The irregular Cantor sets C^e ([0, 1]) and C^π ([0, 1]), and the Cantor-Lebesgue irregular functions G^e and G^π

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Abstract: - In this article, we introduce and study a new class of perfect nowhere-dense sets, which are not self-similar in any subset, also, we constructed the correspondent singular functions. We construct a two-dimensional irregular Cantor set $C^{e,\pi}$ ([0, 1]) on the real plane.

Keywords: - Cantor set, Cantor function, irregular Cantor set, irregular Cantor function, fractal.

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1. Introduction

In 1883, G. Cantor constructed the first widely known example of a perfect nowhere-dense fractal set in the real line, similarly, sets were constructed by V. Volterra and J. Smith in 1881 and 1875, respectively. The perfect nowhere-dense fractal sets are the base for the construction of many counterexamples in analysis such as singular functions, which are Holder continuous with some degree but not with the others. The Cantor sets have a self-similar, fractal structure since the Cantor set is equal to two of its translated copies after being shrunk by factor three. Since this branch of analysis is intensively developing, there is extensive literature on the subject that we will not review in this short article, some works are given in references [1 - 8].

The main goal of this article is to introduce and study a new type of perfect nowhere-dense sets, which are not self-similar, and correspondent singular non-self-similar functions.

2. Generalized irregular Cantor sets

For any $x \in [0,\ 1]$, there exists a unique decimal expression x in the form of an infinite series

$$x = \sum_{k=1,\dots,} \frac{a_k}{10^k} = \frac{a_1}{10^1} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \frac{a_4}{10^4} + \dots.. \quad (1)$$

where a set $\{a_k\}$ of numbers $a_k \in \{0,\ 1, \dots,\ 9\}$, therefore, each $x \in [0,\ 1]$ can be uniquely presented by infinite decimal series $x = \sum_{k=1,\dots,} \frac{a_k(x)}{10^k}$ with a unique set $\{a_k(x)\}$ of numbers $a_k(x) \in \{0,\ 1, \dots,\ 9\}$.

The numbers e and π are transcendental, and the numbers e and π can be written in the form of expansions

$$e = 2 + 7\times 10^{-1} + 1\times 10^{-2} + 8\times 10^{-3} + 2\times 10^{-4} + \dots \quad (2)$$

and

$$\pi = 3\times 10 + 1\times 10^{-1} + 4\times 10^{-2} + 1\times 10^{-3} + 5\times 10^{-4} + .., \quad (3)$$

therefore, the numbers e and π, we identify with sequences

$$e \leftrightarrow \{2,\ 7,\ 1,\ 8,\ 2,\ 8,\ 1,\ \dots\} = \{\tilde{a}_k{}^e\}_{k=1,\dots.} \quad (4)$$

and

$$\pi \leftrightarrow \{3,\ 1,\ 4,\ 1,\ 5,\ 9,\ 2,\ \dots\} = \{\tilde{a}_k{}^\pi\}_{k=1,\dots.}, \quad (5)$$

respectively.

Definition 1. ***The irregular Cantor sets*** $C^e([0,\ 1])$ ***and*** $C^\pi([0,\ 1])$ ***consist of all real numbers*** $x \in [0,\ 1]$ ***such that***

$$x_e = \sum_{k=1,\dots,} \frac{a_k{}^e}{10^k} = \frac{a_1{}^e}{10^1} + \frac{a_2{}^e}{10^2} + \frac{a_3{}^e}{10^3} + \frac{a_4{}^e}{10^4} + \dots \quad (6)$$

and

$$x_\pi = \sum_{k=1,\dots,} \frac{a_k{}^\pi}{10^k} = \frac{a_1{}^\pi}{10^1} + \frac{a_2{}^\pi}{10^2} + \frac{a_3{}^\pi}{10^3} + \frac{a_4{}^\pi}{10^4} + \dots \quad (7)$$

where $a_k{}^e(x) \in \{0,\ 1, \dots,\ 9\} \backslash \tilde{a}_k{}^e$ ***and*** $a_k{}^\pi(x) \in \{0,\ 1, \dots,\ 9\} \backslash \tilde{a}_k{}^\pi$ ***for each*** k***, respectively.***

Irregular Cantor sets have the following properties:

1. *The Cantor sets* $C^e([0,\ 1])$ *and* $C^\pi([0,\ 1])$ *have a cardinality of the continuum.*

2. *The Cantor sets* $C^e([0,\ 1])$ *and* $C^\pi([0,\ 1])$ *are closed in the topology of the real line.*

3. *The Cantor sets $C^e([0,\ 1])$ and $C^\pi([0,\ 1])$ are compact in the topology of the real line.*

4. *The Cantor sets $C^e([0,\ 1])$ and $C^\pi([0,\ 1])$ are rare in the topology of the real line.*

Theorem 1. ***The Lebesgue measures of the Cantor sets $C^e([0,\ 1])$ and $C^\pi([0,\ 1])$ equal zero.***

Proof.

The Lebesgue measure μ of the Cantor sets $C^e([0,\ 1])$ and $C^\pi([0,\ 1])$ is given by

$$\begin{aligned}&\mu(C^e([0,\ 1])) = \mu([0,\ 1]) - \mu([0,\ 1]\setminus C^e([0,\ 1])) = \\ &= 1 - \left(\tfrac{9}{10} + \tfrac{9}{10}\tfrac{1}{10} + \tfrac{9}{10}\left(\tfrac{1}{10}\right)^2 + ...\right) = \\ &= 1 - \tfrac{9}{10}\sum_k \left(\tfrac{1}{10}\right)^k = 0.\end{aligned}$$

Theorem 1 is proven.

The irregular Cantor set $C^e([0,\ 1])$ can be constructed from the unit compact interval ${C_0}^e = [0,\ 1]$ by performing the following recursive procedure, which is based on correspondence $e \leftrightarrow \{2,\ 7,\ 1,\ 8,\ 2,\ 8,\ 1,\ ...\} = \{{\tilde{a}_k}^e\}_{k=1,....}$: the first iteration consists of the removal of the open interval $\left(\frac{2}{10},\ \frac{3}{10}\right)$ corresponding to ${a_1}^e = 2$,the second iteration consists of the removal of nine subintervals corresponding to ${a_2}^e = 7$, the second iteration consists of the removal eighty-one subintervals corresponding to ${a_3}^e = 1$, and so on. Continuing this infinite procedure, we obtain a sequence $\{{C_k}^e\}$ of strictly decreasing sets C_k such that $C_k \supset C_{k+1}$ with strict inclusions for all numerators, the limited set of points, that remain after infinite numbers of iterations, coincides with an irregular Cantor set $C^e([0,\ 1]) = \lim\limits_{k\to\infty} C_k = \bigcap_{k=0,1,...,} C_k$. Similar considerations can be accomplished for a set $C^\pi([0,\ 1])$ taking $\pi \leftrightarrow \{3,\ 1,\ 4,\ 1,\ 5,\ 9,\ 2,\ ...\}$ as a guiding sequence.

3. The Cantor-Lebesgue irregular Functions

We are going to introduce the concept of the Cantor-Lebesgue irregular functions G^e and G^π for the irregular Cantor sets $C^e([0,\ 1])$ and $C^\pi([0,\ 1])$ through an iterative procedure. We will construct the function G^e, the function G^π can be constructed similarly.

We present the ordinate axis in base nine as

$$y = \sum_{k=1,...,} \frac{b_k}{9^k} = \frac{b_1}{9^1} + \frac{b_2}{9^2} + \frac{b_3}{9^3} + \frac{b_4}{9^4} + ... \qquad (8)$$

where a set $\{b_k\}$ of numbers $b_k \in \{0,\ 1, ...,\ 8\}$.

We construct the sequence $\{\varphi_n\}$ of continuous monotone-increasing functions φ_n, $n = 1,\ 2,\ ,....,$ which converges to the function G^e. The continuous monotone-increasing function φ_1 takes the constant value $\frac{2}{9}$ on the open interval $\left(\frac{2}{10},\ \frac{3}{10}\right)$, which corresponds with the 2 in the expansion for e and has been removed from $[0,\ 1]$, and the continuous monotone-increasing function φ_1 is linear on the remaining intervals. The function φ_2 is given by

$$\varphi_2 = \begin{cases} \frac{7}{9^2}, & \left(\frac{7}{10^2},\ \frac{8}{10^2}\right), \\ \frac{1}{9} + \frac{7}{9^2}, & \left(\frac{1}{10} + \frac{7}{10^2},\ \frac{1}{10} + \frac{8}{10^2}\right), \\ \frac{2}{9}, & \left(\frac{2}{10},\ \frac{3}{10}\right), \\ \frac{3}{9} + \frac{7}{9^2}, & \left(\frac{3}{10} + \frac{7}{10^2},\ \frac{3}{10} + \frac{8}{10^2}\right), \\ \frac{4}{9} + \frac{7}{9^2}, & \left(\frac{4}{10} + \frac{7}{10^2},\ \frac{4}{10} + \frac{8}{10^2}\right), \\ \frac{5}{9} + \frac{7}{9^2}, & \left(\frac{5}{10} + \frac{7}{10^2},\ \frac{5}{10} + \frac{8}{10^2}\right), \\ \frac{6}{9} + \frac{7}{9^2}, & \left(\frac{6}{10} + \frac{7}{10^2},\ \frac{6}{10} + \frac{8}{10^2}\right), \\ \frac{7}{9} + \frac{7}{9^2}, & \left(\frac{7}{10} + \frac{7}{10^2},\ \frac{7}{10} + \frac{8}{10^2}\right), \\ \frac{8}{9} + \frac{7}{9^2}, & \left(\frac{8}{10} + \frac{7}{10^2},\ \frac{8}{10} + \frac{8}{10^2}\right), \\ linear \quad on \quad remaining \quad intervals; \end{cases}$$

which corresponds with the 7, in the expansion for e. We continue this procedure and obtain the sequence $\{\varphi_n\}$ of continuous monotone-increasing functions.

For functions φ_n, we have the uniform estimation

$$|\varphi_{n+1}(x) - \varphi_n(x)| \le 9^n$$

for all $x \in [0,\ 1]$. So that the sequence $\{\varphi_n\}$ converges to the continuous monotone-increasing function which we denote G^e.

Definition 2. ***The limit of the sequence $\{\varphi_n\}$ defined as about is called the Cantor-Lebesgue irregular function G^e.***

Similarly, employing expansion of π we can define the Cantor-Lebesgue irregular continuous monotone-increasing function G^π.

Straightforward considerations yield the following statements.

Lemma 1. ***The Cantor-Lebesgue irregular functions G^e and G^π have the following properties:***

1) G^e and G^π are continuous and monotone-increasing functions however they are not absolutely continuous;

2) G^e and G^π are singular functions;

3) G^e and G^π map the Cantor sets $C^e([0,\ 1])$ and $C^\pi([0,\ 1])$ onto $[0,\ 1]$.

Theorem 2. ***The Cantor-Lebesgue irregular functions $G^e(x)$ and $G^\pi(x)$ are locally concave-convex at point $\tilde{x}$ if and only if $\tilde{x} \in [0,\ 1]\setminus C^e([0,\ 1])$ and $\tilde{x} \in [0,\ 1]\setminus C^\pi([0,\ 1])$, respectively.***

Proof. We are going to prove the theorem for the G^e; in case of the G^π, a similar consideration can be employed.

We denote Θ_{G^e} the set of all points where the function G^e takes constant values. If $x \in \Theta_{G^e}$ then $G^e(x)$ is concave-convex in a neighborhood Υ of x. Let $[0,\ 1]\setminus\Theta$ and let G^e be concave-convex in a neighborhood of the point x. There are points $y,\ z \in [0,\ 1]\setminus\Theta$ and a positive constant δ such that $y < z$,

$$(y - \delta,\ z - \delta) \subset \Upsilon,$$

and $(y,\ z) \subset \Theta$. We have that exist points $\tilde{y},\ \tilde{z}$ such that

$$\tilde{y} \in (y - \delta,\ y) \cap ([0,\ 1]\setminus\Theta)$$

and

$$\tilde{z} \in (z - \delta,\ z) \cap ([0,\ 1] \setminus \Theta),$$

and we apply a standard argument since G^e is the increasing function, the pair of straight lines pass through points $(\tilde{y},\ G^e(\tilde{y}))$, $\left(y + 2^{-1}z,\ G^e\left(y + 2^{-1}z\right)\right)$ and $(\tilde{z},\ G^e(\tilde{z}))$, $\left(z + 2^{-1}y,\ G^e\left(z + 2^{-1}y\right)\right)$, respectively, then the straight line passes through points $(\tilde{y},\ G^e(\tilde{y}))$, $\left(y + 2^{-1}z,\ G^e\left(y + 2^{-1}z\right)\right)$ is passing under point $(y,\ G^e(y))$, and the straight line passes through points $(\tilde{z},\ G^e(\tilde{z}))$, $\left(z + 2^{-1}y,\ G^e\left(z + 2^{-1}y\right)\right)$ is passing over point $(z,\ G^e(z))$, therefore, the restriction of G^e to $(\tilde{y},\ \tilde{z})$ is not concave-convex, this contradiction proves the statement of the theorem.

Lemma 2. ***The Cantor-Lebesgue irregular functions*** G^e ***and*** G^π ***satisfy the Banach conditions*** T_1.

The proof is straightforward.

4. Two-dimensional irregular Cantor set $C^{e,\pi}$ ([0, 1])

Now, we are going to construct the irregular analog of the Sierpinski carpet, a fractal-like two-dimensional structure produced by an irregular iterated system with a constant base of ten.

We start with a unit square $[0,\ 1] \times [0,\ 1]$ and present its points $(x,\ y) \in [0,\ 1] \times [0,\ 1]$ in form of an expansion in base ten

$$x = \sum_{k=1,\ldots,} \frac{a_k}{10^k} = \frac{a_1}{10^1} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \frac{a_4}{10^4} + \ldots..$$

$$y = \sum_{k=1,\ldots,} \frac{b_k}{10^k} = \frac{b_1}{10^1} + \frac{b_2}{10^2} + \frac{b_3}{10^3} + \frac{b_4}{10^4} + \ldots...$$

We assume that abscissa and ordinate axes govern by numbers e and π, respectively. A two-dimensional irregular Cantor set $C^{e,\pi}([0,\ 1])$ is constructed by an iterative procedure of deleting squares, whose coordinates expressed in the base ten do not both have the respective index k digits of respective numbers e and π, respectively $(x,\ y) \in [0,\ 1] \times [0,\ 1]$. The first deleted square is $\left(\frac{2}{10},\ \frac{3}{10}\right) \times \left(\frac{3}{10},\ \frac{4}{10}\right)$, next we delete 99 squares governing by 1 and 1 from e and π, respectively. The process is infinite removing squares, and as a result, we obtain a fractal irregular two-dimensional set, which we called a two-dimensional irregular Cantor set $C^{e,\pi}([0,\ 1])$.

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