Alternative representations of some arithmetic functions

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Abstract: - This article presents some results of the attempt to simplify the writing of arithmetic functions on the computer so that users can apply them without additional operations, such as summing after a set whose elements must be calculated, such as the set of numbers prime. The important role of the remainder function in defining most arithmetic functions is highlighted. Defining algorithms for the prime factorization of natural numbers highlights the possibility of representing natural numbers in a basis as "natural" as possible for natural numbers, namely the basis of prime numbers. The disadvantage of this natural basis is, for the time being, that it is infinitely dimensional. For now, this representation provides advantages but also disadvantages. Among the arithmetic functions proposed in the article, there are also statistical characterizations of the distribution of prime numbers, given with the hope of helping a better knowledge of the set of prime numbers. The importance of the remainder function in the computational definitions of arithmetic functions leads to reflections on the importance of fundamental operations - addition and multiplication - of natural numbers and the importance of inverse functions - subtraction and division. In turn, these operations can be seen as functions of two variables on the set of natural numbers. From here, readers are invited to reflect on the problem of the origin of natural numbers, the origin based on revelation or the origin provided by set theory, although this may also be a revelation.

Key-Words: - arithmetic, functions, remainder, infinite, basis, numbers, prime

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1 Introduction

"God made the natural numbers; all else is the work of man." This is the statement of the mathematician called by some of the most prominent representatives of the pre-intuitionist current. I also quoted this interesting statement in [2]. Natural numbers, as useful tools in the daily practices of human life, have a dual role: estimating quantity and quality (order). In [4] it is stated that „Numbers make up the foundation of mathematics. The first numbers people used were natural or counting numbers, consisting of 1, 2, 3, ... When 0 is added to the set of natural numbers, the set is called the whole numbers". Dedekind's opinion, [5], does not contradict Kronecker's statement, " I regard the whole of arithmetic as a necessary, or at least natural, a consequence of the simplest **arithmetic act**, that of counting, and counting itself as nothing else than the successive creation of the infinite series of positive integers in which each individual is defined by the one immediately preceding; the simplest act is the passing from an already-formed individual to the consecutive new one to be formed". The arithmetic act can be interpreted by readers as an autonomous fact of the human species or as a revelation or vision of divine origin. Dedekind does not specify the meaning of the notion of the arithmetic act. It is further stated in [5]: „The chain of these numbers forms in itself an exceedingly useful instrument for the human mind; it presents an inexhaustible wealth of remarkable laws obtained by the introduction of the four fundamental operations of arithmetic. The addition is the combination of any arbitrary repetitions of the above-mentioned simplest act into a single act; from it in a similar way arises multiplication". Dedekind designates the two fundamental operations of: addition and multiplication, both of which can be seen as functions of two arguments applied to the Cartesian product of natural numbers with themselves. Unconsciously introduced to the functional characterization of the two operations, it generates through their inverses the appearance of integers and rational numbers. Later, the search for solid foundations of mathematics, put before numbers, set theory and thus, natural numbers appeared in their quantitative (cardinal) and ordinal (ordinal).

Functions defined on the set of natural numbers, subsets of it, or various Cartesian products of the set of natural numbers, are used to address important issues of natural number theory, such as the divisibility or distribution of prime numbers.

The functions defined in this article are part of a category of functions called in the literature, arithmetic functions. A number of such functions are defined in [6] - some of which are among the functions proposed in this article, with a slight change, especially regarding the summary indices used. The formulations proposed in this article allow the direct calculation of the values of these functions, while the formulations in [6] imply a pre-calculation of the set of indices after which the summation is made. The functions proposed in this article are, perhaps, a little more explicit than those used in the theoretical literature, easier to apply on the computer. To better understand the meaning of this "more explicit", compare the definition given to Euler's function in [7] or the expression of the same function given by the theorem which provides the values of Euler's function using prime factorization of number (Theorem 62 of [7]), with the formulation given in this article using formulas (39)-(42).

Arithmetic functions are defined in all the fundamental works of number theory, their purpose being that of useful tools for knowing natural numbers, for solving some fundamental problems of them. Among these works, we mention [7-10]. The author [9] defines simple arithmetic functions as "A function whose domain is the set of positive integers is called theoretic or arithmetic number". Two of the important functions of number theory are designated in [9] as the number of divisors, respectively the sum of the divisors of a given natural number. The definition of these functions is formulated by summing the divisors of the number for which the values of the searched functions are calculated. For the same functions, we used a sum with natural summation for the functions *sump1*, *numdiv1* and *prodp*. The basic function for the calculation is for all three of the above functions, the remainder function. Also, an expression of the Euler index, based on the remainder function, is given as an application in the article.

2 Problem Formulation

In [1] we have proposed a representation for the characteristic function of prime numbers in the set of natural numbers, $\mathbb{N}$. In [3], we reformulated the representation of the characteristic function of prime numbers in the set of natural numbers, using the remainder function defined from the theorem of division by the remainder in the set of natural numbers. I also gave a representation of the function number of prime numbers smaller than a given natural number. Also in [3] representations were given for the functions: density function of prime numbers, the sum function of the divisors of a natural number, the Euler's function or Euler's indicator.

In order to develop tools to help simpler expression or highlight properties of prime numbers, we have built some functions that will be defined below.

I recall, after [3], the *remaining function*:

$$r: \mathbb{N} \times \mathbb{N} \to \mathbb{N}, r(x, y) = x - y\left[\frac{x}{y}\right] \tag{1}$$

where the symbol $[z]$, represents the function of the integer part of the number z (the largest integer less than, see [10]). The *division function*, which indicates whether or not a number is divided by another number, is defined by formula (2):

$$u: \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad u(x, y) = \begin{cases} 1, r(x, y) > 0 \\ 0, r(x, y) = 0 \end{cases} \tag{2}$$

The *characteristic function of prime numbers in the set of numbers* is naturally defined by formula (3):

$$p: \mathbb{N} \to \mathbb{N}, \qquad p(x) = \begin{cases} p_0(x), x > 2 \\ 1, x = 2 \\ 0, x < 2 \end{cases} \tag{3}$$

where p_0 is the function defined by formula (4):

$$p_0: \mathbb{N} \to \mathbb{N},$$
$$p_0(x,y) = \prod_{y=2}^{[\sqrt{x}]+1} u(x,y) \tag{4}$$

The *function indicating the number of prime numbers less than a given number, greater than or equal to* 2, np, is defined using the function np_0, defined by formula (5):

$$np_0: \mathbb{N} \to \mathbb{N}, \qquad np_0(x) = \begin{cases} 1, p(x) > 0 \\ 0, p(x) = 0 \end{cases} \tag{5}$$

according to formula (6):

$$np: \mathbb{N}^\circ \to \mathbb{N},\ np(x) = \sum_{k=2}^{x} np_0(k) \tag{6}$$

where $\mathbb{N}^\circ$ is the set of natural numbers without the numbers 0 and 1. Be the *sum function of prime numbers less than a given number* x, *given, greater than or equal to* 2 by formula (8), using the function defined in (7):

$$np_1: \mathbb{N}^\circ \to \mathbb{N}, \qquad np_1(x) = \begin{cases} x, p(x) > 0 \\ 0, p(x) = 0 \end{cases} \tag{7}$$

$$S_p: \mathbb{N}^\circ \to \mathbb{N},\ S_p(x) = \sum_{k=2}^{x} np_1(k). \tag{8}$$

Similarly, the *product function of all prime numbers less than a given number*, x, *greater than or equal to* 2, is defined according to formula (9):

$$P_p: \mathbb{N}^\circ \to \mathbb{N},\ P_p(x) = \prod_{k=2}^{x} np_1(k). \tag{9}$$

The *attenuated prime number function* is introduced, according to definition (10):

$$p_a: \mathbb{N}^\circ \to \mathbb{N},\ p_a(x) = \prod_{y=2}^{[\sqrt{x}]+1} \frac{r(x,y)}{y}. \tag{10}$$

The *probability function of finding a prime number among natural numbers less than a number* x, *greater than or equal to 2*, is defined according to formula (11):

$$p_p: \mathbb{N}^\circ \to \mathbb{N}, p_p(x) = \frac{np(x)}{x} \tag{11}$$

The *entropy of the distribution of prime numbers less than a given number*, x, *greater than or equal to* 2, is defined by definition (12):

$$E_p: \mathbb{N}^\circ \to \mathbb{R},$$
$$E_p(x) = -\sum_{k=2}^{x} \left(p_p(k) \cdot log\left(p_p(k)\right)\right) \tag{12}$$

where $\mathbb{R}$ is the set of real numbers.

The *function of the number of prime numbers between two natural numbers* x *and* y, *greater than or equal to* 2, is defined by formula (13):

$$vp: \mathbb{N}^\circ \times \mathbb{N}^\circ \to \mathbb{N},\ vp(x,y) = np(x) - np(y),\ x \geq y. \tag{13}$$

The *sum function of the divisors of a given number, greater than 1, including the number*, can be defined as in (14):

$$sump: \mathbb{R}^\circ \times \mathbb{N}^\circ \to \mathbb{N}^\circ sump(x,y) = \sum_{y=2}^{x} u^*(x,y),\ x \geq y \tag{14}$$

where u^* is defined by the relationship (15):

$$u^*: \mathbb{N}^\circ \times \mathbb{N}^\circ \to \mathbb{N},$$
$$u^*(x,y) = \begin{cases} 0, r(x,y) > 0 \\ y, r(x,y) = 0 \end{cases} \tag{15}$$

Similarly, the *product function of the divisors of a given number greater than* 1, *including the number*, is defined according to (16):

$$prodp: \mathbb{N}^\circ \times \mathbb{N}^\circ \to \mathbb{N}^\circ$$
$$prodp(x,y) = \prod_{y=2}^{x} u^{**}(x,y),\ x \geq y \tag{16}$$

with the specification from (17):

$u^{**}: \mathbb{N}^\circ \times \mathbb{N}^\circ \to \mathbb{N},$

$$u^{**}(x,y) = \begin{cases} 1, r(x,y) > 0 \\ y, r(x,y) = 0 \end{cases} \quad (17)$$

Using the functions defined by the relations (1) - (16), we can define some functions whose variation is interesting for the study of prime numbers. The *unit characteristic function of prime numbers* is defined by (18):

$$prim: \mathbb{N}^\circ \to \mathbb{N}, prim(x) = \frac{sump(x)}{prodp(x)} \quad (18)$$

the *arithmetic average of prime numbers less than a given number* x:

$$meanp: \mathbb{N}^\circ \to \mathbb{N}, meanp(x) = \frac{Sp(x)}{sp(x)} \quad (19)$$

the *geometric mean function of prime numbers less than a given number* x*:*

$$gmeanp: \mathbb{N}^\circ \to \mathbb{N},$$
$$gmeanp(x) = (Pp(x))^{\frac{1}{sp(x)}} \quad (20)$$

function average value of prime numbers between the numbers y *and* x, $x < y$:

$$meanpxy: \mathbb{N}^\circ \times \mathbb{N}^\circ \to \mathbb{N}, meanpxy(x,y) = \frac{sump(y) - sump(x)}{sp(y) - sp(x)} \quad (21)$$

the *ratio function between the arithmetic mean and the geometric mean of prime numbers less than a given number,* x, *greater than or equal to* 2:

$$rmeanp: \mathbb{N}^\circ \to \mathbb{N},$$
$$rmeanp(x) = \frac{meanp(x)}{gmeanp(x)} \quad (22)$$

For convenience in computing, logarithm can be applied to the study of function (22), which facilitates numerical calculation, and computational effort. Another function of statistical origin that can be defined on the set of prime numbers is the *function probable average deviation of prime numbers less than a given number* x, *greater than or equal to* 2, according to the relation (23):

$$As(x) = \sqrt{\frac{\sum_{k=2}^{x} [as(k) \cdot (k - menp(x))]^2}{sp(x)}} \quad (23)$$

where:

$$as(x) = \begin{cases} 1, p(k) = 1 \\ 0, p(k) \neq 1 \end{cases} \quad (24)$$

3 Results and applications

This chapter gives some simple results that help to understand the behaviour of some of the functions described above.

The remainder function, graphical representation and alternative functions

The remainder function, defined in (1), is a function of two variables and can be represented graphically as in Fig. 1, by the set of integer coordinate points on the planes that form the configuration of the built surface.

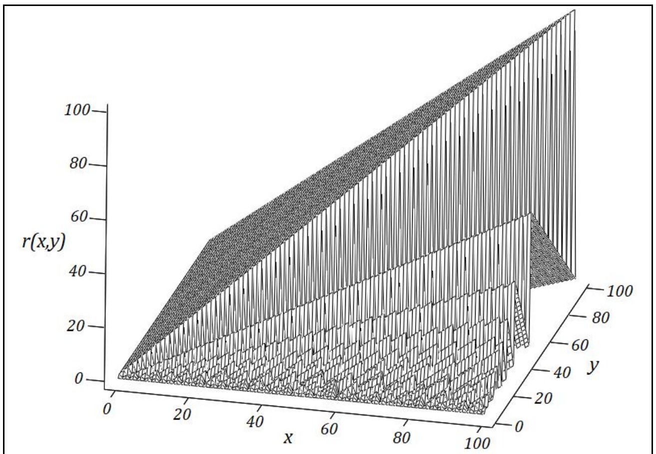

Fig. 1 Three-dimensional graphical representation of the remainder function.

The graph of this function consists of the points with integer coordinates located on the portions of inclined planes as seen in fig. 1. The following functions can be considered for various applications:

$$rs: \mathbb{N} \times \mathbb{N} \to \mathbb{Q},$$
$$rs(x,y) = \frac{r(x,y) + r(y,x)}{2} \quad (25)$$

respectively:

$$ra : \mathbb{N} \times \mathbb{N} \to \mathbb{Q},$$
$$ra(x,y) = \frac{r(x,y) - r(y,x)}{2} \quad (26)$$

assimilated with the symmetric part and the anti-symmetric part of the remainder function, whereby $\mathbb{Q}$ we noted the set of rational numbers. A graphical representation of the characteristic function, defined in (3), for numbers between 0 and 100, is given in fig. 2.

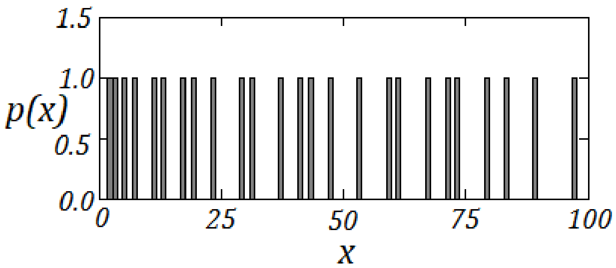

Fig. 2 Graphical representation of the characteristic function of prime numbers, for numbers between 0 and 100.

For the function of the number of prime numbers smaller than a given number, defined by (6), a similar graphical representation is possible, given in fig. 3.

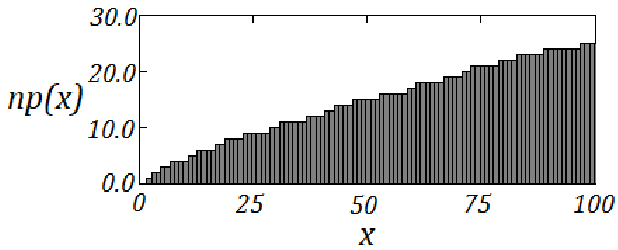

Fig. 3 Graphical representation of the function of the number of prime numbers less than a given number, for numbers between 0 and 100.

The sum and product functions of prime numbers smaller than a given number x, defined by relations (8) and (9), grow monotonously, nonlinearly, in steps and have no remarkable graphical representations. The attenuated prime number function, defined in (10), decreases rapidly, nonlinearly and un-monotonous. The entropy of the distribution of prime numbers, defined in (12), is an increasing function, which can be easily proved by the difference between the values on any two consecutive arguments. The graphical representation in fig. 4 shows the seemingly linear increase in the entropy of the distribution of prime numbers for numbers less than 102.

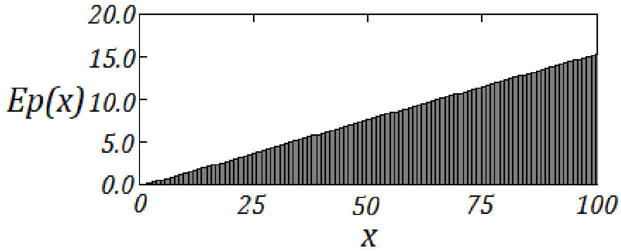

Fig. 4 Graphical representation of the entropy of the distribution of prime numbers less than 102.

In [7] is defined the function that gives for each natural number, the number of natural numbers smaller than the given number and prime with it, noted by the author, φ. There is still a construction alternative with the advantage that the summary index follows a string with a constant pitch, in fact, consecutive natural numbers, as opposed to the definition in [7], in which the index of summation consists of the set of prime numbers that divide the argument of the function. Be the function that gives the number of common divisors of two numbers:

$$\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N},$$
$$\varphi(x,y) = \sum_{k=2}^{\min(x,y)} \tau(x,y,k) \quad (27)$$

where

$$\tau : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \tau(x,y,z) = \begin{cases} 0, r(x,z) > 0 \vee r(y,z) > 0 \\ 1, r(x,z) = 0 \wedge r(y,z) = 0 \end{cases} \quad (28)$$

Then the function that gives the number of smaller and prime numbers with a given number is defined by the relations:

$$\Phi : \mathbb{N} \to \mathbb{N}, \Phi(\mathrm{x}) = \sum_{k=2}^{x} \gamma(x,k) \quad (29)$$

where:

$$\gamma : \mathbb{N} \to \mathbb{N}, \gamma(x,z) = \begin{cases} 0, \varphi(x,z) > 0 \\ 1, \varphi(x,z) = 0 \end{cases} \quad (30)$$

The function Φ defined in relation (29) corresponds to the function φ defined in [7].

4 Comments

An interesting application for knowing the multiplicative structure of natural numbers (implicitly of the set of integers) is the one that generates the prime factorization of natural numbers. An algorithm for multiplying natural numbers is constructed using the functions defined in 1. Let X be a natural number, and the next function:

$$\begin{gathered}\Psi : (\mathbb{N} - \{0,1\}) \times \Pi \to N, \\ \Psi(X, y) = \sum_{j=0}^{\left[\frac{ln(X)}{ln(y)}\right]} \psi\left(r\left(X, y^j\right)\right) - 1, \\ \psi(X) = \begin{cases} 0, X > 0 \\ 1, X = 0 \end{cases}\end{gathered} \tag{27}$$

where Π represents the set of prime numbers. In practical calculus, it is considered instead of Π, only its subset bounded by X. For any natural number X, the function ψ has a finite number of non-zero values, because the multiplicative decomposition of any natural number greater than or equal to 1, it has a finite number of factors. To represent the decomposition of some natural number X in the multiplicative form, the vector function Θ, defined by (28), is used.

$$\begin{gathered}\Theta : (\mathbb{N} - \{0,1\}) \times \Pi \to \Pi \times \mathbb{N}, \\ \Theta(X, y) = (y, \Psi(X, y))\end{gathered} \tag{28}$$

According to (28), for each natural number X, greater than zero, having the multiplicative representation given by the formula (29),

$$\begin{gathered}X = \bar{y}_0^{\Psi(X,\bar{y}_0)} \cdot \bar{y}_1^{\Psi(X,\bar{y}_1)} \cdot \ldots \cdot \bar{y}_m^{\Psi(X,\bar{y}_m)}, \\ m \in \mathbb{N}\end{gathered} \tag{29}$$

m being the index of the largest prime number from the multiplicative representation of X, whose exponent is nonzero, we can consider the infinite vector representation:

$$\begin{matrix} y_0 & y_1 & y_2 & y_3 & \cdots & y_m & \cdots \\ \Psi(X,y_0) & \Psi(X,y_1) & \Psi(X,y_2) & \Psi(X,y_3) & \cdots & \Psi(X,y_m) & \cdots \end{matrix} \tag{30}$$

or finite vector representation:

$$\begin{matrix} \bar{y}_0 & \bar{y}_1 & \bar{y}_2 & \bar{y}_3 & \cdots & \bar{y}_m \\ \Psi(X,\bar{y}_0) & \Psi(X,\bar{y}_1) & \Psi(X,\bar{y}_2) & \Psi(X,\bar{y}_3) & \cdots & \Psi(X,\bar{y}_m) \end{matrix} \tag{31}$$

In (30), the sequence of prime numbers (the first row of the matrix) and the string of exponents (the second row of the matrix) have an infinite number of terms, so they are infinitely dimensional vectors. In the representation (31) of the natural number X, both rows of the matrix are finite-dimensional vectors (with a finite number of components), but with possibly different dimensions from one number to another. Prime numbers generate 1-dimensional vectors, i.e. scalar. For example, the finite and infinite representations of the number 13 are given in (32), and the finite and infinite representations of the number 4056 are given in (33).

$$\begin{matrix} 13 \\ 1 \end{matrix}, \quad \begin{matrix} 2 & 3 & 5 & 7 & 11 & 13 & 17 & 19 & \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \end{matrix} \tag{32}$$

$$\begin{matrix} 2 & 3 & 13 \\ 3 & 1 & 2 \end{matrix}, \quad \begin{matrix} 2 & 3 & 5 & 7 & 11 & 13 & 17 & 19 & \\ 3 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & \cdots \end{matrix} \tag{33}$$

These representations have been addressed before, but not in terms of their calculation, in [2]. We have shown in [2] that the multiplication of natural numbers is easy to perform in the representation of type (30) or (31). According to the results obtained in this paper, the product or multiplication of two numbers in representations (30) or (31) can be expressed as a representation of the same type that has as the vector of prime numbers the vector formed by the reunion of prime number vectors corresponding to the representations. numbers that multiply, and as a vector of exponents, the sum of the corresponding vectors per component (i.e. a classical vector sum, which is related to the fact that the size of the vector can increase). For example, the sum of the two numbers with representations (32) and (33) has representation (34).

$$\begin{matrix} 13 & 313 \\ 1 & 1 \end{matrix}, \quad \begin{matrix} 2 & 3 & \ldots & 11 & 13 & 17 & \ldots & 311 & 313 & 317 & \ldots \\ 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 1 & 0 & \ldots \end{matrix} \tag{34}$$

To illustrate the concordance of the definitions and results obtained with the functions defined in this article, with the definitions and results in the literature, we give the following example. [9] defines the functions of the number of divisors of a given number, the sum of the divisors of a given number, and the product of the divisors of a given number. The number of divisors of a given natural number (considering also 1), is defined similarly to the sump function (noted τ in [9]) defined by (14):

$$numdiv1: \mathbb{R}^{\circ} \times \mathbb{N}^{\circ} \to \mathbb{N}^{\circ}, \quad numdiv1(x,y) = \sum_{y=1}^{x} w^{*}(x,y)\,,\ x \geq y \tag{35}$$

where:

$$w^{*}: \mathbb{N}^{\circ} \times \mathbb{N}^{\circ} \to \mathbb{N}, \quad w^{*}(x,y) = \begin{cases} 0, r(x,y) > 0 \\ 1, r(x,y) = 0 \end{cases} \tag{36}$$

The sum function of the divisors of a natural number x, noted σ in [9], considering also the number 1 is defined by the relations:

$$sump1: \mathbb{N}^{\circ} \times \mathbb{N}^{\circ} \to \mathbb{N}^{\circ}, \quad sump1(x,y) = \sum_{y=1}^{x} w^{**}(x,y)\,,\ x \geq y \tag{37}$$

where:

$$w^{**}: \mathbb{N}^{\circ} \times \mathbb{N}^{\circ} \to \mathbb{N}, \quad w^{**}(x,y) = \begin{cases} 0, r(x,y) > 0 \\ y, r(x,y) = 0 \end{cases} \tag{38}$$

The product function of the divisors of the natural number x, is defined as in (16) - (17), formally, instead of $y = 2$ we put $y = 1$, which, however, is a neutral element when multiplied. With these specifications, it is possible to make the numerical verification from [9]: $numdiv1(12)= 6$, $sump1(12)= 28$ and $prodp(12)= 1728$. Also as an application, is obtaining the function Euler indicator (which count the positive integers less than or equal to n and prime with it), [11], first defining the function:

$$E1: \mathbb{N}^{\circ} \times \mathbb{N}^{\circ} \to \mathbb{N}^{\circ}, \quad E1(x,y) = \sum_{k=2}^{\min(x,y)} e^{*}(x,y,k)\,,\ x \geq y \tag{39}$$

where

$$e^{*}: \mathbb{N}^{\circ} \times \mathbb{N}^{\circ} \times \mathbb{N}^{\circ} \to \mathbb{N}, \quad e^{*}(x,y,k) = \begin{cases} 0, r(x,k) > 0 \vee r(y,k) > 0 \\ 1, r(x,k) = 0 \wedge r(y,k) = 0 \end{cases} \tag{40}$$

The function of the Euler indicator is defined by the relation:

$$E: \mathbb{N}^{\circ} \to \mathbb{N}^{\circ}, \qquad E(x) = \sum_{k=2}^{x} e^{**}(x,k)\,, \tag{41}$$

where:

$$e^{**}: \mathbb{N}^{\circ} \times \mathbb{N}^{\circ} \to \mathbb{N}, \quad e^{**}(x,y) = \begin{cases} 0, E1(x,y) > 0 \\ y, E(x,y) = 0 \end{cases} \tag{42}$$

To compare the definition (39) - (42) of Euler's function with the definitions in the literature, we further reproduce only the form deduced in most literature, after [7, 12, 13,14, 15,16] for example:

$$\phi(m) = m \prod_{p/m} \left(1 - \frac{1}{p}\right) \tag{43}$$

where $m \in \mathbb{N}$ is the argument of the function, and the product is made after all the prime factors from the decomposition of the number m. In essence, many of the functions defined in this article are included as stages, in what many authors call algorithms for obtaining the values of some arithmetic functions, [17]. Since the string of prime numbers has a character, at least apparently, random, we considered interesting the characterization of this string by statistical functions, (12), (19) - (23). Such approaches exist in the literature, [18], [19].

5 Conclusion

The representations of the arithmetic functions present in this paper use the summation by explicit indices in relation to the formulas in the literature, which usually, before being introduced on the computer, still need a stage of explication of the summation index. Thus, the arithmetic functions became easier to introduce in the numerical calculation performed with computers.

The remainder function, a function of two variables, is deeply involved in defining many arithmetic functions. The remainder function is obtained from the theorem of division with remainder. This means that most of the problems of natural and integer numbers arise as a result of the introduction of inverse operations to the two fundamental operations, addition and multiplication, that is, subtraction and division. Reverse operations are those that give the first extensions to the set of natural numbers, the number 0, the negative integers and the rational numbers.

The representation of natural numbers in various numbering bases is common in the case of finite bases. The most common numbering base is 10. In the comments of this article, we presented a natural system of representation of natural numbers, whose base has an infinite number of elements, namely the set of prime numbers. Due to the multiplicative decomposition underlying the infinite representation of natural numbers, the operation of multiplying two numbers written in this base is very simple. However, the addition operation is complicated and we could not find an algorithm that does not go through finite representations (i.e. conversion from the infinite base of prime numbers to base 10 or another finite basis, known addition in this basis and then reconversion to the infinite basis of the prime numbers).

In addition to exploiting the forms proposed in this article for some arithmetic functions, for educational purposes, in the future one can try to use the same representations in order to study some properties of natural numbers and find others. One of the future directions remains the investigation of the existence of an algorithm as short as possible for calculating the addition of natural numbers in the infinite basis of multiplicative representation. Last but not least, there is the hope of obtaining new information about the distribution of prime numbers in the set of natural numbers.

References:

[1] Cardei P., A Representation of the Characteristic Function of the Prime Numbers, *Preprint*, DOI: 10.13140/RG.2.2.33888.12805, 2019.
[2] Cardei P., Numbers are not what they seem to be, *Preprint*, DOI: 10.13140/RG.2.2.35269.70888, 2019.
[3] Cardei P., Elemental formulas for calculating the values of some common functions in arithmetic, *Preprint*, DOI: 10.13140/RG.2.2.33994.70083, 2021.
[4] Bluman A.G., Pre-Algebra DeMYSTiFieD, *The McGraw-Hill Companies, Inc.*, 2004.
[5] Dedekind R., Theory of numbers, *Chicago The Open Court Publishing Company London Agents, Kegan Paul, Trench, Trubner & Co., Ltd.*, 1901.
[6] Niven I. M.; Zuckerman, H. S., An introduction to the theory of numbers (3rd Edition), *John Wiley & Sons*, ISBN 0-471-64154-5, 1972.
[7] Hardy G.H., Wright E.M., An introduction to the theory of numbers, *Oxford at the Clarendon Press*, 1975.
[8] Popovici C.P., Teoria numerelor, *Editura Didactica si Pedagogica, Bucuresti*, 1973.
[9] Tattersall J.J., Elementary Number Theory in Nine Chapters, *Cambridge University Press*, 1999.
[10] Andrei Gh., Caragea C., (2018), Partea intreaga [x] Partea fractionara {x}, vol. I, *Editura Paralela 45*, 2018.
[11] Euler L., Theoremata arithmetica nova methodo demonstrata, *Novi commentarii academiae scientarium imperialis Petropolitanae*, 8, 74-104, 1763.
[12] Le Veque W.J., Fundamentals of Number Theory, *Dover Publications, Inc. Neew York*, 1977.
[13] Shirali S., Yogananda C.S. (Editors), Number Theory, *University Press (India) Private Limited*, 2003.
[14] Moser L., An Introduction to the Theory of Numbers, *The Trillia Group, West Lafayette, Indiana*, USA, 2011.
[15] Bourbaki N., Elements of Mathematic Algebra I, Chapter 1 – 3, *Hermann Publishers*, 1974.
[16] Banks W.D., Ford K., Luca F., Pappalardi F., Shparlinski I.E., Values of the Euler Function in Various Sequences, *Monatshefte fur Mathematik, 146(1):1-19*, 2005.
[17] Stein W., Elementary Number Theory: Primes, Congruences, and Secrets, https://wstein.org/ent/ent.pdf, (last accessed, 22.04.2022), 2017.
[18] Feldman P.P., Prime Numbers and Their Statistical Properties, https://phillipmfeldman.org/mathematics/primes.html, 2020.
[19] Abdel-Mageed M., Salim A., Osamy W., Khedr A.M., A Study of statistical Properties of the Primes Numbers using the Classical and Superstatistical Random Matrix Theory, Advances in Mathematical Physics, 2021.