Approximate Solution of Initial Boundary Value Problem of One-Dimensional Heat Conduction for the Thermal Shock of Thin Plate

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Abstract: - The article represents a new approximate analytical solution of the initial boundary value problem of one-dimensional heat conduction with boundary conditions of the third kind. It describes the temperature field of a homogeneous thin plate under thermal shock more accurately than previous solutions. The obtained solution was determined by the ANSYS mathematical package. The results of the analysis can be used to construct control laws for a small spacecraft taking into account the thermal shock of its solar panels.

Key-Words: - thermal shock, thin plate, one-dimensional heat conduction problem, approximate analytical solution, temperature field, ANSYS mathematical package.

Received: May 5, 2024. Revised: November 13, 2024. Accepted: December 7, 2024. Published: December 31, 2024.

1 Introduction

The development of modern mathematical packages and applications does not reduce the relevance of searching for approximate analytical solutions of initial-boundary value problems. The results and the ability to solve a problem are undoubted and indisputable advantages of modeling in modern mathematical packages and applications. However, in a number of situations, the ability to use computer resources is limited. Spacecraft is one example. Its onboard computing resources are limited and used mainly to solve target problems, [1]. At the same time, a modern spacecraft is a complex mechanical system. This system consists of a central solid body (spacecraft body) and elastic bodies (solar panels, antennas, etc). Fairly complex algorithms are developed to control such systems, [2], [3], [4]. However, there are target problems that impose special requirements on control. Such problems include remote sensing of the Earth [5] and the implementation of gravity-sensitive processes, [6]. To obtain high-resolution images, it is necessary to impose strict requirements on the orientation of the spacecraft at the time of photographing, [7], [8]. The implementation of gravity-sensitive processes is associated with the limitation of micro-accelerations in the working area of the technological equipment, [9], [10]. In both cases, to construct a control law that ensures an effective solution to target problems, it is necessary to take into account a large number of factors. One of these factors is the thermal shock of solar panels when the spacecraft enters and exits the Earth's shadow, [11], [12]. It is impossible to solve initial-boundary value problems of heat conductivity on an onboard computer. Therefore, it is necessary to construct an approximate analytical dependence of the solution.

The purpose of this research is to construct an approximate analytical dependence of the temperature field for a thin homogeneous plate under thermal shock. A one-dimensional heat conduction problem is considered. In this case, the influence of thermal shock on the motion of the spacecraft is maximal. The more appropriate solution is obtained in this research. Taking into account the thermal shock allows development of a more effective law for controlling the spacecraft. The results of the research can be used in designing control systems for small spacecraft with solar panels.

2 Problem Formulation

First of all, the solar panel of a spacecraft can be represented as a homogeneous thin plate with constant thermal properties. This formulation is widespread and can be found in a number of well-known researches, [13], [14], [15], [16], [17]. A one-dimensional heat conduction problem with boundary conditions of the third kind is considered. In this case, the solar radiation flux will be perpendicular to the solar panel. The panel has an undeformed shape at the moment of thermal shock. This formulation of the problem ensures maximum heating of the surface layer of the solar panel due to solar radiation. Therefore, the effect of thermal shock will be maximum in this case. The research of the maximum effect allows us to answer the question of the advisability of taking into account the thermal shock while modeling the motion of a small spacecraft. For the described formulation of the problem, the initial boundary value problem of one-dimensional heat conduction will take the following form, [15]:

$$\begin{cases} \dfrac{\partial T(z,t)}{\partial t} = a\dfrac{\partial^2 T(z,t)}{\partial z^2},\ 0 \le z \le h,\ t>0; \\ \left(\lambda \dfrac{\partial T(h,t)}{\partial z}\right) = Q - e\Theta\left(T^4(h,t) - T_C^4\right),\ z=h,\ t>0; \\ \left(\lambda \dfrac{\partial T(0,t)}{\partial z}\right) = -e\Theta\left(T^4(0,t) - T_C^4\right),\ z=0,\ t>0; \\ T(z,0) = T_0 = const,\ 0 \le z \le h,\ t=0. \end{cases} \quad (1)$$

In the initial boundary value problem (1) the following values are used: $T = T(z, t)$ is the temperature field of the plate; a is the temperature conductivity coefficient; λ is the thermal conductivity coefficient; Q is the value of the incident heat flux; e is the degree of blackness of the plate material; Θ is the Stefan-Boltzmann constant; T_C is the temperature of the environment surrounding the plate; h is a plate thickness.

In the system of equations (1), the first equation is a one-dimensional heat conduction equation. It assumes the validity of the Fourier's law. The boundary conditions are determined by equations 2 and 3. The solar flux Q falls on one surface of the plate ($z = h$), and radiation is taken into account according to the Stefan-Boltzmann law (second equation). The second surface only radiates energy into space (third equation). Radiation through the side faces of the plate is neglected because a thin plate is considered. The fourth equation (1) is a homogeneous initial temperature field. An approximate analytical solution to the initial boundary value problem (1), which can be used to optimize the control laws for small spacecraft, will be considered.

3 Problem Solution

To obtain an approximate solution to the initial boundary value problem of heat conductivity posed in the previous section, the expansion proposed in [18], [19] will be considered:

$$T_n(z,t) = \sum_{i=0}^{n} \xi_i(t) z^i. \quad (2)$$

Actually, it is a well-known expansion of a function of two variables as a product of two functions. Each of these functions depends on only one variable. Four terms of the expansion will be considered (2):

$$T_3(z,t) = \xi_0(t) + \xi_1(t)z + \xi_2(t)z^2 + \xi_3(t)z^3. \quad (3)$$

The remaining values (2) will be considered negligibly small due to the thinness of the plate. Then the partial derivatives included in the first equation (1) will take the following form:

$$\begin{cases} \dfrac{\partial T_3(z,t)}{\partial t} = \dfrac{d\xi_0(t)}{dt} + \dfrac{d\xi_1(t)}{dt}z + \dfrac{d\xi_2(t)}{dt}z^2 + \dfrac{d\xi_3(t)}{dt}z^3 \\ \dfrac{\partial^2 T_3(z,t)}{\partial z^2} = 2\xi_2(t) + 6\xi_3(t)z \end{cases}. \quad (4)$$

Substitute (4) into the first equation (1):

$$\frac{d\xi_0(t)}{dt} + \frac{d\xi_1(t)}{dt}z + \frac{d\xi_2(t)}{dt}z^2 + \frac{d\xi_3(t)}{dt}z^3 = 2a[\xi_2(t) + 3\xi_3(t)z]. \quad (5)$$

Let us group the values in (5), neglecting the values containing z to a degree higher than the first:

$$\left[\frac{d\xi_0(t)}{dt} - 2a\xi_2(t)\right] + z\left[\frac{d\xi_1(t)}{dt} - 6a\xi_3(t)\right] = 0. \quad (6)$$

Square brackets on the left side that are equal to zero in (6) are required to satisfy the heat equation. Then:

$$\begin{cases} \dfrac{d\xi_0(t)}{dt} = 2a\xi_2(t) \\ \dfrac{d\xi_1(t)}{dt} = 6a\xi_3(t) \end{cases}. \quad (7)$$

Thus, equalities (7) guarantee the satisfaction of the heat conduction equation (1) with an accuracy of up to terms containing z^2 within the framework of expansion (3).

The boundary conditions (the second and third equations (1) will be considered. The expressions included in it, taking into account the expansion (3) at $z = 0$, will take the following form:

$$\begin{cases} \dfrac{\partial T_3(z,\, t)}{\partial z} = \xi_1(t),\ z = 0;\ t \geq 0; \\ T_3(0,\, t) = \xi_0(t),\ z = 0;\ t \geq 0. \end{cases} \tag{8}$$

Neglecting the ambient temperature, substituting (8) into (1), will obtain:

$$\lambda \xi_1(t) = -e\Theta \xi_0^4(t). \tag{9}$$

Similarly, the boundary conditions (the fourth equation (1)). The expressions included in it, taking into account the expansion (3) at $z = h$, will take the form:

$$\begin{cases} \dfrac{\partial T_3(z,\, t)}{\partial z} = \xi_1(t) + 2\xi_2(t)h + 3\xi_3(t)h^2; \\ T_3(h,\, t) = \xi_0(t) + \xi_1(t)h + \xi_2(t)h^2 + \xi_3(t)h^3. \end{cases} \tag{10}$$

Neglecting the ambient temperature, substituting (10) into (1), will obtain:

$$\begin{aligned} &\lambda\left[\xi_1(t) + 2\xi_2(t)h + 3\xi_3(t)h^2\right] = \\ &= Q - e\Theta\left[\xi_0(t) + \xi_1(t)h + \xi_2(t)h^2 + \xi_3(t)h^3\right]^4. \end{aligned} \tag{11}$$

Linearizing (11), leaving only the first degree of h:

$$\lambda\left[\xi_1(t) + 2\xi_2(t)h\right] = Q - e\Theta\left[\xi_0(t) + \xi_1(t)h\right]^4. \tag{12}$$

The bracket on the right side of (12) that contains the fourth degree will be removed:

$$\begin{aligned} &\left[\xi_0(t) + \xi_1(t)h\right]^4 = \xi_0^4(t) + 4\xi_0^3(t)\xi_1(t)h + \\ &+ 6\xi_0^2(t)\xi_1^2(t)h^2 + 4\xi_0(t)\xi_1^3(t)h^3 + \xi_1^4(t)h^4. \end{aligned} \tag{13}$$

Linearizing (13) with respect to h:

$$\left[\xi_0(t) + \xi_1(t)h\right]^4 \approx \xi_0^4(t) + 4\xi_0^3(t)\xi_1(t)h. \tag{14}$$

Taking into account (14), the boundary condition (12) will take the following form:

$$\lambda \xi_1(t) + 2\lambda \xi_2(t)h = Q - e\Theta \xi_0^4(t) - 4e\Theta \xi_0^3(t)\xi_1(t)h. \tag{15}$$

From the first equation (7) will be obtained:

$$\xi_2(t) = \frac{1}{2a}\frac{d\xi_0(t)}{dt}. \tag{16}$$

From equation (9) will be obtained:

$$\xi_1(t) = -\frac{e\Theta}{\lambda}\xi_0^4(t). \tag{17}$$

(16) and (17) are substituted into (15) then, taking into account (9), after elementary transformation will be obtained:

$$\frac{d\xi_0(t)}{dt} = \frac{Q\,a}{\lambda\, h} + 4\frac{e^2\Theta^2 a}{\lambda^2}\xi_0^7(t). \tag{18}$$

By integrating (18), using (7) and (9), the remaining functions included in the expansion (3) can be found. It should be noted that equation (18) is not integrated analytically. Its solution can only be obtained approximately.

4 Numerical Modeling and Validation

Numerical modeling in the ANSYS package will be conducted. For numerical modeling, the parameters presented in Table 1 will be represented, [20].

Table 1. The main parameters of the simulated plate, [20]

Parameter	Designation	Value	Dimension
Solar panel frame material	–	MA2	–
Coefficient of thermal conductivity	λ	96,3	$W/(m\cdot K)$
Stefan-Boltzmann constant	Θ	$5{,}67\cdot 10^{-8}$	$W/(m^2\cdot K^4)$
External heat flux	Q	1400	W/m^2
Vacuum temperature	T_C	3	K
Initial temperature of the solar panel frame	$T_0 = T(z,\, 0)$	200	K
Degree of blackness	e	0,2	-
Specific heat	c	1130,4	$J/(kg\cdot K)$
Density	ρ	1780	kg/m^3
Young's Module	E	$4\cdot 10^{10}$	Pa
Shift modulus	μ	$1{,}6\cdot 10^{10}$	Pa
Poisson 's Ratio	v	0,3	-
Solar panel length	l	1	m
Solar panel width	b	0,5	m
Solar panel frame thickness	h	1	mm

The article [20] represents the results of a comparison of the approximate analytical dependence with numerical modeling in the ANSYS package for the same parameter values. This approximate dependence looked as follows:

$$T(z,t) = Cz\frac{t}{t+a} + T_0,\ 0 \le z \le h,\ t > 0. \tag{19}$$

Dependence (19) describes the dynamics of the temperature field during thermal shock, according to researchers [20]. However, this dependence does not satisfy the heat conductivity equation (1). The discrepancy of expression (19) while substituting it into the first equation (1) will be considered. Obviously, the right side will be equal to zero, since the temperature dependence (19) is linear. The left side will take the following form:

$$\frac{\partial T}{\partial t} = Cz\frac{a}{(t+a)^2}. \tag{20}$$

The time of the characteristic phase of the thermal shock is limited to several seconds. To determine the maximum discrepancy of the dependence (19) the integration of (20) will be performed:

$$\Delta T = \int_0^{10} Cz\frac{a}{(t+a)^2}\,dt.$$

With the parameter values: $C = 200$ K/m; $a = 1$ s; $z = h = 1$ mm (they were used in the research [20]), this discrepancy will be:

$$\begin{aligned}\Delta T &= 200\cdot 10^{-3}\cdot 1\left(-\frac{1}{t+1}\right)\Bigg|_0^{10} = \\ &= 0{,}2\cdot\left(-\frac{1}{11}+1\right) \approx 0{,}18\ K\end{aligned}. \tag{21}$$

In fact, (21) is an estimate of the method error while using (19) as an approximate solution to the initial boundary value problem (1). The method error (21), for example, is significantly lower than the measurement error of the Pt500 resistive temperature sensor (approximately 0.6 K). However, expression (3) obtained in this research is a solution to the initial boundary value problem of heat conductivity (1). Its method error lies in the limitation of the number of terms taken into account in the expansion (2) and the linearization of expressions (11) and (13).

Let us compare how much the accuracy of the temperature field description has increased using the results obtained in this article. Figure 1 demonstrates the temperature field dependencies obtained in research [20] and using expansion (3).

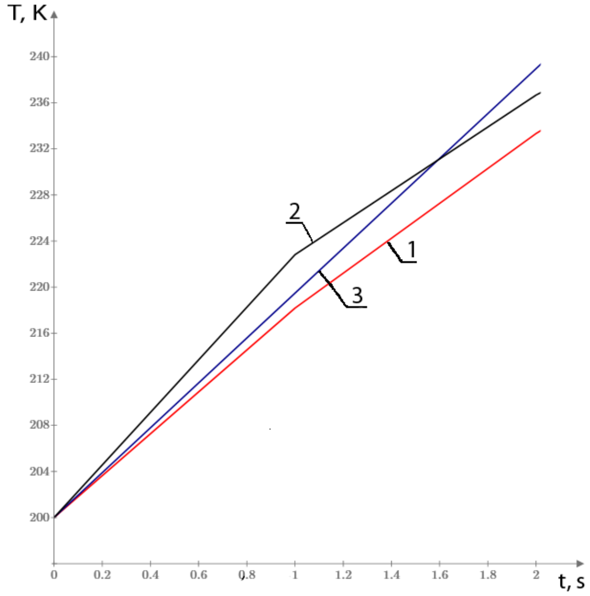

Fig. 1: Temperature dynamics of the plate surface layer: 1 is ANSYS approximation; 2 is approximated dependence (19) from [20]; 3 is approximated dependence (3) from this paper

As can be seen from Figure 1, during the thermal shock period, expansion (3) describes the temperature field better than the dependence in [20]. Then, a temperature gradient is formed inside the plate. Therefore, the linearization of expressions (11) and (13) provides more noticeable errors than at the moment of thermal shock. The discrepancy (20) accumulates more slowly than the linearization error. Starting from $t = 2$ s, the temperature dependence [20] describes the model data better than expansion (3).

5 Conclusion

Thus, the approximate solution of the initial boundary value problem of one-dimensional heat conductivity for the thermal shock of a thin plate obtained in this article allows us to improve the description of the temperature field during the thermal shock itself (approximately the first two seconds from the moment of its occurrence). Then, due to the linearization of expressions (11) and (13) relative to the plate thickness, an error accumulates. The temperature field determined by expansion (3) begins to diverge from the results of numerical simulation. However, for the characteristic time of the thermal shock (until a large temperature gradient has formed across the plate thickness), the results are more accurate than for other models, [20].

Further, increase in the accuracy of the temperature field description within the framework of the considered formulation may be associated with the refusal to linearize expressions (11) and (13). It is also possible to take into account a larger number of expansion terms (2). In this case, the accuracy of the approximation may increase insignificantly. Therefore, for most practical problems, the approximate solution presented in this article may be quite sufficient.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

- A.V. Sedelnikov: conceptualization, methodology, formal analysis, investigation, resources, data curation, writing–original draft preparation, writing–review and editing, supervision, project administration, funding acquisition.
- A.S. Nikolaeva: methodology, software, validation, formal analysis, data curation, writing–original draft preparation, writing–review and editing, visualization.
- V.V. Serdakova: software, validation, writing–original draft preparation, writing–review and editing, visualization.
- M.A. Evtushenko: software, formal analysis, writing–original draft preparation, writing–review and editing, visualization.

All authors have read and agreed to the published version of the manuscript.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

This work is supported by the Ministry of Education and Science of the Russian Federation in the framework of the State Assignments to higher education institutions and research organizations in the field of scientific activity (the project FSSS-2023-0007).

Conflict of Interest

The authors have no conflicts of interest to declare.