Solution of Integral Equations Using Local Splines of the Second Order

I.G.BUROVA, G. O. ALCYBEEV
Dept. of Computational Mathematics
St. Petersburg State University
St. Petersburg, RUSSIA

Abstract— Splines are an important mathematical tool in Applied and Theoretical Mechanics. Several Problems in Mechanics are modeled with Differential Equations the solution of which demands Finite Elements and Splines. In this paper, we consider the construction of computational schemes for the numerical solution of integral equations of the second kind with a weak singularity. To construct the numerical schemes, local polynomial quadratic spline approximations and second-order nonpolynomial spline approximations are used. The results of the numerical experiments are given. This methodology has many applications in problems in Applied and Theoretical Mechanics

Keywords—Splines, integral equation of the second kind, weak singularity, local spline, polynomial spline, nonpolynomial spline

Received: January 13, 2022. Revised: November 14, 2022. Accepted: December 13, 2022. Published: December 31, 2022.

1. Introduction

A lot of mathematical formulations of physical phenomena contain integral and/or integro-differential equations. These equations occur in many applications such as in the transport of air and ground water pollutants, oil reservoir flow, in the modeling of semiconductors etc. Currently many papers have been devoted to the numerical solution of integral equations with a weak singularity. Let's mention some papers published recently.

In paper [1] an iterative scheme to approach the solution of nonlinear integro-differential Fredholm equation with a weakly singular kernel using the product integration method is developed.

Cubic trigonometric B-spline functions are used in the paper [2] to solve the convection-diffusion type partial integro-differential equation (PIDE) with a weakly singular kernel. Cubic trigonometric B-spline (CTBS) functions are used for interpolation in both methods. The first method is the CTBS based collocation method which reduces the PIDE to an algebraic tridiagonal system of linear equations. The other method is the CTBS based differential quadrature method which converts the PIDE to a system of ODEs by computing spatial derivatives as weighted sum of function values.

A new orthogonal basis for the space of cubic splines has been used in paper [3] for obtaining the numerical solutions of a partial integro-differential equation with a weakly singular kernel.

In paper [4] a meshless method in local setting and Laplace transform are coupled to approximate partial integro-differential equations (PIDEs).

In paper [5] a numerical scheme is developed to solve the Volterra partial integro-differential equation of the second order having a weakly singular kernel. The scheme uses cubic trigonometric B-spline functions to determine the weighting coefficients in the differential quadrature approximation of the second order spatial derivative.

In the paper [6] the trigonometric cubic B-spline collocation method is extended to the solution of a second order partial integro-differential equation with a weakly singular kernel.

Splines are often used to solve various problems: interpolation; solving the Cauchy problem; Image compression; and enlargement (see, for example, [8]-[11]). In paper [11], splines were used to solve Volterra integral equations of the second kind with a smooth kernel. This methodology has many applications in problems in Applied and Theoretical Mechanics

In section 2 of this paper, for the approximate calculation of integrals with a weak singularity, we use polynomial and nonpolynomial splines of the second order of approximation. In section 3 the numerical examples are given.

2. The construction of the method

In this paper, we consider the numerical solution of the Fredholm integral equation of the second kind with a weak singularity

$$u(x) - \int_a^b K(x,s)u(s)ds = f(x), \qquad x \in [a,b].$$

We assume that the kernel $K(x,s)$ has the form:

$$K(x,s) = p(x,s)g(x,s) = \frac{g(x,s)}{|x-s|^\alpha}, \qquad \alpha \in (0,1),$$

where $p(x,s)$ is a weight function, $p(x,s) = \frac{1}{|x-s|^\alpha}$, $\alpha \in (0,1)$. We assume that the function $g(x,s)$ is bounded function on $[a,b]$. In this paper, to construct calculation formulas, we use quadratic basis splines and a Gaussian-type quadrature formula.

Let an ordered grid of nodes $\{x_k\}$, be constructed on the interval $[a,b]$: $a = x_0 < \cdots < x_n = b$. We represent the integral $\int_a^b K(x,s)u(s)ds$ as the sum of integrals over the grid segments:

$$\int_a^b p(x,s)g(x,s)u(s)ds = \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} p(x,s)g(x,s)u(s)ds .$$

Applying local interpolation splines of the second order of approximation (see [8]), we obtain $\tilde{u} \approx u$, where

$$\tilde{u}(s) = \sum_{i=k}^{k+1} u(x_i)\omega_i(s), s \in [x_k, x_{k+1}],$$

$$\omega_k(s) = \frac{s - x_{k+1}}{x_k - x_{k+1}},$$
$$\omega_{k+1}(s) = \frac{s - x_k}{x_{k+1} - x_k}.$$

Recall that we have proved the approximation error theorem (see paper [7]).
Denote the norm:

$$\| u \|_{[a,b]} = \max_{x\in[a,b]} |u(x)|.$$

Theorem. If x_k, x_{k+1} are the nodes, $h = x_{k+1} - x_k$, $s \in [x_k, x_{k+1}]$, and $u \in C^2[x_k, x_{k+1}]$ then the next estimation is valid

$$|u(s) - \tilde{u}(s)| \le \frac{1}{8} h^2 \| u'' \|_{[x_k, x_{k+1}]}.$$

Denote $v_i(s) = g(x,s)\omega_i(s)$. Let us construct a quadrature formula of the Gaussian type with two nodes:

$$\int_{x_k}^{x_{k+1}} p(s)v_i(s)ds \approx A_{1,k}v_i(y_1) + A_{2,k}v_i(y_2).$$

It is necessary to calculate the nodes y_i and the coefficients $A_{i,k}$, $i = 1,2$. The nodes and coefficients of the quadrature formula of the Gaussian type with the weight $p(s)$ can be found in the traditional way. Let the function $\omega(x)$ be such that $\omega(x) = (x - y_1)(x - y_2)$, where $y_i, i = 1,2$, are the nodes of the quadrature formula of the Gaussian type.

For the convenience of these calculations, we will write the polynomial $\omega(s)$ in the form:

$$\omega(s) = (s - y_1)(s - y_2) = s^2 + qs + r.$$

First, we calculate the moments c_i:

$$c_i = \int_{x_k}^{x_{k+1}} p(s)s^i ds, \qquad i = 0, 1, 2, 3.$$

Next , we solve the system of the equations and find the unknowns q, r:

$$c_2 + qc_1 + rc_0 = 0,$$
$$c_3 + qc_2 + rc_1 = 0.$$

Now, we can solve the quadratic equation $s^2 + qs + r = 0$ and find its roots y_i. These roots are the nodes of the quadrature formula. Now we determine the coefficients $A_{i,k}$ of the quadrature formula by solving the system of equations:

$$A_{1,k} + A_{2,k} = c_0,$$
$$A_{1,k}y_1 + A_{2,k}y_2 = c_1.$$

Recall the theorem on the remainder term of a quadrature formula of the Gaussian type. In the case of two nodes, the remainder term R,

$$R = \int_a^b p(s)v(s)ds - A_{1,2}v(y_1) - A_{2,2}v(y_2)$$

takes the form:

$$R = \frac{v^{(4)}(\xi)}{4!} \int_a^b p(x)\omega^2(x)dx,$$

$\omega = (s - y_1)(s - y_2)$, $\xi \in [a, b]$, $v \in C^4[a, b]$.

When calculating sequentially, we get a chain of equalities:

$$\int_{x_k}^{x_{k+1}} p(x,s) \sum_{i=k}^{k+1} v_i(s)\, ds =$$

$$\sum_{i=k}^{k+1} u(x_i) \int_{x_k}^{x_{k+1}} p(x,s)\, g(x,s)\omega_i(s)ds \approx$$

$$u(x_k)(A_{1,k}g(x,y_1)\omega_k(y_1) + A_{2,k}g(x,y_2)\omega_k(y_2) +$$

$$u(x_{k+1})(A_{1,k}g(x,y_1)\omega_{k+1}(y_1) + A_{2,k}g(x,y_2)\omega_{k+1}(y_2) .$$

Now we are constructing a system of linear algebraic equations by setting $x = x_k, k = 0,1,\dots,n$. Solving the system of equations, we obtain the solution values $u(x_k)$ at the grid nodes x_k , $k = 0,1,\dots,n$. If necessary, we can connect the found values using the piecewise linear basis splines.

Remark. If functions $\varphi_1(s)$ and $\varphi_2(s)$ form a Chebyshev system, then the basis functions $\omega_j(s)$ and $\omega_{j+1}(s)$ can be determined by solving the system of equations

$$\varphi_1(x_j)\, \omega_j(s) + \varphi_1(x_{j+1})\, \omega_{j+1}(s) = \varphi_1(s),$$
$$\varphi_2(x_j)\, \omega_j(s) + \varphi_2(x_{j+1})\, \omega_{j+1}(s) = \varphi_2(s),$$
$$s \in [x_j, x_{j+1}].$$

Let us assume that the determinant of the system is not equal to zero. Let us study the case when $\varphi_1(s) = 1$ and $\varphi_2(s) = \varphi(s)$, where $\varphi(s)$ is a continues function. Let us construct a nonpolynomial approximation of the function $u(s)$ on each grid interval $[x_j, x_{j+1}]$ in the form:

$$\tilde{u}(s) = \sum_{i=k}^{k+1} u(x_i)\omega_i(s), \qquad s \in [x_k, x_{k+1}],$$

where

$$\omega_k(s) = \frac{\varphi(s) - \varphi(x_{k+1})}{\varphi(x_k) - \varphi(x_{k+1})},$$

$$\omega_{k+1}(s) = \frac{\varphi(s) - \varphi(x_k)}{\varphi(x_{k+1}) - \varphi(x_k)}.$$

Depending on the choice of the function $\varphi(s)$, different error estimates are obtained. Approximation errors of the solution obtained with the non-polynomial splines are discussed in papers [10]-[11].

3. The Results of the Numerical Experiments

Example 1. Let us start with solving the integral equation with a weak singularity

$$u(x) - \int_0^1 p(x,s)exp(x+s)u(s)ds = f(x),$$
$$x \in [0,1].$$

Here $p(x,s) = \frac{1}{\sqrt{|x-s|}}$, the function $f(x)$ is constructed using the functions $p(x,s), exp(x+s)$ and the exact solution $u(x) = exp(-x)$.

We construct the set of equidistant nodes with the step $h = 1/n$. We develop a program in Maple with Digits=10. First consider the use of the approximation with the polynomial splines.
Fig. 1 shows the plot of the exact and approximate solutions of the integral equation when $n = 3$.

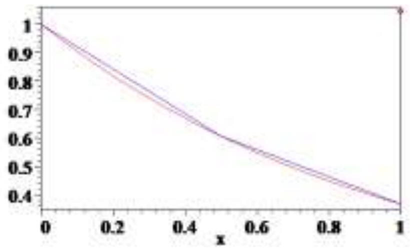

Fig.1. The plot of the exact and approximate solutions of the integral equation when $n = 3$

Fig. 2 shows the plot of the error of the solution of the integral equation when $n = 3$.

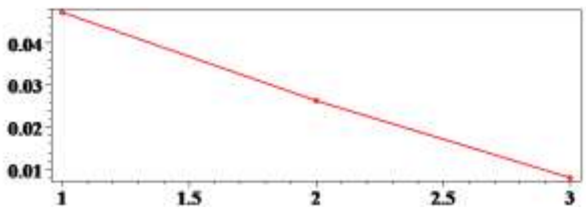

Fig.2. The plot of the errors of the solutions of the integral equation when $n = 3$

Figs. 3-5 show the plots of the errors of the solutions of the integral equation. Fig. 3 shows the plots of the errors of the solutions of the integral equation when $n = 16$. Fig. 4 shows the plots of the errors of the solutions of the integral equation when $n = 32$. Fig. 5 shows the plots of the errors of the solutions of the integral equation when $n = 256$.

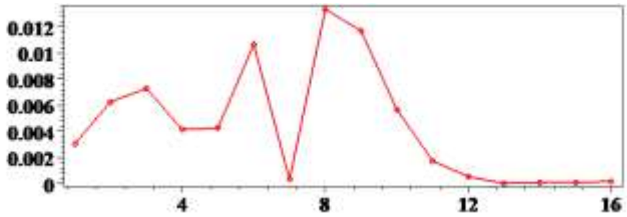

Fig.3. The plot of the errors of the solutions of the integral equation when $n = 16$

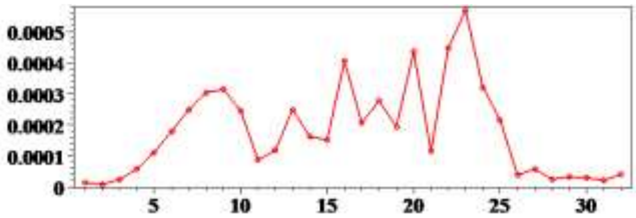

Fig.4. The plot of the errors of the solutions of the integral equation when $n = 32$

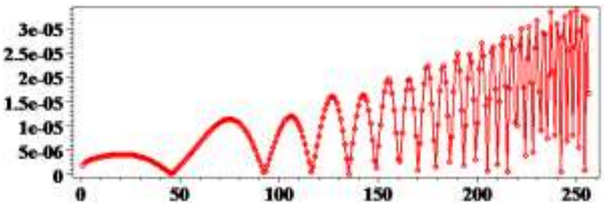

Fig.5. The plots of the errors of the solutions of the integral equation when $n = 256$

It can be seen that with an increase in the number of nodes, the solution error decreases. Now consider the use of approximation with the nonpolynomial (exponential) splines. We take $\varphi(s) = \exp(-s)$. Fig. 6 shows the plot of the exact and approximate solutions of the integral equation when $n = 3$. Fig. 7 shows the plot of the error of the solution of the integral equation when $n = 3$.

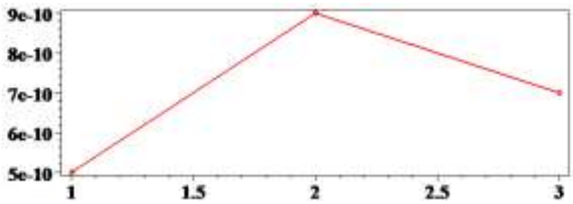

Fig.6. The plot of the exact and approximate solutions of the integral equation when $n = 3$ (exponential splines)

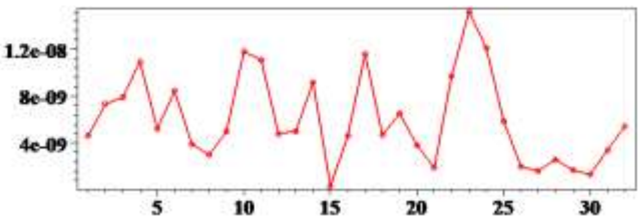

Fig.7. The plot of the errors of the solutions of the integral equation when $n = 32$ (exponential splines)

The results of numerical experiments show that well-chosen basis functions can significantly reduce the error of the solution.

Example 2. Let us continue with solving the next integral equation with a weak singularity

$$u(x) - \int_0^1 p(x,s)\sin(xs)u(s)ds = f(x)\,, \qquad x \in [0,1]\,.$$

Here $p(x,s) = \frac{1}{\sqrt{|x-s|}}$, the function $f(x)$ is constructed using the functions $p(x,s)$, $\sin(xs)$ and the exact solution $u(x) = \exp(-x)$.

The plot of the error of the solution obtained with the exponential splines when $n = 10$ is given in Fig.8.

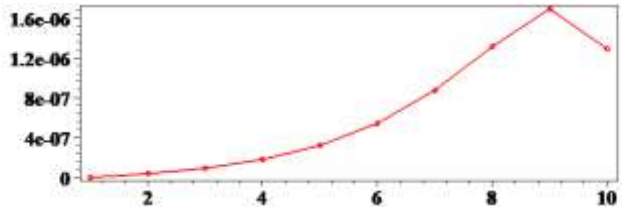

Fig.8. The plot of the errors of the solutions of the integral equation when $n = 10$ (*Example 2*, exponential splines)

The plot of the error of the solution obtained with the polynomial splines when $n = 10$ is given in Fig.9.

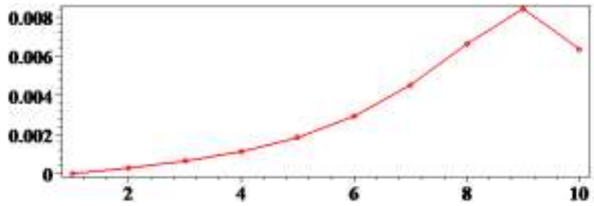

Fig.9. The plot of the errors of the solutions of the integral equation when $n = 10$ (*Example 2*, polynomial splines)

Example 3. Let us continue with solving the next integral equation with a weak singularity

$$u(x) - \int_0^1 p(x,s)\sin\left(\sqrt{1-s^2x^2}\right)u(s)ds = f(x)\,,$$
$$x \in [0,1]\,.$$

Here $p(x,s) = \frac{1}{\sqrt{|x-s|}}$, the function $f(x)$ is constructed using the functions $p(x,s), \sin\sqrt{1-s^2x^2}$ and the exact solution $u(x) = \exp(-x)$.

The plot of the error of the solution obtained with the exponential splines when $n = 10$ is given in Fig.10.

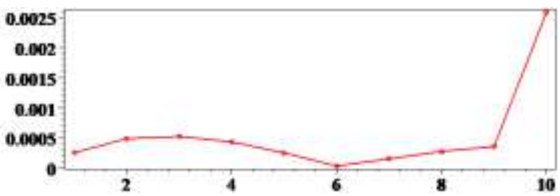

Fig.10. The plot of the errors of the solutions of the integral equation when $n = 10$ (*Example 3*, exponential splines)

The plot of the error of the solution obtained with the polynomial splines when $n = 10$ is given in Fig.11.

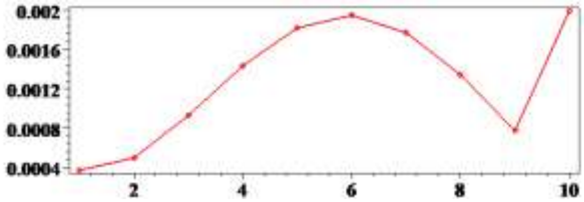

Fig.11. The plot of the errors of the solutions of the integral equation when $n = 10$ (*Example 3*, polynomial splines)

4. Conclusion

Splines are an important mathematical tool in Applied and Theoretical Mechanics. Several Problems in Mechanics are modeled with Differential Equations the solution of which demands Finite Elements and Splines. In this paper, we considered the construction of computational schemes for the numerical solution of integral equations of the second kind with a weak singularity. To construct the numerical schemes, local polynomial quadratic spline approximations and second-order nonpolynomial spline approximations are used. The results of the numerical experiments are given.

ACKNOWLEDGMENT

The authors are gratefully indebted to St. Petersburg University for their financial support in the preparation of this paper (Pure ID 94029567), as well as to a resource center for providing the package Maple.

References

[1] S. Touati, , M.Z. Aissaoui, S.Lemita, H.Guebbai, "Investigation Approach for a Nonlinear Singular Fredholm Integro-differential Equation," Boletim da Sociedade Paranaense de Matematica, vol.40, 2022.

K.Khan, A. Ali, Fazal-I-Haq, I.Hussain, N. Amir. "A comparative numerical study of parabolic partial integro-differential equation arising from convection-diffusion," CMES - Computer Modeling in Engineering and Sciences, vol.126 (2), pp. 673-692, 2021.

[2] J.Alavi, H.Aminikhah, "Orthogonal cubic spline basis and its applications to a partial integro-differential equation with a weakly singular kernel," Computational and Applied Mathematics, vol.40 (2), art. no. 55, 2021.

[3] G.Ali, Kamran, J.F.Gómez-Aguilar, "Approximation of partial integro differential equations with a weakly singular kernel using local meshless method," Alexandria Engineering Journal, vol.59 (4), pp. 2091-2100, 2020.

[4] Siraj-Ul-Islam, A.Ali, A. Zafar, I. Hussain, "A differential quadrature based approach for volterra partial integro-differential equation with a weakly singular kernel," CMES - Computer Modeling in Engineering and Sciences, vol.124 (3), pp. 915-935, 2020.

[5] A.Ali, K.Khan, F.Haq, S.I.A. Shah, "A Computational Modeling Based on Trigonometric Cubic B-Spline Functions for the Approximate Solution of a Second Order Partial Integro-Differential Equation," Advances in Intelligent Systems and Computing, 930, pp. 844-854, 2019.

[6] S.A.Karim, A.Saaban, V. Skala,. A.Ghaffar, K.S. Nisar,. D.Baleanu, "Construction of New Cubic Bézier-Like Triangular Patches with Application in Scattered Data Interpolation," Advances in Difference Equations, vol.151, pp.1-22, Springer Open, U.K., 2020, DOI: 10.1186/s13662-020-02598-w

[7] S.A.A.Karim, A.Saaban, V.Skala, "Range-Restricted Surface Interpolation Using Rational Bi-Cubic Spline Functions with 12 Parameters," IEEE Access, vol.7, pp.104992-105006, IEEE, 2019 DOI: 10.1109/ACCESS.2019.2931454 .

[8] I.G.Burova, Yu.K.Dem'yanovich, A.N. Terekhov, A.Yu.Altynova, A.D.Satanovskiy, A.A.Babushkin," Image compression and

enlargement algorithms," International Journal of Circuits, Systems and Signal Processing, 15, pp. 836-846, 2021.

[9] I.G.Burova, "On left integro-differential splines and Cauchy problem," International Journal of Mathematical Models and Methods in Applied Sciences, 9, pp. 683-690, 2015.

[10] I.G.Burova, G.O.Alcybeev, "Application of splines of the second order approximation to Volterra integral equations of the second kind. Applications in systems theory and dynamical systems," International Journal of Circuits, Systems and Signal Processing, 15, pp. 63-71, 2021.

[11] Abd Elhakim Lamairia, "Nonexistence Results of Global Solutions for Fractional Order Integral Equations on the Heisenberg Group", WSEAS Transactions on Systems, vol. 21, pp. 382-386, 2022.

[12] Nongluk Viriyapong, "Modification of Sumudu Decomposition Method for Nonlinear Fractional Volterra Integro-Differential Equations,"WSEAS Transactions on Mathematics, vol. 21, pp. 187-195, 2022.