The vacuum as imaginary space. The unreasonable effectiveness of complex numbers.

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Abstract: - The background to the article is the classic and quantum understandings of the vacuum and the use of imaginary numbers in quantum models. The purpose of the article is to outline the possible understanding of the vacuum as imaginary space always coupled with the real space in the complex space of complex numbers. This understanding relates to the duality real-potential, collapsed–collapsible, and superimpositions of waves-phenomena as in quantum mechanics. The incomputability of the imaginary parts may represent the physical meaning of the permanent potential pending nature of the vacuum. The presence of imaginary numbers in models may be intended as warranty that it is not possible to compute definitive results, but it is possible to have pending multiple equivalences and superimpositions as in quantum physics and emergent collective processes in complexity. We consider how much the complexity (i.e., the study of emergence and chaos) may be considered related to and represented by complex numbers (i.e., properties of their dual variables and their collapsibility in real numbers). The usage of imaginary numbers may also be intended as the expression or manifestation of something we do not understand yet, as it was for the indemonstrability of the fifth Euclidian postulate and the unavailability of a distribution law for prime numbers. We conclude that a new global understanding is necessary and capable of explaining what we understand as the unreasonable effectiveness of complex numbers.

Key-Words: - Collapse, Computability, Emergence, Field, Imaginary, Matter, Quantum, Vacuum.

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1 Introduction

This article is finalized to consider concepts, problems, characteristics, and particularly possible representations of the vacuum in classical and quantum physics having aspects of compatibility with the complex space in mathematics using imaginary numbers.

The classic naïve representation of the vacuum, considered in section 2, reduces the vacuum to the emptiness of something specific up to the possible ideal lack of *everything*, which encounters conceptual inconsistencies such as the problem of the boundaries and coexistence between vacuum and non-vacuum. In contrast with this supposed absolutely nothing, the vacuum can be considered as relating to the *non-material* presence of implicit potential properties such as incompatibilities, metastability, and mutual exclusivity of alternative properties such as the uncertainty and complementary principles, remote synchronization, and the *theoretical incompleteness* necessary for processes of emergence. Emergence is indeed intended, in short, to be a continuous, irregular, undesigned, and unpredictable but also coherent multiple processes of acquisition of coherent, *new* -compared to those already owned-, *non-equivalent* -that is, not linearly convertible one to another- *properties* of complexity, such as topological and behavioral (e.g., the properties acquired by the climate system, collective behaviors of flocks and swarms, and whirlpools) [1].

We consider the concept of *vacuum domains* as having properties. Such properties are active independently by the materiality *observable*: they are *pending* properties ready to *collapse*/to *decide* between equivalences such as in bifurcations (changes in the topological structure of the system and in the number or type of attractors due to small, smooth changes in parameter values); to keep long-range correlations and remote synchronizations between elements without direct structural connections or intermediate mediating entities; and in quantum mechanics (QM) when the wave function initially in a superposition of several eigenstates collapses, it reduces to a single eigenstate as a consequence of the interaction with the external world. We may use the term *materialize*, and the vacuum is defined as the state with the lowest possible energy, namely the zero-point energy (ZPE). We consider the idea that under the usual naiveness attributed in the classical understanding of the vacuum, it may be possible to find problems, characteristics, and representations allowing conceptual forms of continuity between classic and non-classic physics, considered here representable by complex variables (see section 3) and relating to the duality real-potential, collapsed–collapsible, superimpositions of waves-phenomena, as in QM, and imaginary-real.

One of these possible continuities considered in the article relates to the representation of the vacuum as imaginary space of complex variables connecting real space with the usage of imaginary numbers in quantum physics and the consideration of imaginary time as in approaches to special relativity and QM.

Possible aspects of representations allowing conceptual forms of continuity between classic and non-classic physics may also be useful to represent processes of emergence (i.e., the discontinuity between acquired, non-equivalent properties) and how emergence emerges [1, 2] and, in case, to represent processes of phase transitions as considered in section 4.

In section 5, we present the results and concluding remarks on different issues such as how the presence of imaginary numbers and their theoretical incomputablity in models may be intended as *warranty* that it is not possible to have *objectivistic final* results, but it is possible to have multiple pending multiple equivalences and superimpositions as in quantum physics and emergent collective processes in complexity. We may have materialization only when there is collapse, as represented when the imaginary numbers become real through some computations and combinations. We mention how the phenomenological collapse can be represented but not reduced to computability, intended only as the *result* of a computation, as in correspondence with the similar incomputability of processes of emergence.

We conclude by mentioning some related research issues.

The purpose of the article is to outline the possible understanding of the vacuum as imaginary space always coupled with a non-imaginary space in the complex space of complex numbers. The real space as the real part of the dual complex space is composed of the real and the unavoidable pending, implicit, imaginary part representing the vacuum. *This representation relates to the duality real-potential, collapsed–collapsible, and superimpositions of waves-phenomena as in QM*. The incomputability of the imaginary part may represent the physical meaning of the permanent potential pending nature of the vacuum allowing superimpositions and potential properties. Another purpose is to outline the correspondence between quantum and emergence processes, between pervasive vacuum and environment. As a methodology we will start by considering how the vacuum has been considered in physics up to the modern quantum understanding. Considerations on the reality of incomputable and imaginary numbers follow. Finally, the vacuum is considered as an imaginary space (not only incomputable) in relation to the wide use of imaginary numbers in quantum physics. This is followed by a section of concluding remarks in which resulting understandings and research issues are dealt with.

2 A short overview of the classic vacuum

In this opening section, we mention some of the classic understanding of the vacuum.

The vacuum problem has been considered since ancient times, as by Aristotle, Democritus, and Leucippus.

The theme was then considered by Galilei (1564–1642) and his school until the realization of 'Torricelli's famous experiment in 1644.

We mention the corresponding idea of the existence of the ether supposed as having the character of an immovable material substance in absolute space. As is well known, this idea was definitively refuted by the experiments of J.C. Maxwell (1831–1879) and A.A. Michelson (1852–1931).

We will then come to Rutherford (1871–1937) and his famous experiment in 1909, relevant that the atom is largely made of a vacuum and his introducing the idea of matter as discontinuous.

We mention now some specific cases.

- The vacuum represented by *total absence* is conceptually fragile, raising questions such as those related to the edges of vacuum, ignoring its pervasiveness, and assuming and admitting its possible locality, its *observability*, its relation with implicit, potential, and pending properties (see point 2.6) such as for metastability. How can the vacuum coexist with the non-vacuum? There is coexistence in first-order phase transitions (e.g., among phases such as liquid and gas phase of matter), whereas there is not in second-order phase transitions (e.g., among phases such as ferromagnetic and magnetic). What coexistence, if any, exists between the vacuum and non-vacuum? Where does one begin and the other end? We should consider *open intervals* of the vacuum and the non-vacuum, such as between irrational numbers.

- We mention the related concept of *empty set* considered in mathematics. However, the concept of the empty set suffers from paradoxes (e.g., the empty set may contain another empty set and can be introduced in different ways). For instance, by using a semantic definition, a rule to determine what the elements are and membership rules that allow us to decide whether an element belongs to the set or not (the rule can be even a list of possible elements). Set theory may be defined by axioms; however, according to 'Gödel's incompleteness theorems, it is not possible to demonstrate that axiomatic set theory is free from paradox. The empty set is usually denoted by the symbol $\varnothing$. It is supposed to state that the sets we were considering include an empty one *of them*. Then we can use it in computations as the zero in number theory. The empty set seems to be a *generic zero set* unavoidably specified in *any* collection of specific sets. It seems to make sense if it is the zero of a collection of sets. The empty set is both open and closed for any set and topology. In fact, the empty set is open by definition in any topological space because the complement of an open set is closed. Moreover, the closure of the empty set is empty.
- In thermodynamics, in the nineteenth century it was believed that a degree of temperature corresponds to a well-defined amount of energy supplied. It could reasonably be assumed that it would always take the same amount of energy to produce a one-degree change in temperature. The German chemist W.H. Nernst (1864–1941), thanks to the progress of cryogenics, in 1906 measured the specific heats of substances up to temperatures of the order of 5 absolute degrees. He noted that the specific heats, far from being constant, became smaller and smaller as the temperature decreased. Therefore, it had to be explained how it was possible to make the same energy leap (i.e., raising the temperature by one degree) by supplying smaller and smaller quantities of energy from the outside. It was necessary to identify a physical subject, different from the material bodies in question, capable of supplying the missing supplement of energy. Nernst affirmed that this subject was the vacuum, not coinciding with nothing but intended as a physical entity, not analyzable into atoms and not separable from bodies, but, however, able to influence the temperature.
- In chemistry, we mention how subsequent sequences of dilutions of chemical elements lead to dilutions where the chemical initial element is no longer present as stated by the Avogadro (1776–1856) number. In relation to the diluted chemical product, the vacuum, intended as the total absence of the initial element, is reached. The diluent passes from being gradually predominant to being the only entity present and detectable. The pervasiveness of the diluent corresponds conceptually to that of the vacuum. "In chemistry, the limit for high dilution is represented by Avogadro's number. However, there is an intense debate about possible properties acquired by elements diluted beyond Avogadro's number, studied by the physics of high dilutions" [3, pp. 342-349]. The topic, once accepted conceptually, is explored by considering approaches of quantum physics to explain the properties of high dilutions of matter [4-6].
- The appearance of the vacuum as a physical object (i.e., a special unavoidable environment) undermined the concept of *isolated* body at its root, leading to the *entanglement* considered in quantum physics. In this regard, we mention the ideal correspondence between the vacuum and the environment. "Regarding the separability of systems from the environment, a simple example of the inapplicability of this assumption is given by ecosystems where the differentiation between external and internal is unsuitable. In these cases, the environment pervades the elements which produce, in their turn, an active environment. This environment, if we can still call it such, is active and not an amorphous, abstract space-hosting processes. It is interesting to consider eventual conceptual correspondences with the quantum vacuum pervading everything." [7, p. 13]. This description also corresponds to the concept of multiple systems when the same components establish at the same time different, superimposed systems [7, pp. 161-170]. We also mention the concept of *systems propagation* related, for instance, to "synchronizations and remote synchronizations occurring when nonadjacent pairs of entities become substantially synchronized in spite of the absence of direct structural connections between them or intermediate mediating entities such as in the brain and networks [8, 9]; and those belonging to the basin of an attractor." [10]. Furthermore, we may consider the exchange of information without direct exchange between elements of collective behaviors keeping coherences such as in swarms

and flocks in long-range correlations [7, pp. 271-272].

The Aristotelian *horror vacui* has long since been abandoned, accepting the possible pervasive but also *local* nature of the vacuum.

- Another way to consider the vacuum is as an ideal place of implicit potentialities ready to materialize/collapse (see point 2.8) in non-equivalent real material events. Examples are given by meta-stabilities and phase transitions. Furthermore, collective conditions of behavioral adequacy, admissibility, compatibility, equivalence, and interchangeability of agents make possible, induce, and facilitate long-range correlations and the emergence of coherent collective behaviors. The occurrence of such collective conditions establishes immaterial *domains* influencing any entering entities and consisting of real dynamic degrees of freedom, however *immaterially prescribed* such as in ecosystems and remote synchronization. We may consider these domains as properties of active vacuums, pervading the collective system. Pre-established environmental conditions may be considered as the implicit pre-existence of domains, completely different, for instance, from real, material fields of physics. This also relates to singularities in catastrophe theory, high sensitivity to initial conditions, and change of attractors in chaos theory. The entering by an entity in these *vacuum domains*, implicitly full of unlimited but specific possibilities allowed by degrees of freedom and constraints, involves it being significantly affected. In short, we consider *vacuum immaterial domains* [10].
- It is possible to deduce and suppose the existence of the vacuum as a strange physical entity [11] that is difficult to detect and measure but nevertheless indisputable as it is for quantum physics. In some approaches, it seems that the vacuum is considered as a physical entity lacking measurability but allowed to have quasi-localization (such as quasi-particles), no significant edges, and low detectability and being constant over time. We may consider issues such as the local and general percentages of vacuum versus the non-vacuum, the variability and the changing nature of the balance and ways to change it, the possible invariance of the vacuum or the replacing dynamics between vacuum and non-vacuum, the possibility of *transferring* the vacuum, the possibility of *differentiating* between vacuums, and how the vacuum takes place. Should we consider only one kind of vacuum, that is, are all vacuums equivalent? Is it possible to differentiate and transform one kind of vacuum from another? Is it possible to consider the *quasi-vacuum*, a situation of unstable dynamics between vacuum and non-vacuum?

As we know, most answers have come from quantum physics.

We conclude this section by mentioning the concept of *collapse* of the vacuum, that is, in short, the shift from implicit, potential, pre-existing domains, pre-phase transitions, configurations, and metastable states into detectable and measurable effects. This is in conceptual correspondence, in quantum physics, with conceptual collapse, for instance, of a wave—in a superposition of several eigenstates (possible values of the *observable*)—that then collapses, reduces to a single eigenstate. About the *observable*, we mention that in classical physics, almost any quantity may be considered observable (e.g., energy, mass, and momentum), starting from the introduction of electromagnetism the obviousness of the situation changed. As a matter of fact, quantities considered in electromagnetism (e.g., fields and potentials) are not directly measurable. With the introduction of QM, the concept of observable is further fuzzified because, over and above the conceptual measurement limits imposed by Heisenberg's uncertainty principle, some fundamental quantities introduced by QM are not only not observable but are not even real quantities and are described using complex numbers. The idea is that matter/particles can be intended as excited states of the underlying quantum vacuum [12]. Furthermore, the properties of matter can be intended as vacuum fluctuations arising from interactions of the zero-point field [13] and macroscopic manifestations of quantum field theory (QFT) [14], which differently from QM (see section 4.2) considers particles as excited states of their underlying quantum fields as in statistical field theory. Furthermore, a quantum system and the external environment may interact in such a way as to destroy the quantum coherence. When the *decoherence* (by which exposure to and entanglement with any macroscopic environment converts quantum information into classical information) *time* is short enough, macroscopic coherence due only to QM becomes unobservable. The states that are a superposition of basic states can no longer exist, because the interaction with the environment selects, decides one particular basic state among all the various possible ones, and then the system falls into it with a probability equal to *1*, and its dynamics loses its quantum character. This limits the effectiveness of QM to specific cases (e.g., the world of atoms and molecules, very low temperatures, etc.) [15, pp 230-239].

3 On the reality of incomputable and imaginary numbers

Reality in mathematics is mostly understood as *effective computability*. Effective computable numbers are intended to be real numbers [16] that can be computed by a Turing machine terminating in a finite amount of time and with arbitrary precision, where the Turing machine is specified as a quadruple $T = (Q, \Sigma, s, \delta)$ where Q is a finite set of states q_i; Σ is a finite set of symbols s_j, (e.g., an alphabet); s is the initial state $s \subset Q$, being Q the set of all the possible states; and δ is a transition function that determines the next acquired state occurring from computation state q_i to computation state q_{i+1} in finite time and with arbitrary finite precision. Different versions of the Turing machine are all computationally equivalent [17].

3.1 Incomputability

Real numbers that are incomputable, due to the unavailability of an algorithm that computes in finite time and with arbitrary precision, include algebraical irrational, trigonometric (based on Euler's formula), and transcendent numbers, endless processes of convergence, and some special numbers, such as the so-called 'Chaitin omega number' [18]. Here, we term this incomputability as *non-Turing computability*.

Furthermore, there are forms of *non-explicit computability*, that is non-symbolic computability, such as for artificial neural networks (ANN) and cellular automata (CA) performing emergent computation [19] and dealing with the reality of incomputable real numbers [20].

We consider here the incomputability of imaginary numbers as *theoretical incomputability*, for which no resolutive procedure is conceivable/admissible, much less a Turing machine. We may term this theoretical incomputability as *t-incomputability*. This is matter of numbers containing the imaginary unity $i = \sqrt[2]{-1}$ such as in physical equations, which are actually anything but imaginary in the common sense. More precisely,

- In the case of non-Turing computability, the existence of a hypothetical Turing machine is admissible, but it is not effectively available because at least one of the definitory requests listed above is not satisfied (e.g., to end the computation in finite time and with arbitrary finite precision). This issue is related to irrational numbers. The partial Turing computability is, however, admitted, that is, admitting operative, acceptable approximations occurring in finite time (i.e., renouncing to arbitrary finite precision in finite time).
- In the case of non-explicit computability, the computational processing is non-analytically representable, and that is why it is called sub-symbolic, whereas the computational process is performed by an explicit computable algorithm. The computational process's whole set of weights and levels used in ANN and transition rules used in CA cannot be zipped [19] (i.e., analytically represented into individual general formulae or functions), being instead a dynamical computational process to be subsequentially and completely performed to reach the result.
- In the case of t-incomputability, the existence of whatsoever resolutive procedure is not admissible, such as for i. However, the t-incomputability also relates to physical phenomena and the unpredictability, indeterminism, and randomness of measurements such as for 'Heisenberg's uncertainty principle, whose equations, coincidentally, use imaginary numbers. This condition relates in general to situations of quasi-ness and theoretical incompleteness, incompletability, uniqueness, and equivalences [3] when, for instance, a collective system is not always a system, not always the same system, and not only a system in the dynamics of maintaining, losing, and resuming variable levels of coherence.

3.2 Complex numbers

It is well known that every complex number, see Figure 1, z_i can be expressed in the form $x + iy$, where x and y are real numbers. Each complex number can be then represented by the couple $(x, y) \subset R \times R$. C is the set of complex numbers and is the plane $R \times R = R^2$ equipped with *complex addition* and *complex multiplication* making it the *complex field*.

The n-dimensional complex coordinate space is the set of all ordered n-tuples of complex numbers. It is denoted C^n, and is the n-fold Cartesian product of the complex plane C with itself. Symbolically,

$$C^n = (z_1, \ldots, z_n), \text{ where } z_i \subset C. \tag{1}$$

In the Cartesian plane, the point (x, y) can also be represented in polar coordinates such as the following

$$(x, y) = (r \cdot cos\theta, r \cdot sin\theta) \tag{2}$$

where the module r and the phase θ are obtained from the formulas

$$r = \sqrt{x^2 + y^2}\ ;\ \theta = arctan\,\frac{y}{x}. \tag{3}$$

In mathematics, a first bridge between imaginary numbers and their representations in the plane is allowed by the so-called Euler identity, which relates e, i, π, 1, and 0:

$$e^{i\pi} + 1 = 0. \quad (4)$$

The geometric interpretation of the formula allows complex numbers to be viewed as points in the plane. Furthermore, Euler's formula states that, for any real number x, we have

$$e^{ix} = \cos x + i \sin x \quad (5)$$

where e is the base of the natural logarithms, i is the imaginary unit, and *sine* and *cosine* are trigonometric functions.

C Complex numbers with the form $x + iy$, where x and y are real numbers and i is the imaginary unit, imaginary solution to the equation $x^2 = -1$. **C** is not orderable.

R Real numbers are not all algebraic (algebraic irrational are obtained as solutions to polynomial equations with integer coefficients, e.g., $x^2 - 2 = 0$).

R = Q U I, R ⊂ C

Q Rational numbers,
- expressible as a ratio of two integers, that is, by means of a fraction.

U

I Irrational numbers,
- all and only algebraic, non-periodic, unlimited, decimal numbers, therefore, not expressible by means of a fraction;
- transcendent numbers are irrational but not algebraic numbers, i.e., they are not solutions to any polynomial equation, e.g., the Euler number e and π.

Furthermore, irrational, decimal unlimited numbers are not computable (in a finite amount of time).

Fig. 1. Complex numbers

We note that C and R have the same cardinality. Therefore, a bijection $f: C \rightarrow R$ is possible.

We consider now the 'collapse' of complex numbers into real numbers as the result of proper computations prescribed by equations and formulas. Cases of such collapsing, that is *mutation* of imaginary numbers into real numbers, are given by

1. The squaring of complex variables, when $(iy)^k$ with $k = 2n$, n real integer.
2. The effects of the conjugation of two complex numbers, z and its conjugate $\bar{z}$. Let $z = x + iy$. Then considering $\bar{z} = x - iy$, one is the conjugate of the other. The equality of two conjugate complex numbers $x + iy = x - iy$ implies that $iy = -iy$ (i.e., $y = 0$).
3. The effect given by the sum of two conjugate complex numbers $(x + iy) + (x - iy) = 2x$.
4. The product between two conjugate complex numbers $(x + iy) \cdot (x - iy) = x^2 - i^2y^2 = x^2 + y^2$.

A complex variable function is defined as a function defined on a subset of the complex numbers having values in this same set. In complex analysis, there is the study of the theory of functions of several complex variables.

We conclude this section by mentioning how the collapsing of imaginary numbers into real numbers in some way corresponds to the collapsing of some Turing-incomputable real numbers into computable ones. The latter is the case, for instance, for all non-computable roots of numbers that are not exact powers, that is, $(\sqrt[2]{x})^2$, where x is not a power of two, as instead it is in the case, for instance, of $\sqrt[2]{x^2}$. Where $(\sqrt[2]{x})^2$ and $\sqrt[2]{x^2}$ are formally but not computationally equivalent because $(\sqrt[2]{x})^2 = \sqrt[2]{x} \cdot \sqrt[2]{x} \neq \sqrt[2]{x^2}$, as for $\sqrt[2]{2} = 1.41421356237\ldots \cdot \sqrt[2]{2} = 1.41421356237\ldots = 1.99999999999\ldots \neq \sqrt[2]{2^2} = 2$.

Correspondingly, the computation of $\sqrt[2]{-1}$ is impossible, and $(\sqrt[2]{x})^2$ is also impossible and incomputable when x is not a power of two. When considering this problem formally and not computationally, $\sqrt[2]{-1} = i$, then $(\sqrt[2]{-1})^2 = i^2 = -1$.

Computation is mostly symbolic calculus that delays as much as possible, as optional or after assigning values to the variables, effective numerical computation. In the case of imaginary numbers, this last step is impossible/not feasible.

In analogy with the Schrödinger's cat having superimposed the states of being dead and alive, computation may be like a metaphorical card game that does not always end by *seeing*, but often, after *changing* cards, by *passing* or *raising*. With imaginary numbers, we can only play by passing, raising, and never seeing. However, we still do play.

The *possible* mutation between incomputable, non-Turing solvable to computable, for instance through exponentiation, may be intended to have prevalent if not only mathematical aspects, whereas the mutation between imaginary to non-imaginary may be intended to have a significant physical meaning/interpretation due to its use in models and physical equations, as mentioned below. In this regard, *we stress that we do not speak of the space of non-Turing computable numbers, while we speak of the space of t-incomputable complex numbers*.

4 The vacuum as imaginary space

How can the imaginary space C^n be related to the vacuum and the collapsing of C^n in R^n related to the collapsing of the vacuum into detectable and measurable effects?

In the quantum physics literature, the quantum vacuum is intended as an entity that precedes matter, so it must also precede space and time.

This situation is related to models of quantum physics considering that it is the quantum vacuum giving properties to matter, such as that of being always connected, and not a lack of matter being the vacuum [12, 13]. Imaginary, complex variables are regularly used in quantum models (see below). However, it seems there are no fully developed theoretical reasons for this usage.

An initial possible generalizing idea may be to represent the vacuum as a general t-incomputable domain of possibilities specified by the imaginary space, with complex variables, imaginary models (i.e., models using imaginary variables), and collapsing mechanisms that may be represented by suitable symbolic collapse to turn imaginary numbers into real ones (see point 'a' in section 5.1). Furthermore, non-imaginary models may be intended as particular complex models having the imaginary part equal to zero.

As mentioned at the end of the previous section, we focus here on incomputability as *t-incomputability*. The *t-incomputability* identifies a space compatible with research approaches considering/implementing imaginary and non-imaginary models for the classic vacuum, for instance imaginary attractors (see, for instance [21-23]) considered in chaos theory; t-incomputable dependence from initial conditions; multiplicity of domains in ecosystems with effects of systems propagation, remote synchronizations, and long-range correlation; metastability; phenomena represented with complex random probabilities; conditions for the occurrence of symmetry-breaking (when a symmetry transformation leaves invariant the form of the evolution equations but changes the form of their solutions) in phase transitions; and complex variables as in statistics.

For instance, regarding the last case, *complex random variables* considered in statistics and probability theory generalize real-valued random variables to complex numbers, that is, the possible values of complex random variables may take complex numbers. Complex random variables can then be considered as pairs of real random variables corresponding to their real and imaginary parts. Accordingly, the distribution of one complex random variable is intended as the joint distribution of two real random variables. Complex random variables are used in digital signal processing (e.g., biomedical, and information theory) [24-26].

On the other hand, we may consider phenomena and processes of complexity by which there is an acquisition by systems of properties non-equivalent to those already possessed such as in self-organization and emergence (i.e., conceptual *mutations*). We may ask how much the science of complexity (i.e., the study of emergence and the *emergence of emergence*) [1] and chaos [27] may be considered to be related to and representable by complex numbers (i.e., properties of their dual variables and their collapsibility into real numbers) [28-31]. How much is emergence *non-Turing computable* and *t-incomputable*? Is complexity *non-Turing computable* or *t-incomputable*? Probably, we should consider a mix of possibilities, such as in the occurrence of superconductivity and superfluidity, transitions from paramagnetic to ferromagnetic phases, and some order-disorder transitions, which have a mix of very complicated transient dynamics and classical and quantum aspects [32, 33].

4.1 The quantum case

In QM [34, 35], the crucial idea is of non-commutativity, when the position and velocity of a particle (at the subatomic scale) are non-commuting. QM operates with manifolds of quantities, such as matrices. As for imaginary models in this conceptual context, we may refer to their large usage in quantum physics.

As is well known, matrix mechanics is a formulation of QM interpreting the physical properties of particles as matrices that evolve in time. This is an alternative to using usual scalar values and then replacing the classic continuity with discretization (i.e., possible *admissible* values).

In matrix mechanics intended as a formulation of QM observables, when considering pairs of observables an important quantity is the commutator. For instance, for a pair of operators $\hat{A}$ and $\hat{B}$, one defines their commutator as

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}. \quad (6)$$

In the case of position and momentum considered by Heisenberg's uncertainty principle, the commutator is the canonical commutation relation using *i*:

$$[\hat{x}, \hat{p}] = i\hbar. \quad (7)$$

We may figure out the physical meaning of the non-commutativity when considering the effect of the commutator on the position and momentum eigenstates.

For instance, in a simplified, conceptual version of (7),

$$PM\text{-}MP = i\hbar, \text{ where} \quad (8)$$

- *P matrix* – is the possible, admissible positions of a particle;
- *M matrix* – is the possible, admissible momentum of the particle, equal to its mass times its velocity ($PM \neq MP$ because they are matrices);
- i – imaginary number, $\sqrt[2]{-1}$;
- $\hbar$ – is the Planck constant $\frac{h}{2\pi}$.

Examples of other usages of *i* relate to *imaginary time* considered in approaches, for instance, based on special relativity.

We mention that in QM, the wave function $\Psi(x, t)$ is a complex variable function, as in the Schrödinger equation using imaginary numbers.

Furthermore, there are several kinds of relationships between traditional and non-traditional models (see Table 1).

4.2 Quantum versus stochastic

Different approaches are available in the literature dealing with the role of complex numbers in modeling quantum versus stochastic processes, such as [36]. In this regard, we mention the case of the so-called *Fokker–Planck equation* (FPE), [37] initially introduced for the statistical description of the Brownian motion of a particle in a fluid [38]. The FPE is a partial differential equation in the unknown function *P*. Such an equation describes the time evolution of probability in the presence of random noise.We may consider $P(x, t)$ as the probability of transition of an element from an initial value x_0, present at time $t = 0$, to a value x at time t. This represents the probability that the stochastic process $x(t)$ gives rise, at time t, just to the value x. It has been shown that, if the stochastic process obeys the stochastic differential equation:

$$dx/dt = g(x) + h(x)\ \xi(t) \qquad (9)$$

where $\xi(t)$ is a Gaussian white noise process (stationary stochastic process whose spectral power density is constant over all frequencies considered for mathematical convenience), then the associated transition probability $P(x, t)$ is represented by the so-called *Fokker-Planck equation* (FPE) considered in [39; 7, pp. 273-279]:

$$\partial_t P = -\partial_x [g(x)\ P] + (\sigma^2/2)\ \partial_x \{h(x)\ \partial_x [h(x)\ P]\} \qquad (10)$$

where:

- $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$;
- σ^2 is the noise intensity;
- The FPE is a partial differential equation in the unknown function *P*.

Such equation describes the time evolution of probability in the presence of random noise.

Furthermore, it has been observed that the FPE [15, pp. 273-279; 39] has a *form* strongly resembling that of the Schrödinger equation used in QM. It is well known that the two can be made identical by introducing an imaginary time given by $\tau = i \cdot t$ into the Schrödinger equation, which immediately becomes a sort of FPE such as in [40, 41, pp. 252-253] and in equations (42–43) in [42].

This transformation is a mathematical trick, but physical interpretations are necessary. We mention the possibility of considering as a physical interpretation a hypothesis introduced in the 1970s as in [37]. It has been mathematically shown how the formalism of QM could be used to describe a system using classical mechanics and embed it into a stochastic, noisy background. The probabilistic features of QM can be then intended as a consequence of the fact that the ground state of the universe is merely a noisy state, and this prevents the existence of truly deterministic phenomena [15, pp. 279-281]. Furthermore, it has been demonstrated how a stochastic many-body system could be modeled by using the formalism of the so-called *second quantization* [43], one of the main technical tools of QFT. Due to the probabilistic features of QM and QFT, the latter, contrarily to what happens in QM, assumes that the main physical entities are fields (of force) and not particles [7, p. 36; 11, pp. 230-239]. From this line of research arose the so-called *statistical field theory* [44-47].

At this point, we notice the central probabilistic character of —especially quantum— so-called physical 'laws', when considering implicit, unaware, but effective, approximations and interpolations [48].

Results as the ones mentioned above support the idea that, from a formal point of view involving or not imaginary numbers and complex variables, a number of stochastic models could be suitably reformulated in such a way to transform in QFT-based models (see below for the difference between QM and QFT). Moreover, this requires to redefine in a suitable way the *Planck constant* of the system. Once the presence of the three fundamentals ingredients, i.e., non-linearity, spatial extension and fluctuations, for the occurring of *radical emergence*, i.e., when the new system's phase requires a new description level for its behaviors, such as for the protein folding, acquisition of superconductivity, and superfluidity, has been granted, then the theories can be found equivalent to one another, at least with regard to their formal structure. This possibly considering imaginary variables such as imaginary time, imaginary mass and imaginary dimension. Accordingly, a non-ideal model endowed with noisy fluctuations, should have good probabilities of being equivalent to a QFT model, without the need for quantizing it [7, pp. 245-279].

Ideal models characterized by a top-down structure, based on general assumptions assumed to be largely valid and then covering the widest possible spectrum of phenomena. This feature allows to deduce particular consequences and to forecast only if suitable mathematical tools are available.	Non-ideal characterized by a bottom-up approaches, based on opposite assumptions considering 'lucky' choices and studied through computer simulations.
Examples	Examples
Chaos	Dissipative structures
Noise-induced phase transitions	Cellular Automata
Spontaneous Symmetry Breaking in Quantum Field Theory	Agent-based models
Network Science (ideal scale-free networks)	Artificial Life

We distinguish between

Homogeneity-based models neglecting any differences between the components and treat them all being equivalent to one another
Heterogeneity-based models when components are distinguishable

Table. 1 - Some hints at the relationship between traditional and non-traditional, ideal and non-ideal models.

We conclude this section by summing up how "Emergence could be intended as coming first, as a property of pre-matter, of the vacuum. The quantum vacuum could thus be intended as a kind of field of potentialities ready to collapse but always pervasive as are the probabilistic features of QM…" [7, p. 130].

5 Concluding remarks

Could the imaginary units introduced by René Descartes (1596–1650) in the treatise *La Dioptrique, Les Météores, et La Géométrie* first published in 1637, subsequently elaborated by mathematicians such as Euler, Gauss, and Cardano (the first to introduce complex numbers into algebra) represent the conceptual and physical distance between the classic and non-classic, quantum representations?

If *i* were not available, several equations would be impossible to write and symbolically solve, intractable, and meaningless.

When dealing with real, *non-Turing computable* numbers, the strategy is to process them symbolically and postpone as much as possible the computation requiring necessary levels of acceptable approximation.

This is not suitable for *t-incomputable* imaginary numbers only symbolically treatable ones. Complex numbers may then be partially computable, only for their real part.

Going back to the vacuum, we may consider it as imaginary space always coupled with the non-imaginary space in the complex space of complex numbers. In other words, the real space is a real part of the dual complex space composed of real and imaginary parts (particular cases occur when the imaginary part is zero or when the real part is zero). *This is intendable related to the duality real-potential, collapsed–collapsible, and superimpositions of something such as /waves/phenomena/planes/states/spaces] as in QM.*

However, by considering the *pervasiveness* of the vacuum in physical models, as introduced above, the first possibility (i.e., imaginary part as zero) would be considered unlikely.

Furthermore, this intrinsic duality could represent the perennial, intrinsic state of implicit interaction and emergence of the so-called 'matter' ready to collapse into entities acquiring properties such as the mass (mass intended as vacuum collapsed). This relates, for instance, to the fact that in QFT [49]

"It is the quantum vacuum giving properties to matter, such as that of being always connected, and not a lack of matter being the vacuum. The approach based on considering material entities as fields (of force) and not as particles has a long tradition in physics, from Faraday and Maxwell, and onwards to general relativity. Within this conceptual framework, the concept of particle is considered to denote regions of space where a field has a particularly high intensity. The subject of such matter considered as a condensation of emergent properties acquired by the quantum vacuum will be considered below. Higher levels of emergence acquire properties, …, such as dimensionality, weight, volume and mass." [7, p. 54].

Based on the discussion above, we conclude that it is possible to consider the usage of the mathematics of complex variables not only in (typically quantum) physics but also to consider and represent the implicit properties of matter and the dualities between classic and non-classic approaches.

5.1 Resulting understandings

It is interesting to compare the possible physical meanings of the two corresponding collapsing processes from imaginary, *t-incomputable* to real and from *non-Turing computable* to Turing computable. We considered how these cases occur particularly in equations of quantum physics. The symmetric reverse processes (i.e., from real to *t-incomputable* imaginary and from computable to *non-Turing computable*) seem less promising of physical significance but have prevalent computational aspects instead.

- The presence of imaginary numbers and their t-incomputablity in models may be intended as *guarantee* that it is not possible to *see* (as in an imaginary card game with Nature), but it is possible to have *pending multiple equivalences and superimpositions as in quantum physics and emergent collective processes in complexity*. We can *see* only when there is *collapse* (see the following point 5.1.2), as represented in simple cases when the imaginary number becomes real through some computations and combinations such as in the occurrences 1–4 considered in section 3.
- We consider the *phenomenological* nature of the quantum collapsing of a wave function as a superposition of several eigenstates, reducing to a single one as an effect of the interaction with the external world. We say *phenomenological* only ideally *computable*, a contrast that is, incidentally, also valid for the occurrence of processes of emergence. The occurrence of phenomenological incomputability is mostly represented by the t-incomputablity of models using imaginary numbers. On the other hand, the phenomenological collapsing is *represented, corresponding even if not modeled* by the reducing of t-incomputabilities into computability due, for instance, to the occurrence of computational combinations with complex random variables when modeling unpredictability, indeterminism, and randomness of measurements of physical phenomena [50]. A similar situation occurs with processes of emergence intrinsically, theoretically incomputable and requiring incompleteness until interactions with the environment lead to the collapse of incompleteness, understood as equivalences, to the establishment of configurations endowed with variable and dynamic coherences [1, 7, 19]. However, the incomputability considered does not affect the possibilities of simulation, at a level of description under the responsibility of the researcher, considering parametrically definable cases and configurations.

5.2 Research issues

Interdisciplinary research could occur *within* the same discipline (i.e., within physics) allowing interchanged usage of approaches, redefinitions of variables, meanings of constants, elaborate analogies, and correspondences between non-quantum and quantum modeling as considered in Table 1. This could contribute to approaches and concepts allowing forms of unification resulting in a unified, more general theory. However, multiple, non-equivalent superimposed representations is a feature of QM, as in the duality real-potential and collapsed–collapsible, conceptually transposable to complexity for processes of emergence and multiple, pending, or actual systems of ecosystems.

It may be interesting to research the complexity of models of classical physics generalizable through the usage of complex variables as in the cases mentioned and for the FPE. We may research how much the complexity (i.e., the study of emergence and the *emergence of emergence* and chaos) may be considered as being related to and representable by complex numbers (i.e., properties of their dual variables and their collapsibility in real numbers). How much emergence is *non-Turing computable* and/or *t-incomputable*?

Properties of complex numbers and of mechanisms of mutation in real numbers may be intended to conceptually correspond closely to models of collapse in physics. The mathematical trick often used to replace variables with *i*, as for the FPE, considering complex variables and imaginary time, should be given physical interpretations, probably all related to probability and superimposed dualities intrinsically open to collapse and the impossibility to *see*.

- However, even the possible or limited possibility of finding physical meanings should be interpreted (i.e., *it is curious that the issue comes after modeling, as result, and not before as an assumption: we do not start by considering complex numbers*).
- We may also consider the Turing-incomputability of real numbers [51], as such or of real parts of complex numbers, and the t-incomputability as

representations of *quasi-ness*, continuous negotiation between extremes, and of *theoretical incompleteness*, as necessary ingredients for the occurrence of equivalences to be collapsed in emergence and complexity, as in [3, 7, 52].

However, it is always a matter of the *usage* of incomputable, real, or complex results.

Paraphrasing 'the unreasonable effectiveness of mathematics' introduced by Wigner [53] and elaborated, for instance, in [54], we may consider *the unreasonable effectiveness of complex numbers*. In this regard, we mention how difficult it usually is to model complex phenomena. In some cases, as for the usage of imaginary, complex numbers, the case seems to have reverse aspects. We use imaginary, complex numbers to modify models, sometimes as a mathematical trick, or we accept models *implying* imaginary, complex numbers, and *then* we look for the physical meaning.

- It may be intended as if the usage of imaginary, complex numbers is not only *guarantee that it is not possible to compute definitive results, having pending multiple equivalences and superimpositions only phenomenologically solvable as in quantum physics and emergent collective processes in complexity*, but also the expression, manifestation of something we do not understand yet. It may be related to the impossibility in mathematics to demonstrate inconsistent issues however initially assumed to be admissible, if not evident. Such a situation is considered as evidence of an *intrinsic* consistency/logical robustness, contrasting with the issues assumed to be evident but, however, resistant to any attempt of demonstration. In such cases, the issue to be considered is the impossibility to demonstrate the supposed evident, as signal indeed that something unexpected must be realized, incompatible with the demonstration pursued. There are also significant methodological aspects. The typical example is the impossibility to demonstrate that in Euclidian geometry only one line parallel to another one passes through the same point (the fifth postulate). Such an impossibility opened the doors to non-Euclidian geometry revealing that what was supposed to be evident and then provable was inconsistent/incoherent.

Another case is the impossibility of finding a complete general explicit distribution law of prime numbers, only asymptotically approximated in the prime number theorem (PNT) proved independently by Hadamard and Poussin in 1896 using ideas introduced by Riemann, that is, his zeta function. Such impossibility has been considered when considering what is computable/decidable and what is not. Indeed, the possibility of finding such a distribution was intended to be equivalent to the availability of an impossible algorithm…

"…able to compute the general properties of the presumed primes' distribution law *without computing such distribution*. The link between the conceptual availability of a distribution law of primes and decidability is given by considering how to decide whether a number is prime without computing. *Factorial properties of numbers, such as their property of primality, require their factorization (or equivalent, e.g., the sieves), that is, effective computing. However, we have factorization techniques available, but there are no (non-quantum) known algorithms that can effectively factor arbitrary large integers. Then, factorization is undecidable. We consider the theoretical unavailability of a distribution law for factorial properties, as being prime, equivalent to its non-computability/undecidability*." [55].

The solvability of the problem of finding the hypothetical law of a distribution of prime numbers is then considered as undecidable (i.e., non-Turing computable) and equivalent to the availability of a general algorithm of factorization. *When the numbers are sufficiently large, no efficient integer factorization algorithm is known. Such non-efficiency in the face of unlimited large numbers is then de facto equivalent to the non-Turing computability, because it admits computability in finite but, however, non-limited time*. This actually rules out effective computability even though it has not been proven that such an efficient integer factorization algorithm does not exist.

However, there are new approaches and results that may allow us to consider the problem from new points of view such as by considering the existence of bounded gaps between primes [56] proving that

$$\liminf_{n\to\infty} (p_{n+1} - p_n) < 7 \times 10^7, \text{ where } p_n \text{ is the } n\text{-th prime} \quad (11)$$

and primes in tuples (i.e., proving that consecutive primes exist that are closer than any arbitrarily small multiple of the average spacing) [57, 58]. Furthermore, works on the still unreached proof of the Riemann hypothesis can demonstrate, in the absence of a regular cadence, the existence or otherwise of a logic in the distribution of prime numbers. This could have important effects on cryptography using integers whose factorization into prime numbers cannot be calculated in acceptable times. Such knowledge of the distribution of prime numbers could facilitate the factorization. The alternative of using quantum cryptography for the moment seems unassailable [59].

6 Conclusions

Following the considerations introduced above, we can conclude by stressing some themes correlating the concept of the vacuum, imaginary numbers, and their t-incomputability; traditional and non-traditional, ideal and non-ideal models; and research topics. In particular,

- The vacuum as a general t-incomputable domain of possibilities specified by the imaginary space, with complex variables, imaginary models (i.e., models using imaginary variables), and collapsing mechanisms that may be represented by a suitable symbolic computational collapse.

The vacuum as imaginary space always coupled with the non-imaginary space in the complex space of complex numbers. In other words, the real space as the real part of the dual complex space is composed of real and imaginary parts. This representation relates to the duality real-potential, collapsed–collapsible, and superimpositions of something such as /waves/phenomena/planes/states/spaces as in QM.

- The use of imaginary, complex numbers, and their t-incomputability may be intended as guarantee and representation that it is not possible to compute definitive results, but it is possible to have pending multiple equivalences and superimpositions as in quantum physics and emergent collective processes in complexity.
- The phenomenological nature of the quantum collapsing of a wave function in a superposition of several eigenstates, reducing to a single one as an effect of the interaction with the external world. We say phenomenological only ideally *computable*, when the phenomenological collapse cannot be reduced to be a result of a computational process. This contrast is in correspondence with the phenomenological nature of processes of emergence not reducible to computational, as in the science of complexity. It is represented (not reduced to) by the *non-Turing computability*, *non-explicit computability*, and *t-incomputability* of models using imaginary numbers. Furthermore, we may consider the role of the incomputability to represent systemic *irreducibility* such as between systemic and non-systemic properties, incoherences as the manifestation of non-equivalences, irreducible multidimensionality, incompleteness, and complexity of the world only approximated by non-linearity as in simulations.

From a formal point of view, a number of stochastic models could be suitably reformulated in such a way as to transform into QFT-based models, which sometimes involves the use of imaginary numbers as a trick, but require, however, a physical interpretation.

The effectiveness of imaginary numbers may then be, moreover, in their role of ensuring the t-incomputability of phenomenological collapsing and in reformulating stochastic models as QFT-based models.

Examples of related research lines to be considered are

- The usage of the mathematics of complex variables not only in (typically quantum) physics but also to consider and represent the implicit pending properties of matter; dualities between classic and non-classic approaches; and theoretically incomplete and incomputable processes of emergence from the predominance of collective multiple remote synchronization effects [60, 61];
- The relations between physical, biological, and mathematical processes of reduction of t-incomputabilities into computability. Such processes of reduction include the meaning of the fact that imaginary t-incomputability disappears as soon as we properly symbolically compute and combine complex numbers;
- The meaning of i as theoretically incomputable and, however, expressed by singularly computable expressions as in (7).
- The use of imaginary, complex numbers as the expression/manifestation of something we do not understand yet, as was for the indemonstrability of the fifth postulate of Euclid, an indemonstrability that covered for centuries the non-Euclidean geometries.

The usage of imaginary, complex numbers, complex variables, their *t-incomputability*, and their unreasonable effectiveness in physic modes, would be an *outstanding* interdisciplinary project between mathematics, physics, and philosophy of science.

Proper modeling is necessary, probably based on new approaches such as the so-called complex-valued neural networks (CVNN) as in [62] and on the so-called sub-symbolic, emergent computation [18].

The present article is dedicated to the memory of Professor Eliano Pessa with whom these issues were under study and to celebrate his valuable interdisciplinary contribution and expertise in the science of complexity.

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