensional problem was solved in an analytical way with the use of the Rayleigh-Ritz method. Whitcombe [22] implemented FEM to the solution of two dimensional plate problems. In the area local buckling with delaminations a broder review of literature is presented by Smitses [23] and Muc et al [24], [25].

Fig.1 Buckling of plates subjected to transverse shear loads

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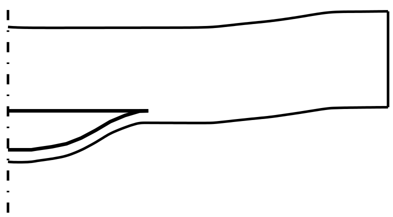
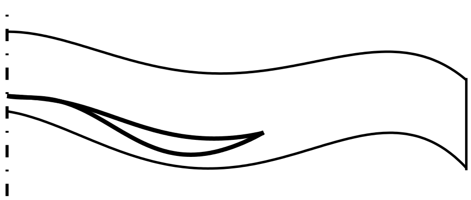
2. A Brief description of contact problems associated with local buckling

The description of contact problems for composite structures can be divided into two classes:

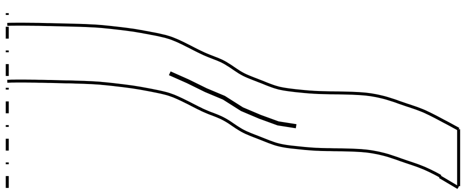
- One dimensional contact problems - the delamination is represented by a single line that separates (or not) the sublaminates – Fig. 2
- Two dimensional contact problems – the delamination is described by a surface – Fig.3.



a) $\rho = 4.5$



b) $\rho = 6$



c) $\rho = 10$

Fig.2 Buckling modes of axisymmeteic spherical shell with a single delamination being the function of the shallowness parameter ρ defined in the next sections

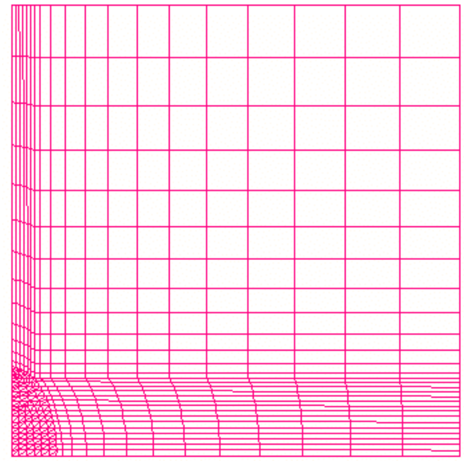


Fig.3 Two dimensional elliptical delamination (top view)

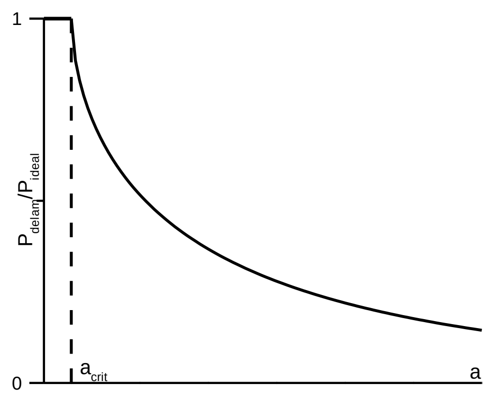


Fig. 4 The effects of the delamination length a on the buckling pressures p.

Fig. 4 shows two characteristic cases: when the delamination length a $<a_{crit}$ is not the critical length and does not cause local sublaminate stability loss, and the second, when the sublaminate stability loss above the critical length occurs. Therefore, it is obvious that the construction of models characterizing the local loss of stability by the sublaminate should allow for the determination of both the critical length of a_{crit} delamination as a function of geometrical and material parameters, as well as the assessment (after local buckling) of the degree of decrease in the value of the critical load as a function of length of the gap a (or its surface area A for two-dimensional delamination) - see Fig. 3.

3. Variational formulations of contact problems with local buckling

The composite structure represents 3D body (laminates or functionally graded material - FGM) that is a space occupied by layered structures $\sum_{k=1}^{N} V_k$ where V_k is a space occupied by the individual k-th layer.description of the layered structures.

The functional characterizing composite structures J can be divided into three parts:

$$J = J_{en} + J_{bbc} + J_{ubc} \tag{1}$$

The first part J_{en} corresponds to the strain (internal) energy. The classical approaches are presented below:

-the Hu-Washizu

$$\begin{aligned} J_{en}(\sigma, e, u) = \sum_{k=1}^{N} \Bigg[\int_{V^{(k)}} \Bigg(\frac{1}{2}\left[e^{(k)}\right]^{Tr}\left[Q^{(k)}\right]\left[e^{(k)}\right] - \left[\sigma^{(k)}\right]^{Tr}\left[e^{(k)}\right] \\ + \left[\sigma^{(k)}\right]^{Tr}\left[e^{(k)}\right]^{Tr}\left[\left[e^{(k)}\right]\left(u_{\alpha}^{(k)}\right)\right] \Bigg) \Bigg] dV \end{aligned} \tag{2}$$

-the Hellinger-Reissner

$$\begin{aligned} J_{en}(\sigma, e) = \sum_{k=1}^{N} \Bigg[\int_{V^{(k)}} \Bigg(-\frac{1}{2}\left[\sigma^{(k)}\right]^{Tr}\left(\left[Q^{(k)}\right]^{-1}\right)^{Tr}\left[\sigma^{(k)}\right] \\ \left[\sigma^{(k)}\right]^{Tr}\left[\left[e^{(k)}\right]\left(u_{\alpha}^{(k)}\right)\right] \Bigg) \Bigg] dV \end{aligned} \tag{3}$$

-the Lagrange

$$J_{en}(u) = \sum_{k=1}^{N} \Bigg[\int_{V^{(k)}} \Bigg(\frac{1}{2}\left[\left[e^{(k)}\right]\left(u_{\alpha}^{(k)}\right)\right]^{Tr}\left[Q^{(k)}\right]\left[e^{(k)}\left(u_{\alpha}^{(k)}\right)\right] \Bigg) \Bigg] dV \tag{4}$$

where σ is the stress tensor, e the strain tensor and u denotes the components of the displacements. The relation between stresses and strains is described by the linear elastic relation. For laminated composite the traditional relation takes the classical form presented e.g. by Jones [26]. For the FGMs the physical relation takes the following form: Considering the thickness stretching the stiffness matrix components Q_{ij} are 3D relations given by:

$$[Q(z)] = \frac{E(z)}{(1+\nu)} \begin{bmatrix} \frac{1-\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-2\nu} & \frac{1-\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & \frac{1-\nu}{1-2\nu} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \end{bmatrix} \tag{5}$$

The elastic modulus E variation characterizes the distribution of porosity along the thickness direction z and is defined in the following way:

$$E(z)/E_b = [(E_t/E_b - 1)f(z) + 1] \quad f(z) = \left(\frac{z}{h} + \frac{1}{2}\right)^n \tag{6}$$

where the symbols *t* and *b* refer to the material properties on top and bottom surfaces, *n* is power index. ν (const) is the Poissons ratio.

The second term in the functional (1) corresponds to the bilateral boundary conditions formulated in the equality form:

$$\sigma_{(\alpha\beta)}^{(k)} n_{\beta}^{(k)} = s_{\alpha}, u_{j\alpha}^{(k)} = W_{j\alpha} \text{ on } S, k = 1 \text{ or } \mathrm{k} = \mathrm{N} \tag{7}$$

They determine values of the displacement field W or the distributed external loads s.

The third component in the functional J (1) represents the unilateral boundary conditions, i.e. the kinematic boundary conditions between layers :

$$u_{\alpha}^{(k)}(x, y, z_l) = u_{\alpha}^{(k-1)}(x, y, z_l) \tag{8}$$

and transverse shear and normal stress continuity conditions at the contact interfaces:

$$\sigma_{\alpha 3}^{(k)}(x, y, z_l) = \sigma_{\alpha 3}^{(k-1)}(x, y, z_l), \ \alpha = 1,2,3 \tag{9}$$

They takes the following form:

-the Hu-Washizu

$$J_{ubc} = -\sum_{l=1}^{N-1}\left(\int_{S^{(l,l+1)}} \left[u^{(l)} - u^{(l+1)}\right]\left[\sigma_3^{(l,l+1)}\right] dS \right) \tag{10}$$

-the Hellinger-Reissner

$$J_{ubc} = \sum_{l=1}^{N-1}\left(\int_{S^{(l,l+1)}} \left[u^{(l)} - u^{(l+1)}\right]\left[\sigma_3^{(l,l+1)}\right] dS \right) \tag{11}$$

-the Lagrange

$$J_{ubc} = -\sum_{l=1}^{N-1}\left(\int_{S^{(l,l+1)}} \left[u^{(l)} - u^{(l+1)}\right][\lambda] dS \right) \tag{12}$$

where λ denotes the Lagrange multiplier

4. Compuational Models

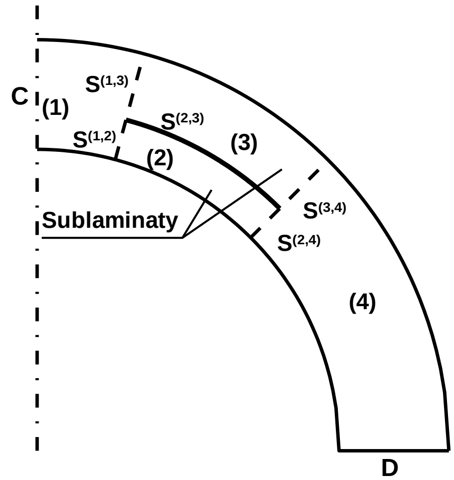


Fig.5 Single, axisymmetric delamination – the division of the structure into sublaminates

The construction of composite structures with the delamination is presnted in Fig. 5. The total area V is divided into three (Fig. 2a) or four parts (Figs 2b and 2c). Two of areas (1) and (4) represents the domains without delaminations, and the domains (2) and (3) corresponds to division of the thickness along the line of the delamination (unilateral boundary condition). For each of the areas (2) and (3) the independent sets of kinematical relations is formulated, whereas in the areas (1) and (4) the global system of coordinates is used.

Correct numerical modeling of the problem of development of delamination and subsequent loss of stability by sublaminate requires taking into account the following factors in the analysis:

- application of the large displacement (or deformation) option to determine the bifurcation point ,
- structure modeling using 2D or 3D elements; it is necessary for the analysis of the sublaminate buckling state, because the classic FEM packages contain only shell elements based on the 3D or first order transverse shear theory, and as stated previously, in the delamination problems, the theories of higher order shells should be used,
- the area where delamination occurs and its surroundings should be discretized using 2D or 3D elements with a triangular base in order to better approximate, especially at the edge of delamination, the values of the G energy release factors,
- along the thickness of FGMs the division should include d 15 to 20 FE since the material properties vary significantly along the coordinate z – see Eq (6)

5. Numerical Results

Using the above formulations it is possible to solve various for delaminated composite structures with local buckling. The problems shown below deals with a single delamination.

5.1 Spherical Shells under External Pressure

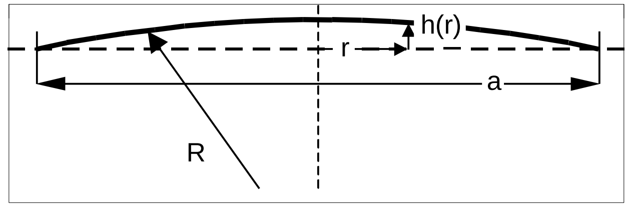


Fig.6 Cross-section and geometry of spherical shells

The cross-section of spherical shells is presented in Fig. 6. The shell geometry is characterized by the shallowness parameter ρ defined below.

$$\rho = \sqrt[4]{12(1-\nu_{12}\nu_{21})}\,\frac{a}{\sqrt{Rh}}\,. \tag{13}$$

For isotropic (quasiisotrpic) material properties the dimensionless value of the external buckling pressure is described by the following relation:

$$p_{\dim} = \frac{2E_{quasi}}{\sqrt{3\left(1-\nu^2\right)}}\left(\frac{h}{R}\right)^2 \tag{14}$$

The variations of buckling pressures with the shallowness parameter are plotted in Fig. 7 for laminated shells. The single delamination reduces values of buckling pressures. The buckling forms of such structures are plotted in Fig. 2 – see also [27].

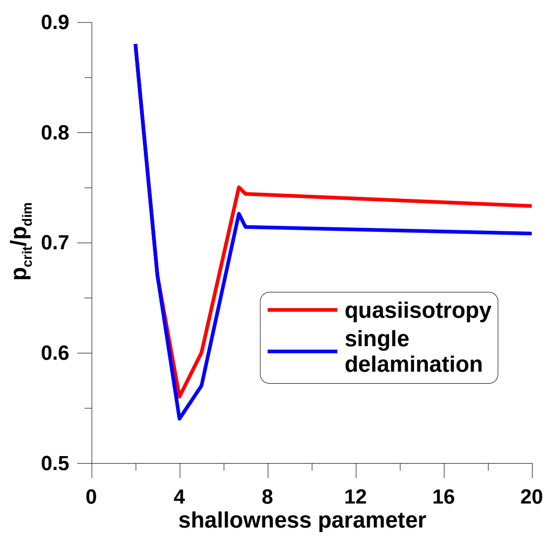


Fig.7 Distributions of buckling pressures for laminated spherical shells with and without of single delamination located at the shell mid-surface.

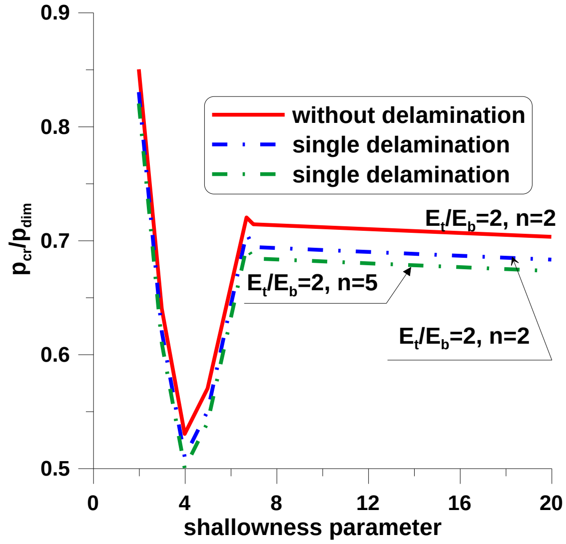


Fig. 8 Distributions of buckling pressures for spherical shells made of FGMs with and without of single delamination located at the shell mid-surface.

The unsymmetric properties of the FGM structures decrease buckling pressures (Fig. 8) since they reduce shell bending stiffnesses being the most significant in the evaluation of buckling loads – see Eq (14). The values of buckling pressures (Fig.8) vary with the change of the E_t/E_b ratio and of the n coefficient – see Eqs (6). The distributions of buckling pressures are similar for both symmetric laminates and unsymmetric configurations of structures made of functionally graded materials. The broader discussion of the results is presented in [28].

5.2 Compressed Rectangular Plates with a Single Delamination

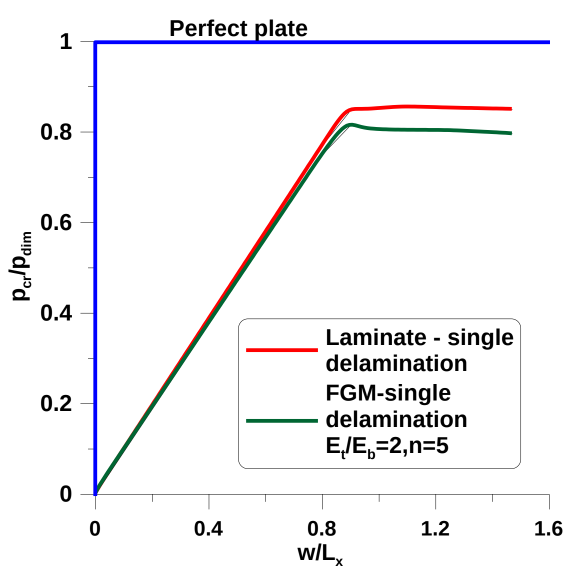


Fig.9 Pre- and post-buckling deformations of compressed plates with delaminations (w is a normal deflection at the moddle of the plate).

Now, let us consider the buckling problem of compressed plates with delamination having the form plotted in Fig.3. Delamination results in the both pre- and post-buckling behaviour of structures – see Fig.9. For perfect plates without delamination the rapid change between pre- and post-buckling deformations is observed. For structures with delaminations the initiation and then the development of delaminated area leads to the nonlinear behaviour (Fig. 9). Similarly as for spherical shells the effect of unsymmetry of FGM configuration is lower than for composite laminates.
The value p_{dim} corresponds to buckling loads of compressed isotropic (quasiisotropic) plates and is equal to:

$$p_{\dim} = \frac{\pi^2 E_{quasi}}{12\left(1-\nu^2\right)} \frac{h^2}{L_x^2} \left(\frac{mL_x}{L_y} + \frac{L_y}{mL_x} \right)^2 \tag{15}$$

where the symbol L denote the length of the plate along x and y directions.
The post-buckling form of the shell is drawn in Fig. 10. The elliptical delamination is located at z=h/3. Such a construction of the delamination is assumed in order to simplify computation and derivation of buckling pressures.

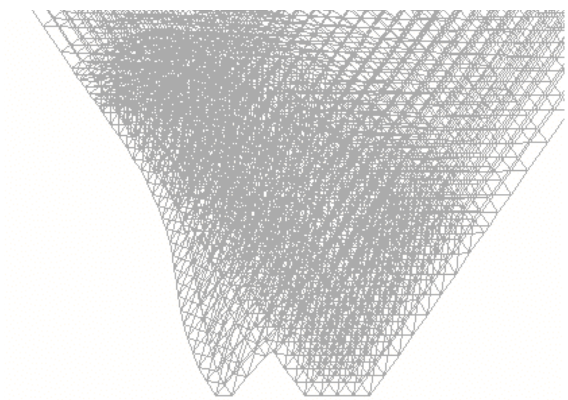


Fig. 10 Post-buckling form of compressed simply supported rectangular plate with a single elliptical delamination.

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