Hyperbolic k-Oresme and k-Oresme-Lucas Quaternions

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Abstract: - In this study, we define hyperbolic k-Oresme and k-Oresme-Lucas quaternions. For these quaternions, we give the Binet Formulas, summation formulas, etc. Then we obtain the generating functions and exponential generating functions of these quaternions. Also, we find relations among the hyperbolic k-Oresme and k-Oresme-Lucas quaternions and their conjugates. In addition, we calculate the special identities of these quaternions. Moreover, we examine the relationships between the hyperbolic k-Oresme and k-Oresme-Lucas quaternions. Finally, the terms of the k-Oresme and k-Oresme-Lucas sequences are associated with their hyperbolic quaternion values.

Key-Words: - Oresme number, Quaternions, Lucas Number, Catalan identity, Generating function

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1 Introduction

Sequences of numbers play a vital role in understanding the complexity of any problem consisting of some patterns. An example of this is the Rabbit problem in Leonardo Fibonacci's classic book Liber Abaci. Inspired by the rabbit problem, the Fibonacci sequence was developed and the relationship between the terms of this sequence became the golden ratio. The Fibonacci and Lucas sequences are famous sequences of numbers. These sequences have intrigued scientists for a long time. Fibonacci and Lucas sequences have been applied in various fields such as Algebraic Coding Theory [1], Phyllotaxis [2], Biomathematics [3], Computer Science [4], etc. Many generalizations of the Fibonacci sequence have been given. The known examples of such sequences are the Bronze Fibonacci, Bronze Lucas, k-Fibonacci, k-Lucas, Oresme, k-Chebsyhev, Jacobsthal-Lucas, Pell, Lenardo, Narayana, Padovan sequences, etc (see for details in [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17]).

For $n \in \mathbb{N}$, the Fibonacci numbers F_n and Lucas numbers L_n are defined by the recurrence relations, respectively,

$$F_{n+2} = F_{n+1} + F_n, \text{ and } L_{n+2} = L_{n+1} + L_n,$$

with the initial conditions $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$.

Binet formulas Fibonacci numbers F_n and Lucas numbers L_n are given by relations, respectively,

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n,$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation $r^2 - r - 1 = 0$. Here the number α is the known golden ratio.

In [18], for $n \in \mathbb{N}$, he defined the Oresme sequence O_n and Oresme-Lucas sequence H_n by the recurrence relations, respectively;

$$O_{n+2} = O_{n+1} - \frac{1}{4} O_n, \text{ and } H_{n+2} = H_{n+1} - \frac{1}{4} H_n,$$

with the initial conditions $O_0 = 0$, $O_1 = \frac{1}{2}$ and $H_0 = 2$, $H_1 = 1$.

With the help of the recurrence relation of the Fibonacci sequence, k-sequences were introduced and these sequences had an important place in number theory [19].

In [20], for $n \in \mathbb{N}$, they defined the k-Oresme sequence $O_{k,n}$ and k-Oresme-Lucas sequence $P_{k,n}$ the recurrence relations, respectively;

$$O_{k,n+2} = O_{k,n+1} - \frac{1}{k^2} O_{k,n}, \text{ and}$$
$$P_{k,n+2} = P_{k,n+1} - \frac{1}{k^2} P_{k,n},$$

with the initial conditions $O_{k,0} = 0$, $O_{k,1} = \frac{1}{k}$ and $P_{k,0} = 2$, $P_{k,1} = 1$.

Binet formulas of the k-Oresme and the k-Oresme-Lucas sequence are given by relations, respectively,

$$O_{k,n} = \frac{r_1^n - r_2^n}{(r_1 - r_2)k} \text{ and } P_{k,n} = r_1^n + r_2^n,$$

where $r_1 = \frac{k+\sqrt{k^2-4}}{2k}$ and $r_2 = \frac{k-\sqrt{k^2-4}}{2k}$ are the roots of the characteristic equation $r^2 - r + \frac{1}{k^2} = 0$.

In mathematics, quaternions (or quadruplets) are a number system that expands the complex numbers into one real and three imaginary dimensions.

The quaternions were first described by Hamilton in 1843. In addition, quaternions are used to control rotational movements, especially in Kinematics [21], 3D games [22], mechanics [23], Eulerian angles [24], and Chemistry [25]. In [26], Horadam defined Complex Fibonacci and Fibonacci quaternions, and various features were found.

The algebra of hyperbolic quaternions is an algebra that is not related to the elements of the form over the real numbers.

$$q = xi_1 + yi_2 + zi_3 + ti_4, \; x, y, z, t \in \mathbb{R}$$

He gave the properties of the q components defined in Table 1.

$\cdot$	i_1	i_2	i_3	i_4
i_1	i_1	i_2	i_3	i_4
i_2	i_2	i_1	i_4	$-i_3$
i_3	i_3	$-i_4$	i_1	i_2
i_4	i_4	i_3	$-i_2$	i_1

Table 1. Hyperbolic Quaternions Units

In [27], he did a lot of research on hyperbolic quaternions and their properties. An expression of the general form of hyperbolic quaternions is

$$\hbar = \hbar_1 i_1 + \hbar_2 i_2 + \hbar_3 i_3 + \hbar_4 i_4 = (\hbar_1, \hbar_2, \hbar_3, \hbar_4).$$

Here, $\hbar_1, \hbar_2, \hbar_3, \hbar_4$ are the terms of the sequence and i_1, i_2, i_3, i_4 are hyperbolic quaternions.

In [28,] he defined the hyperbolic k-Fibonacci and k-Lucas quaternions and he found properties of these quaternions. Also, they conducted a study on the quaternions and obtained many features related to these quaternions [29]. In addition, they introduced the Jacobsthal and Jacobsthal-Lucas quaternions [30]. Moreover, many applications of sequences have been made on quaternions (see for details in [31], [32], [33], [34], [35], [36], [37], [38], [39], [40]).

As seen above, many generalizations of hyperbolic quaternions of sequences have been given so far. In this study, we give new generalizations inspired by the hyperbolic k-Fibonacci and Jacobsthal quaternions. We call these quaternions the hyperbolic k-Oresme and k-Oresme quaternions and denote them as $\breve{H}O_{k,n}$, and $\breve{H}P_{k,n}$, respectively.

We separate the article into three parts.

In Chapter 2, we define the hyperbolic k-Oresme and k-Oresme-Lucas quaternions and the terms of these quaternions are given. Then, we find some properties of these quaternions.

In Chapter 3, information is given about the characteristic equations of hyperbolic k-Oresme and k-Oresme-Lucas quaternions. Then, we obtain the Binet formulas, generating functions and summation formulas of these quaternions. In addition, we find the relationship of hyperbolic k-Oresme and k-Oresme-Lucas quaternions, Catalan identity, Cassini identity, D'ocagne identity, Vajda's identity, etc. Finally, the terms of the sequence are associated with their hyperbolic values.

2 Hyperbolic k-Oresme and k-Oresme-Lucas Quaternions

For $n \in \mathbb{N}$, the hyperbolic k-Oresme $\breve{H}O_{k,n}$ and k-Oresme-Lucas quaternions $\breve{H}P_{k,n}$ are defined by, respectively,

$$\breve{H}O_{k,n} = O_{k,n}i_1 + O_{k,n+1}i_2 + O_{k,n+2}i_3 + O_{k,n+3}i_4 = (O_{k,n}, O_{k,n+1}, O_{k,n+2}, O_{k,n+3})$$

and

$$\breve{H}P_{k,n} = P_{k,n}i_1 + P_{k,n+1}i_2 + P_{k,n+2}i_3 + P_{k,n+3}i_4 = (P_{k,n}, P_{k,n+1}, P_{k,n+2}, P_{k,n+3})$$

where $O_{k,n}$, is n^{th} k-Oresme sequence, $P_{k,n}$, n^{th} k-Oresme-Lucas sequence and i_1, i_2, i_3, and i_4 are the hyperbolic quaternion units in table 1.

Let us now give the first three terms of the hyperbolic k-Oresme and k-Oresme-Lucas quaternions, respectively,

- $\breve{H}O_{k,0} = \frac{1}{k}i_2 + \frac{1}{k}i_3 + \frac{1}{k^3}(k^2-1)i_4$,
- $\breve{H}O_{k,1} = \frac{1}{k}i_1 + \frac{1}{k}i_2 + \frac{1}{k^3}(k^2-1)i_3 + \frac{1}{k^3}(k^2-2)i_4$,
- $\breve{H}O_{k,2} = \frac{1}{k}i_1 + \frac{1}{k^3}(k^2-1)i_2 + \frac{1}{k^3}(k^2-2)i_3 + \frac{1}{k^5}(k^4-3k^2+1)i_4$,

and

- $\breve{H}P_{k,0} = 2i_1 + i_2 + \frac{1}{k^2}(k^2-2)i_3 + \frac{1}{k^2}(k^2-3)i_4$,
- $\breve{H}P_{k,1} = i_1 + \frac{1}{k^2}(k^2-2)i_2 + \frac{1}{k^2}(k^2-3)i_3 + \frac{1}{k^4}(k^4-4k^2+2)i_4$,

- $\breve{H}P_{k,2} = \frac{1}{k^2}(k^2-2)i_1 + \frac{1}{k^2}(k^2-3)i_2 + \frac{1}{k^4}(k^4-4k^2+2)i_3 + \frac{1}{k^4}(k^4-5k^2+5)i_4.$

Definition 2.1. For $n \in \mathbb{N}$, the conjugate of hyperbolic k-Oresme $\breve{H}O^*_{k,n}$ and k-Oresme-Lucas $\breve{H}P^*_{k,n}$ quaternions are defined by, respectively,

$$\breve{H}O^*_{k,n} = O_{k,n}i_1 - O_{k,n+1}i_2 - O_{k,n+2}i_3 - O_{k,n+3}i_4 = (O_{k,n}, -O_{k,n+1}, -O_{k,n+2}, -O_{k,n+3})$$

and

$$\breve{H}P^*_{k,n} = P_{k,n}i_1 - P_{k,n+1}i_2 - P_{k,n+2}i_3 - P_{k,n+3}i_4 = (P_{k,n}, -P_{k,n+1}, -P_{k,n+2}, -P_{k,n+3}).$$

Definition 2.2. For $n \in \mathbb{N}$, the norms of the hyperbolic k-Oresme $\|\breve{H}O_{k,n}\|$ and k-Oresme-Lucas quaternions $\|\breve{H}P_{k,n}\|$ are defined by, respectively,

$$\|\breve{H}O_{k,n}\| = \sqrt{O^2_{k,n} + O^2_{k,n+1} + O^2_{k,n+2} + O^2_{k,n+3}}$$

and

$$\|\breve{H}P_{k,n}\| = \sqrt{P^2_{k,n} + P^2_{k,n+1} + P^2_{k,n+2} + P^2_{k,n+3}}.$$

In the following theorems, we examine the relations between the $\breve{H}O_{k,n}$, $\breve{H}O^*_{k,n}$, $\breve{H}P_{k,n}$, and $\breve{H}P^*_{k,n}$ quaternions.

Theorem 2.1. Let $n \in \mathbb{N}$. The following equations are true:

i. $\breve{H}O_{k,n+2} = \breve{H}O_{k,n+1} - \frac{1}{k^2}\breve{H}O_{k,n}$,

ii. $\breve{H}P_{k,n+2} = \breve{H}P_{k,n+1} - \frac{1}{k^2}\breve{H}P_{k,n}$,

iii. $\breve{H}O^*_{k,n+2} = \breve{H}O^*_{k,n+1} - \frac{1}{k^2}\breve{H}O^*_{k,n}$,

iv. $\breve{H}P^*_{k,n+2} = \breve{H}P^*_{k,n+1} - \frac{1}{k^2}\breve{H}P^*_{k,n}$.

Proof. i. If the definition is used, we have

$$\breve{H}O_{k,n+1} - \frac{1}{k^2}\breve{H}O_{k,n} = (O_{k,n+1}i_1 + O_{k,n+2}i_2 + O_{k,n+3}i_3 + O_{k,n+4}i_4) - \frac{1}{k^2}\left(O_{k,n}i_1 + O_{k,n+1}i_2 + O_{k,n+2}i_3 + O_{k,n+3}i_4\right)$$
$$= \left(O_{k,n+1} - \frac{1}{k^2}O_{k,n}\right)i_1 + \left(O_{k,n+2} - \frac{1}{k^2}O_{k,n+1}\right)i_2 + \left(O_{k,n+3} - \frac{1}{k^2}O_{k,n+2}\right)i_3 + \left(O_{k,n+4} - \frac{1}{k^2}O_{k,n+3}\right)i_4.$$

Since, $O_{k,n+2} = O_{k,n+1} - \frac{1}{k^2}O_{k,n}$. Thus, we obtain

$$\breve{H}O_{k,n+2} = \breve{H}O_{k,n+1} - \frac{1}{k^2}\breve{H}O_{k,n}.$$

The proofs of the others are shown similarly. □

Theorem 2.2. We obtain

i. $\breve{H}O_{k,n} + \breve{H}O^*_{k,n} = 2O_{k,n}i_1$,

ii. $\breve{H}O^2_{k,n} = 2O_{k,n}\breve{H}O_{k,n} + (\|\breve{H}O_{k,n}\|^2 - 2O^2_{k,n})i_1$,

iii. $\breve{H}P_{k,n} + \breve{H}P^*_{k,n} = 2P_{k,n}i_1$,

iv. $\breve{H}P^2_{k,n} = 2P_{k,n}\breve{H}P_{k,n} + \left(\|\breve{H}P_{k,n}\|^2 - 2P^2_{k,n}\right)i_1$,

v. $\breve{H}O_{k,n}\breve{H}O^*_{k,n} = \frac{P_{k,2n}+P_{k,2n+2}+P_{k,2n+4}+P_{k,2n+6}-\frac{2}{k^{2n}}-\frac{2}{k^{2n+2}}-\frac{2}{k^{2n+4}}-\frac{2}{k^{2n+6}}}{k^2-4}$,

vi. $\breve{H}P_{k,n}\breve{H}P^*_{k,n} = P_{k,2n+6} + P_{k,2n+4} + P_{k,2n+2} + P_{k,2n} - \frac{2}{k^{2n}} - \frac{2}{k^{2n+2}} - \frac{2}{k^{2n+4}} - \frac{2}{k^{2n+6}}$.

Proof. ii. If the definition is used, we have

$$\breve{H}O^2_{k,n} = (O_{k,n}i_1 + O_{k,n+1}i_2 + O_{k,n+2}i_3 + O_{k,n+3}i_4)(O_{k,n}i_1 + O_{k,n+1}i_2 + O_{k,n+2}i_3 + O_{k,n+3}i_4)$$
$$= \left(O^2_{k,n} + O^2_{k,n+1} + O^2_{k,n+2} + O^2_{k,n+3}\right)i_1 + (2O_{k,n}O_{k,n+1})i_2 + (2O_{k,n}O_{k,n+2})i_3 + (2O_{k,n}O_{k,n+3})i_4$$
$$= 2O_{k,n}\left(O_{k,n}i_1 + O_{k,n+1}i_2 + O_{k,n+2}i_3 + O_{k,n+3}i_4\right) + (-O^2_{k,n} + O^2_{k,n+1} + O^2_{k,n+2} + O^2_{k,n+3})i_1.$$

Thus, we obtain

$$\breve{H}O^2_{k,n} = 2O_{k,n}\breve{H}O_{k,n} + (\|\breve{H}O_{k,n}\|^2 - 2O^2_{k,n})i_1.$$

The proofs of the others are shown similarly. □

Theorem 2.3. We obtain

i. $i_1\breve{H}O_{k,n} - i_2\breve{H}O_{k,n+1} - i_3\breve{H}O_{k,n+2} - i_4\breve{H}O_{k,n+3} = (O_{k,n+6} - O_{k,n+4} - O_{k,n+2} + O_{k,n})i_1$.

ii. $i_1\breve{H}P_{k,n} - i_2\breve{H}P_{k,n+1} - i_3\breve{H}P_{k,n+2} - i_4\breve{H}P_{k,n+3} = (P_{k,n+6} - P_{k,n+4} - P_{k,n+2} + P_{k,n})i_1$.

Proof. i. If the definition is used, we have

$$i_1\breve{H}O_{k,n} - i_2\breve{H}O_{k,n+1} - i_3\breve{H}O_{k,n+2} - i_4\breve{H}O_{k,n+3} = \left(O_{k,n} - O_{k,n+2} - O_{k,n+4} + O_{k,n+6}\right)i_1 + \left(O_{k,n+1} - O_{k,n+1} - O_{k,n+5} + O_{k,n+5}\right)i_2 + \left(O_{k,n+2} + O_{k,n+4} - O_{k,n+2} - O_{k,n+4}\right)i_3 + (O_{k,n+3} - O_{k,n+3} + O_{k,n+3} - O_{k,n+3})i_4$$
$$= (O_{k,n+6} - O_{k,n+4} - O_{k,n+2} + O_{k,n})i_1.$$

The proofs of the others are shown similarly. □

3 Properties of Hyperbolic k-Oresme and k-Oresme-Lucas Quaternions

In this chapter, we obtain properties of the hyperbolic k-Oresme and k-Oresme-Lucas quaternions. Then, we examine the relationships between these quaternions. Also, we calculate the special identities of these quaternions. In addition, we find the terms of the k-Oresme and k-Oresme-Lucas sequences are associated with their hyperbolic quaternion values.

In the following theorem, the Binet formulas of the $\breve{H}O_{k,n}$, $\breve{H}O^*_{k,n}$, $\breve{H}P^*_{k,n}$, and $\breve{H}Q^*_n$ quaternions are expressed.

Theorem 3.1. Let $n \in \mathbb{N}$. We obtain

i. $\breve{H}O_{k,n} = \frac{\overline{\alpha}\alpha^n - \overline{\beta}\beta^n}{(\alpha-\beta)k}$, **ii.** $\breve{H}P_{k,n} = \overline{\alpha}\alpha^n + \overline{\beta}\beta^n$,

iii. $\breve{H}O^*_{k,n} = \frac{\overline{\alpha}^*\alpha^n - \overline{\beta}^*\beta^n}{(\alpha-\beta)k}$, **iv.** $\breve{H}P^*_{k,n} = \overline{\alpha}^*\alpha^n + \overline{\beta}^*\beta^n$,

where

$\overline{\alpha} = i_1 + \alpha i_2 + \alpha^2 i_3 + \alpha^3 i_4 = (1, \alpha, \alpha^2, \alpha^3)$,

$\overline{\alpha}^* = (1, -\alpha, -\alpha^2, -\alpha^3)$,
$\overline{\beta} = i_1 + \beta i_2 + \beta^2 i_3 + \beta^3 i_4 = (1, \beta, \beta^2, \beta^3)$ and $\overline{\beta}^* = (1, -\beta, -\beta^2, -\beta^3)$.

Proof. i. With the help of the characteristic equation, the following results are obtained.

$$r^2 - r + \frac{1}{k^2} = 0 \,, \alpha = r_1 = \frac{\mathrm{k}+\sqrt{k^2-4}}{2\mathrm{k}}$$

$\beta = r_2 = \frac{\mathrm{k}-\sqrt{k^2-4}}{2\mathrm{k}}, \alpha + \beta = 1, \alpha - \beta = \delta = \frac{\sqrt{k^2-4}}{\mathrm{k}}$,
$\alpha^2 + \beta^2 = 1 - \frac{2}{k^2}$ and $\alpha\beta = \frac{1}{k^2}$.

The Binet form of the hyperbolic k-Oresme quaternions is

$$\breve{H}O_{k,n} = x\alpha^n + y\beta^n.$$

With the initial conditions, the following equations are obtained.

$\breve{H}O_{k,0} = \frac{1}{k} i_2 + \frac{1}{k} i_3 + \frac{1}{k^3}(k^2 - 1) i_4 = x + y$

and

$\breve{H}O_{k,1} = \frac{1}{k} i_1 + \frac{1}{k} i_2 + \frac{1}{k^3}(k^2 - 1) i_3 + \frac{1}{k^3}(k^2 - 2) i_4 = x\alpha + y\beta$.

Thus, we obtain

$x = \frac{\breve{H}O_{k,1} - \beta \breve{H}O_{k,0}}{\alpha - \beta} = \frac{i_1 + \alpha i_2 + \alpha^2 i_3 + \alpha^3 i_4}{\alpha - \beta} = \frac{\overline{\alpha}}{(\alpha-\beta)k}$

$y = \frac{\breve{H}O_{k,1} - \alpha \breve{H}O_{k,0}}{\beta - \alpha} = \frac{i_1 + \beta i_2 + \beta^2 i_3 + \beta^3 i_4}{-(\alpha - \beta)} = \frac{-\overline{\beta}}{(\alpha-\beta)k}$.

So, we have

$$\breve{H}O_{k,n} = \frac{\overline{\alpha}\alpha^n - \overline{\beta}\beta^n}{(\alpha-\beta)k}.$$

The proofs of the others are shown similarly. □

In the following theorems, we give special sum formulas of the $\breve{H}O_{k,n}$, and $\breve{H}P_{k,n}$ quaternions.

Theorem 3.2. Let $n \in \mathbb{N}$ and $k \in \mathbb{R}$. We obtain

i. $S\breve{H}O_{k,n} = \sum_{j=0}^{n} \breve{H}O_{k,j} = (1 - k^2)\breve{H}O_{k,n} + \breve{H}O_{k,n-1} + k i_1 + k i_2 - \frac{1+k^2}{k} i_3 + \frac{k^2-2}{k} i_4$,

ii. $S\breve{H}P_{k,n} = \sum_{j=0}^{n} \breve{H}P_{k,j} = (1 + k^2)\breve{H}P_{k,n} + \breve{H}P_{k,n-1} + 2k^2 i_1 + (2k^2 - 2) i_2 + (2k^2 - 3) i_3 + \frac{2k^4-4k^2+2}{k^2} i_4$.

Proof. i. Using the definition, we have

$S\breve{H}O_{k,n} = \sum_{j=0}^{n} \breve{H}O_{k,j} = i_1 \sum_{j=0}^{n} O_{k,j} + i_2 \sum_{j=0}^{n} O_{k,j+1} + i_3 \sum_{j=0}^{n} O_{k,j+2} + i_4 \sum_{j=0}^{n} O_{k,j+3}$.

Since $\sum_{j=0}^{n} O_{k,j} = k + O_{k,n-1} + (1 - k^2) O_{k,n}$, we get

$$S\breve{H}O_{k,n} = [(k + O_{k,n-1} + (1 - k^2) O_{k,n}) i_1 + (k + O_{k,n} + (1 - k^2) O_{k,n+1}) i_2 + \left(-\frac{1}{k} - k + O_{k,n+1} + (1 - k^2) O_{k,n+2}\right) i_3 + \left(-\frac{2}{k} + k + O_{k,n+2} + (1 - k^2) O_{k,n+3}\right) i_4.$$

So, we obtain

$S\breve{H}O_{k,n} = (1 - k^2)\breve{H}O_{k,n} + \breve{H}O_{k,n-1} + k i_1 + k i_2 - \frac{1+k^2}{k} i_3 + \frac{k^2-2}{k} i_4$.

The proof of the other is shown similarly. □

Theorem 3.3. Let $k \in \mathbb{R}$ and $x, y, z \in \mathbb{N}$. We obtain

i. $\sum_{y=0}^{n} \breve{H}O_{k,xy} = \frac{\frac{1}{k^{2x}}\breve{H}O_{k,nx} + \breve{H}O_{k,0} - \breve{H}O_{k,nx+x} + \breve{H}P_{k,0} O_{k,x} - \breve{H}O_{k,x}}{1 - P_{k,x} + \frac{1}{k^{2x}}}$

ii. $\sum_{y=0}^{n} \breve{H}P_{k,xy} = \frac{\frac{1}{k^{2x}}\breve{H}P_{k,nx} + \breve{H}P_{k,0} - \breve{H}O_{k,nx+x} - \breve{H}P_{k,0} P_{k,x} - \breve{H}P_{k,x}}{1 - P_{k,x} + \frac{1}{k^{2x}}}$,

iii. $\sum_{y=0}^{n} \breve{H}O_{k,xy+z} =$

$$\begin{cases} \frac{\frac{1}{k^{2x}}\breve{H}O_{k,nx+z} - \breve{H}O_{k,nx+x+z} - \breve{H}O_{k,z} - \frac{1}{k^{2x}}\breve{H}O_{k,x-z}}{1 - P_{k,x} + \frac{1}{k^{2x}}}, & if\ z < x \\ \frac{\frac{1}{k^{2x}}\breve{H}O_{k,nx+z} - \breve{H}O_{k,nx+x+z} - \breve{H}O_{k,z} - \frac{1}{k^{2x}}\breve{H}O_{k,z-x}}{1 - P_{k,x} + \frac{1}{k^{2x}}}, & othervise \end{cases}$$

,

iv. $\sum_{y=0}^{n} \breve{H}P_{k,xy+z} =$

$$\begin{cases} \frac{\frac{1}{k^{2x}}\breve{H}P_{k,nx+z} - \breve{H}P_{k,z} - \breve{H}P_{k,nx+x+z} - \frac{1}{k^{2x}}\breve{H}P_{k,z-x}}{1 - P_{k,x} + \frac{1}{k^{2x}}}, & if\ z < x \\ \frac{\frac{1}{k^{2x}}\breve{H}P_{k,nx+z} - \breve{H}P_{k,z} - \breve{H}P_{k,nx+x+z} - \frac{1}{k^{2x}}\breve{H}P_{k,z-x}}{1 - P_{k,x} + \frac{1}{k^{2x}}}, & othervise \end{cases}$$

.

Proof. With the help of definitions, Binet formulas, and geometric series, we have

i. $\sum_{y=0}^{n} \breve{H}O_{k,xy} = \sum_{n=0}^{\infty} \frac{\overline{\alpha}\alpha^{xy} - \overline{\beta}\beta^{xy}}{(\alpha-\beta)k}$

$$= \frac{\overline{\alpha}}{(\alpha-\beta)k} \sum_{y=0}^{n} (\alpha^x)^y - \frac{\overline{\beta}}{(\alpha-\beta)k} \sum_{y=0}^{n} (\beta^x)^y$$

$$= \frac{1}{(\alpha-\beta)k} \left(\frac{\overline{\alpha}\alpha^{nx+x} - \overline{\alpha}}{\alpha^x - 1} - \frac{\overline{\beta}\alpha^{nx+x} - \overline{\beta}}{\beta^x - 1} \right) =$$

$$\frac{1}{(\alpha-\beta)k} \frac{\overline{\alpha}\alpha^{nx+x}\beta^x - \overline{\alpha}\beta^x - \overline{\alpha}\alpha^{nx+x} + \overline{\alpha} - \overline{\beta}\beta^{nx+x}\alpha^x + \overline{\beta}\alpha^x + \overline{\beta}\beta^{nx+x} - \overline{\beta}}{1 - xP_{k,a} + \frac{1}{k^2}x^2}$$

Thus, we get

$$\sum_{y=0}^{n} \breve{H}O_{k,xy} = \frac{\frac{1}{k^{2x}}\breve{H}O_{k,nx} + \breve{H}O_{k,0} - \breve{H}O_{k,nx+x} + \breve{H}P_{k,0} O_{k,x} - \breve{H}O_{k,x}}{1 - P_{k,x} + \frac{1}{k^{2x}}}.$$

The proofs of the others are shown similarly. □

In the following theorems, we give special generating functions of the $\breve{H}O_{k,n}$, and $\breve{H}P_{k,n}$ quaternions.

Theorem 3.4. The generating functions for hyperbolic k-Oresme and k-Oresme-Lucas sequences are given as follows, respectively,

$$o(x) = \sum_{n=0}^{\infty} \breve{H}O_{k,n} x^n = \frac{(1-x)k^2\breve{H}O_{k,0} + xk^2\breve{H}O_{k,1}}{k^2 - k^2 x + x^2}$$

and

$$p(x) = \sum_{n=0}^{\infty} \breve{H}P_{k,n} x^n = \frac{k^2 x \breve{H}P_{k,1} + k^2(1-x)\breve{H}P_{k,0}}{k^2 - k^2 x + x^2}.$$

Proof. The following equations are written for the hyperbolic k-Oresme sequence.

$o(x) = \sum_{n=0}^{\infty} \breve{H}O_{k,n} x^n = \breve{H}O_{k,0} + \breve{H}O_{k,1} x + \sum_{n=2}^{\infty} \breve{H}O_{k,n} x^n$

$= \breve{H}O_{k,0} + \breve{H}O_{k,1}x + \sum_{n=2}^{\infty} \breve{H}O_{k,n-1}x^n - \frac{1}{k^2}\sum_{n=2}^{\infty} \breve{H}O_{k,n-2}x^n$

$= \breve{H}O_{k,0} + \breve{H}O_{k,1}x + x\big(-\breve{H}O_{k,0} + \sum_{n=0}^{\infty} \breve{H}O_{k,n}x^n\big) - \frac{x^2}{k^2}\sum_{n=0}^{\infty} \breve{H}O_{k,n}x^n.$

Thus, we obtain

$$o(x) = \frac{(1-x)k^2\breve{H}O_{k,0} + xk^2\breve{H}O_{k,1}}{k^2 - k^2x + x^2}.$$

Similarly to i, $p(x)$ is obtained. ◻

Theorem 3.5. For $a, b \in \mathbb{N}^+$, $b \geq a$ and $n \in \mathbb{N}$, we obtain

i. $\sum_{n=0}^{\infty} \breve{H}O_{k,an}x^n = \frac{\breve{H}O_{k,0} + (\breve{H}P_{k,0}O_{k,a} - \breve{H}O_{k,a})x}{1 - xP_{k,a} + \frac{1}{k^{2a}}x^2}$,

ii. $\sum_{n=0}^{\infty} \breve{H}P_{k,an}x^n = \frac{\breve{H}P_{k,0} + (-\breve{H}P_{k,0}P_{k,a} + \breve{H}P_{k,a})x}{1 - xP_{k,a} + \frac{1}{k^{2a}}x^2}$,

iii. $\sum_{n=0}^{\infty} \breve{H}O_{k,an+b}x^n = \frac{\breve{H}O_{k,b} - x\frac{1}{k^{2a}}\breve{H}O_{k,b-a}}{1 - xP_{k,a} + \frac{1}{k^{2a}}x^2}$,

iv. $\sum_{n=0}^{\infty} \breve{H}P_{k,an+b}x^n = \frac{\breve{H}P_{k,b} + x\frac{1}{k^{2a}}\breve{H}P_{k,a}}{1 - xP_{k,a} + \frac{1}{k^{2a}}x^2}$,

v. $\sum_{n=0}^{\infty} \frac{\breve{H}O_{k,bn}}{n!}x^n = \frac{\overline{\alpha}e^{\alpha^b x} - \overline{\beta}e^{\beta^b x}}{(\alpha-\beta)k}$,

vi. $\sum_{n=0}^{\infty} \frac{\breve{H}P_{k,bn}}{n!}x^n = \overline{\alpha}e^{\alpha^b x} + \overline{\beta}e^{\beta^b x}$.

Proof. With the help of definitions, Binet formulas, and geometric series, we have

i. $\sum_{n=0}^{\infty} \breve{H}O_{k,an}x^n = \sum_{n=0}^{\infty} \frac{\overline{\alpha}\alpha^{an} - \overline{\beta}\beta^{an}}{(\alpha-\beta)}x^n$

$= \frac{\overline{\alpha}}{(\alpha-\beta)k}\sum_{n=0}^{\infty}(\alpha^a)^n x^n - \frac{\overline{\beta}}{(\alpha-\beta)k}\sum_{n=0}^{\infty}(\beta^a)^n x^n$

$= \frac{1}{(\alpha-\beta)k}\left(\frac{\overline{\alpha}}{1-\alpha^a x} - \frac{\overline{\beta}}{1-\beta^a x}\right)$

$= \frac{\frac{\overline{\alpha}-\overline{\beta}}{(\alpha-\beta)k} + [(\overline{\alpha}+\overline{\beta})\left(\frac{\alpha^a-\beta^a}{(\alpha-\beta)k}\right) - \frac{\overline{\alpha}\alpha^a - \overline{\beta}\beta^a}{(\alpha-\beta)k}]x}{1 - xP_{k,a} + \frac{1}{k^{2a}}x^2}.$

Thus, we get

$$\sum_{n=0}^{\infty} \breve{H}O_{k,an}x^n = \frac{\breve{H}O_{k,0} + (\breve{H}P_{k,0}O_{k,a} - \breve{H}O_{k,a})x}{1 - xP_{k,a} + \frac{1}{k^{2a}}x^2}.$$

The proof of the other is shown similarly. ◻

In the next lemma, we give the properties that will be used to prove many theorems.

Lemma 3.1. We have

i. $\overline{\alpha} - \overline{\beta} = k\delta\breve{H}O_{k,0}$,

ii. $\overline{\alpha} + \overline{\beta} = \breve{H}P_{k,0}$,

iii. $\overline{\alpha}^* - \overline{\beta}^* = k\delta\breve{H}P_{k,0}{}^*$,

iv. $\overline{\alpha}^* + \overline{\beta}^* = \breve{H}P_{k,0}{}^*$,

v. $\overline{\alpha}\overline{\beta} = (1 + \alpha\beta + \alpha^2\beta^2 + \alpha^3\beta^3, 1 - \delta\alpha^2\beta^2, \alpha^2 + \beta^2 + \delta\alpha\beta, \alpha^3 + \beta^3 - \delta\alpha\beta)$,

vi. $\overline{\beta}\overline{\alpha} = (1 + \alpha\beta + \alpha^2\beta^2 + \alpha^3\beta^3, 1 + \delta\alpha^2\beta^2, \alpha^2 + \beta^2 - \delta\alpha\beta, \alpha^3 + \beta^3 + \delta\alpha\beta)$,

vii. $\overline{\alpha}\overline{\beta} + \overline{\beta}\overline{\alpha} = 2\breve{H}P_{k,0} - 2 + 2\alpha\beta + 2\alpha^2\beta^2 + 2\alpha^3\beta^3$,

viii. $\overline{\alpha}\overline{\beta} - \overline{\beta}\overline{\alpha} = (0, \alpha^2\beta^3 - \alpha^3\beta^2, 2\alpha^3\beta - 2\alpha\beta^3, -\alpha^2\beta + \alpha\beta^2) = \delta(0, -\frac{1}{k^4}, \frac{2}{k^2}, -\frac{1}{k^2})$,

ix. $\overline{\alpha}^2 = 2\overline{\alpha} - 1 + \alpha^2 + \alpha^4 + \alpha^6$,

x. $\overline{\beta}^2 = 2\overline{\beta} - 1 + \beta^2 + \beta^4 + \beta^6$,

xi. $\overline{\alpha}^2 + \overline{\beta}^2 = 2(\overline{\alpha} + \overline{\beta}) - 2P_{k,0} + P_{k,2} + P_{k,4} + P_{k,6}$,

xii. $\overline{\alpha}^2 - \overline{\beta}^2 = 2(\overline{\alpha} - \overline{\beta}) + \delta(O_{k,2} + O_{k,4} + O_{k,6})$.

Proof. v. If the definition is used, we have

$\overline{\alpha}\overline{\beta} = (i_1 + \alpha i_2 + \alpha^2 i_3 + \alpha^3 i_4)(i_1 + \beta i_2 + \beta^2 i_3 + \beta^3 i_4)$

$= i_1 + \alpha i_2 + \alpha^2 i_3 + \alpha^3 i_4 + \beta i_2 + \alpha\beta i_1 + \alpha^2\beta i_4 - \alpha^3\beta i_3 + \beta^2 i_3 - \alpha\beta^2 i_4 + \alpha^2\beta^2 i_1 + \alpha^3\beta^2 i_2 + \beta^3 i_4 + \alpha\beta^3 i_3 - \alpha^2\beta^3 i_2 + \alpha^3\beta^3 i_1.$

Thus,

$\overline{\alpha}\overline{\beta} = (1 + \alpha\beta + \alpha^2\beta^2 + \alpha^3\beta^3, 1 - \delta\alpha^2\beta^2, \alpha^2 + \beta^2 + \delta\alpha\beta, \alpha^3 + \beta^3 - \delta\alpha\beta).$

Other proofs are shown using definitions. ◻

In the following theorems, we calculate some identities for $\breve{H}O_{k,n}$, and $\breve{H}P_{k,n}$ quaternions.

Theorem 3.6. ***(Cassini Identity)*** Let $n \in \mathbb{N}$. We get

i. $\breve{H}O_{k,n+1}\breve{H}O_{k,n-1} - \breve{H}O_{k,n}^2 = k^{-2n+2}(-1 - \frac{1}{k^2} - \frac{1}{k^4} - \frac{1}{k^6}, -1 - \frac{1}{k^4}, \frac{1}{k^2} + \frac{1}{k^4} + \frac{1}{k^2}O_{k,3} - O_{k,3}, -O_{k,4})$,

ii. $\breve{H}P_{k,n+1}\breve{H}P_{k,n-1} - \breve{H}P_{k,n}^2 = k^{-2n+2}\delta^2(1 + \frac{1}{k^2} + \frac{1}{k^4} + \frac{1}{k^6}, 1 - \frac{1}{k^4}, \frac{1}{k^2}P_{k,2}, 1 - \frac{4}{k^2})$.

Proof. i. With the Binet formula, we get

$\breve{H}O_{k,n+1}\breve{H}O_{k,n-1} - \breve{H}O_{k,n}^2$

$= \left(\frac{\overline{\alpha}\alpha^{n+1} - \overline{\beta}\beta^{n+1}}{\alpha-\beta}\right)\left(\frac{\overline{\alpha}\alpha^{n-1} - \overline{\beta}\beta^{n-1}}{\alpha-\beta}\right) - \left(\frac{\overline{\alpha}\alpha^n - \overline{\beta}\beta^n}{\alpha-\beta}\right)^2$

$= \frac{\overline{\alpha}\overline{\beta}\alpha^n\beta^n + \overline{\beta}\overline{\alpha}\beta^n\alpha^n - \overline{\beta}\overline{\alpha}\beta^{n-1}\alpha^{n+1} - \overline{\alpha}\overline{\beta}\alpha^{n-1}\beta^{n+1}}{(\alpha-\beta)^2}$

$= \frac{\overline{\alpha}\overline{\beta}k^{-2n}\left(\frac{\alpha-\beta}{\alpha}\right) + \overline{\beta}\overline{\alpha}k^{-2n}\left(\frac{\beta-\alpha}{\beta}\right)}{(\alpha-\beta)^2}$

$= \frac{k^{-2n}}{\alpha-\beta}\frac{\beta\overline{\alpha}\overline{\beta} - \alpha\overline{\beta}\overline{\alpha}}{\alpha\beta} = \frac{k^{-2n+2}}{\alpha-\beta}(\beta\overline{\alpha}\overline{\beta} - \alpha\overline{\beta}\overline{\alpha}).$

Then, we have

$\beta\overline{\alpha}\overline{\beta} - \alpha\overline{\beta}\overline{\alpha} = (\beta - \alpha + \alpha\beta^2 - \alpha^2\beta - \alpha^3\beta^2 + \alpha^2\beta^3 - \alpha^4\beta^3 + \alpha^3\beta^4, \beta - \alpha - \alpha^4\beta^2 + \alpha^2\beta^4, \alpha^2\beta - \alpha^3 + \beta^3 - \alpha\beta^2 + \alpha^3\beta^2 - \alpha^4\beta - \alpha\beta^4 - \alpha^2\beta^3, -\alpha^4 + \beta^4).$

So, we obtain

$\breve{H}O_{k,n+1}\breve{H}O_{k,n-1} - \breve{H}O_{k,n}^2 = k^{-2n+2}(-1 - \frac{1}{k^2} - \frac{1}{k^4} - \frac{1}{k^6}, -1 - \frac{1}{k^4}, \frac{1}{k^2} + \frac{1}{k^4} + \frac{1}{k^2}O_{k,3} - O_{k,3}, -O_{k,4}).$

ii. With the Binet formula, we obtain

$\breve{H}P_{k,n+1}\breve{H}P_{k,n-1} - \breve{H}P_{k,n}^2$

$= (\overline{\alpha}\alpha^{n+1} + \overline{\beta}\beta^{n+1})(\overline{\alpha}\alpha^{n-1} + \overline{\beta}\beta^{n-1}) - (\overline{\alpha}\alpha^n + \overline{\beta}\beta^n)^2$

$= k^{-2n+2}\delta(\alpha\overline{\alpha}\overline{\beta} - \beta\overline{\beta}\overline{\alpha}).$

Then, we get

$\left(\alpha\overline{\alpha}\overline{\beta}-\beta\overline{\beta}\overline{\alpha}\right)=\delta(1+\frac{1}{k^2}+\frac{1}{k^4}+\frac{1}{k^6},1-\frac{1}{k^4},\frac{1}{k^2}P_{k,2},1-\frac{4}{k^2})$.

Thus, we have

$\breve{H}P_{k,n+1}\breve{H}P_{k,n-1}-\breve{H}P_{k,n}^2=k^{-2n+2}\delta^2(1+\frac{1}{k^2}+\frac{1}{k^4}+\frac{1}{k^6},1-\frac{1}{k^4},\frac{1}{k^2}P_{k,2},1-\frac{4}{k^2})$. □

Theorem 3.7. ***(Catalan Identity)*** Let $c,n\in\mathbb{N}$. We get

i. $\breve{H}O_{k,n+c}\,\breve{H}O_{k,n-c}-\breve{H}O_{k,n}^2=k^{-2n+2c}O_{k,c}((-1-\frac{1}{k^2}-\frac{1}{k^4}-\frac{1}{k^6})O_{k,c},-O_{k,c}-\frac{1}{k^4}P_{k,c},-\frac{1}{k^4}O_{k,c}+\frac{1}{k^2}P_{k,c},-\frac{1}{k^6}O_{k,c}-\frac{1}{k^2}P_{k,c})$,

ii. $\breve{H}P_{k,n+c}\,\breve{H}P_{k,n-c}-\breve{H}P_{k,n}^2=k^{-2n+2c}O_{k,c}\delta^2((1+\frac{1}{k^2}+\frac{1}{k^4}+\frac{1}{k^6})O_{k,c},O_{k,c}-\frac{1}{k^4}P_{k,c},O_{k,c}+P_{k,2}+\frac{1}{k^2}P_{k,c},O_{k,c}+P_{k,3}-\frac{1}{k^2}P_{k,c})$.

Theorem 3.8. ***(Vajda's Identity)*** Let $a,b,n\in\mathbb{N}$. We obtain

i. $\breve{H}O_{k,n+a}\,\breve{H}O_{k,n+b}-\breve{H}O_{k,n}\,\breve{H}O_{k,n+a+b}$
$=k^{-2n}O_{k,a}((1+\frac{1}{k^2}+\frac{1}{k^4}+\frac{1}{k^6})O_{k,b},O_{k,b}+\frac{1}{k^4}P_{k,b},O_{k,b}+P_{k,2}-\frac{1}{k^2}P_{k,b},O_{k,b}+P_{k,3}+\frac{1}{k^2}P_{k,b})$,

ii. $\breve{H}P_{k,n+a}\breve{H}P_{k,n+b}-\breve{H}P_{k,n}\,\breve{H}P_{k,n+a+b}$
$=k^{-2n}O_{k,a}\delta^2\left(\left(-1-\frac{1}{k^2}-\frac{1}{k^4}-\frac{1}{k^6}\right)O_{k,b},-O_{k,b}-\frac{1}{k^4}P_{k,b},-O_{k,b}P_{k,2}+\frac{1}{k^2}P_{k,b},-O_{k,b}P_{k,3}-\frac{1}{k^2}P_{k,b}\right)$.

Theorem 3.9. ***(d'Ocagne Identity)*** Let $a,b,n\in\mathbb{N}$. $b\geq a+1$. We obtain

i. $\breve{H}O_{k,b}\breve{H}O_{k,a+1}-\breve{H}O_{k,b+1}\breve{H}O_{k,a}=k^{-2a}((1+\frac{1}{k^2}+\frac{1}{k^4}+\frac{1}{k^6})O_{k,b-a},O_{k,b-a}-\frac{1}{k^4}P_{k,b-a},O_{k,b-a}P_{k,2}+\frac{1}{k^2}P_{k,b-a},O_{k,b-a}P_{k,3}-\frac{1}{k^2}P_{k,b-a})$,

ii. $\breve{H}P_{k,b}\breve{H}P_{k,a+1}-\breve{H}P_{k,b+1}\breve{H}P_{k,a}=k^{-2a}\delta^2((-1-\frac{1}{k^2}-\frac{1}{k^4}-\frac{1}{k^6})O_{k,b-a},-O_{k,b-a}-\frac{1}{k^4}P_{k,b-a},-O_{k,b-a}P_{k,2}-\frac{1}{k^2}P_{k,b-a},-O_{k,b}P_{k,3}+\frac{1}{k^2}P_{k,b-a})$.

The proofs of Theorem 3.7.,-3.9., are similar to Theorem 3.6., using Binet formulas, Lemma 3.1., and definitions.

In the following theorems, we examine the relationships between hyperbolic k-Oresme and k-Oresme-Lucas quaternions.

Theorem 3.10. Let $n\in\mathbb{N}$. The following equations are true:

i. $\breve{H}O_{k,n}=\frac{2k}{k^2-4}\breve{H}P_{k,n+1}-\frac{k}{k^2-4}\breve{H}P_{k,n}$,

ii. $\breve{H}O_{k,n}^*=\frac{2k}{k^2-4}\breve{H}P_{k,n+1}^*-\frac{k}{k^2-4}\breve{H}P_{k,n}^*$,

iii. $\breve{H}P_{k,n}=2k\breve{H}O_{k,n+1}-k\breve{H}O_{k,n}$,

iv. $\breve{H}P_{k,n}^*=2k\breve{H}O_{k,n+1}^*-k\breve{H}O_{k,n}^*$.

Proof. i. The following relation is used for proofs;

$$\breve{H}O_{k,n}=a\,x\,\breve{H}P_{k,n+1}+b\,x\,\breve{H}P_{k,n}.$$

For these n values, we obtain;

$$\breve{H}O_{k,0}=a\,x\,\breve{H}P_{k,1}+b\,x\,\breve{H}P_{k,0},$$
$$\breve{H}O_{k,1}=a\,x\,\breve{H}P_{k,2}+b\,x\,\breve{H}P_{k,1}.$$

We find

$$a=\frac{2k}{k^2-4}\text{ and }b=-\frac{k}{k^2-4}.$$

Thus, we obtain

$$\breve{H}O_{k,n}=\frac{2k}{k^2-4}\breve{H}P_{k,n+1}-\frac{k}{k^2-4}\breve{H}P_{k,n}.$$

The proofs of the others are shown similarly. □

Theorem 3.11. For any integer $s\leq t$, we get

i. $\breve{H}O_{k,s}\breve{H}O_{k,t}-\breve{H}O_{k,t}\breve{H}O_{k,s}$
$=k^{-2s}O_{k,t-s}(0,-\frac{1}{k^4},\frac{2}{k^2},-\frac{1}{k^2})$,

ii. $\breve{H}P_{k,s}\breve{H}P_{k,t}-\breve{H}P_{k,t}\breve{H}P_{k,s}$
$=k^{-2s}O_{k,t-s}\delta^2\left(0,-\frac{1}{k^4},\frac{2}{k^2},-\frac{1}{k^2}\right)$,

iii. $\breve{H}O_{k,t}\breve{H}P_{k,s}-\breve{H}O_{k,s}\breve{H}P_{k,t}$
$=k^{-2s}O_{k,t-s}\delta(0,-\frac{1}{k^4},\frac{2}{k^2},-\frac{1}{k^2})$.

Proof. With the Binet formula, we obtain

i. $\breve{H}O_{k,s}\breve{H}O_{k,t}-\breve{H}O_{k,t}\breve{H}O_{k,s}$
$=\left(\frac{\overline{\alpha}\alpha^s-\overline{\beta}\beta^s}{k(\alpha-\beta)}\right)\left(\frac{\overline{\alpha}\alpha^t-\overline{\beta}\beta^t}{k(\alpha-\beta)}\right)-(\frac{\overline{\alpha}\alpha^t-\overline{\beta}\beta^t}{k(\alpha-\beta)})(\frac{\overline{\alpha}\alpha^s-\overline{\beta}\beta^s}{k(\alpha-\beta)})$
$=\frac{(\overline{\alpha}\overline{\beta}-\overline{\beta}\overline{\alpha})(\alpha^s\beta^s)(\alpha^{t-s}-\beta^{t-s})}{k^2(\alpha-\beta)^2}$.

Thus, we get

$\breve{H}O_{k,s}\breve{H}O_{k,t}-\breve{H}O_{k,t}\breve{H}O_{k,s}$
$=k^{-2s}O_{k,t-s}(0,-\frac{1}{k^4},\frac{2}{k^2},-\frac{1}{k^2})$.

The proofs of the others are shown similarly. □

In the following theorems, we associate the terms of the k-Oresme and k-Oresme-Lucas sequences with their hyperbolic quaternion values.

Theorem 3.12. Let $a,b,n\in\mathbb{N}$. $b\geq a+1$. We obtain

i. $\breve{H}O_{k,a+b}+\frac{1}{k^{2a}}\breve{H}O_{k,b-a}=P_{k,a}\breve{H}O_{k,b}$,

ii. $\breve{H}O_{k,a+b}-\frac{1}{k^{2a}}\breve{H}O_{k,b-a}=O_{k,a}\breve{H}P_{k,b}$,

iii. $\breve{H}P_{k,a+b}+\frac{1}{k^{2a}}\breve{H}P_{k,b-a}=P_{k,a}\breve{H}P_{k,b}$.

Proof. i. If Binet formulas are used, we get

$$P_{k,a}\breve{H}O_{k,b}=(\alpha^a+\beta^a)(\frac{\overline{\alpha}\alpha^b-\overline{\beta}\beta^b}{k(\alpha-\beta)})$$
$$=\frac{\overline{\alpha}\alpha^{a+b}-\overline{\beta}\beta^{a+b}+\overline{\alpha}\alpha^b\beta^a-\overline{\beta}\beta^b\alpha^a}{k(\alpha-\beta)}$$
$$=\frac{\overline{\alpha}\alpha^{a+b}-\overline{\beta}\beta^{a+b}}{k(\alpha-\beta)}+\frac{\alpha^a\beta^a(\overline{\alpha}\alpha^{b-a}-\overline{\beta}\beta^{b-a})}{k(\alpha-\beta)}.$$

Thus, we have

$$\breve{H}O_{k,a+b}+k^{-2a}\breve{H}O_{k,b-a}=P_{k,a}\breve{H}O_{k,b}.$$

The proofs of the others are shown similarly in i. □

Theorem 3.13. For all $a,b\geq 1$, we have

i. $\breve{H}O_{k,2a+b+1}=P_{k,2a+1}\breve{H}O_{k,b+1}+\frac{1}{k^2}O_{k,2a}\breve{H}P_{k,b}$,

ii. $\breve{H}P_{k,2a+b+1}=P_{k,2a+1}\breve{H}P_{k,b+1}-\delta^2O_{k,2a}\breve{H}O_{k,b}$,

iii. $\breve{H}O_{k,2a+b+1} = k^2 \frac{P_{k,2a+1}}{k^2-3} \breve{H}O_{k,b+3} + \frac{O_{k,2a-2}}{(k^2-3)k^8} \breve{H}P_{k,b}$,

iv. $\breve{H}P_{k,2a+b+1} = k^2 \frac{P_{k,2a+1}}{k^2-3} \breve{H}P_{k,b+3} + \frac{1}{k^2} \delta^2 \frac{O_{k,2a-2}}{(k^2-3)} \breve{H}O_{k,b}$.

Proof. i. If Binet formulas are used, we get

$$P_{k,2a+1}\breve{H}O_{k,b+1} + \frac{1}{k^2} O_{k,2a}\breve{H}P_{k,b}$$
$$= (\alpha^{2a+1} + \beta^{2a+1})\left(\frac{\overline{\alpha}\alpha^{b+1} - \overline{\beta}\beta^{b+1}}{k(\alpha-\beta)}\right) + \frac{1}{k^2}\left(\frac{\alpha^{2a}-\beta^{2a}}{k(\alpha-\beta)}\right)(\overline{\alpha}\alpha^b + \overline{\beta}\beta^b)$$
$$= \frac{1}{k(\alpha-\beta)}\left[\overline{\alpha}\alpha^{2a+b+1}\left(\alpha + \frac{1}{\alpha k^2}\right) - \overline{\beta}\beta^{2a+b+1}\left(\beta + \frac{1}{\beta k^2}\right)\right].$$

Since, $\alpha + \beta = 1$ and $\alpha\beta = \frac{1}{k^2}$. Thus, we obtain

$$\breve{H}O_{k,2a+b+1} = P_{k,2a+1}\breve{H}O_{k,b+1} + \frac{1}{k^2} O_{k,2a}\breve{H}P_{k,b}.$$

The proofs of the others are shown similarly in i. □

Theorem 3.14. For all $a, b, c, d \in \mathbb{N}$, we obtain

i. $\breve{H}O_{k,a+b} = kO_{k,b}\breve{H}O_{k,a+1} - \frac{1}{k} O_{k,b-1}\breve{H}O_{k,a}$,

ii. $\breve{H}O_{k,a+2b} = P_{k,b}\breve{H}O_{k,a+b} - \frac{1}{k^{2b-2}}\breve{H}O_{k,a}$,

iii. $\breve{H}O_{k,a+bd} = \frac{O_{k,bd}}{O_{k,b}}\breve{H}O_{k,a+b} - \frac{1}{k^{2b-2}}\frac{O_{k,bd-b}}{O_{k,b}}\breve{H}O_{k,a}$,

iv. $k^{-2bd}O_{k,b(c-d)}\breve{H}O_{k,a} = \breve{H}O_{k,a+bd}O_{k,bc} - \breve{H}O_{k,a+bc}O_{k,bd}$.

Proof. i. If Binet formulas are used, we get

$$kO_{k,b}\breve{H}O_{k,a+1} - \frac{1}{k} O_{k,b-1}\breve{H}O_{k,a}$$
$$= k\left(\frac{\alpha^b-\beta^b}{k(\alpha-\beta)}\right)\left(\frac{\overline{\alpha}\alpha^{a+1}-\overline{\beta}\beta^{a+1}}{k(\alpha-\beta)}\right) - \frac{1}{k}\left(\frac{\alpha^{b-1}-\beta^{b-1}}{k(\alpha-\beta)}\right)\left(\frac{\overline{\alpha}\alpha^a-\overline{\beta}\beta^a}{k(\alpha-\beta)}\right)$$
$$= \frac{1}{k^2(\alpha-\beta)^2}\left[\overline{\alpha}\alpha^{a+b}\left(k\alpha - \frac{1}{k\alpha}\right) - \overline{\beta}\beta^{a+b}\left(k\beta - \frac{1}{k\beta}\right)\right].$$

Since, $\alpha + \beta = 1$ and $\alpha\beta = \frac{1}{k^2}$. Thus, we obtain

$$\breve{H}O_{k,a+b} = kO_{k,b}\breve{H}O_{k,a+1} - \frac{1}{k} O_{k,b-1}\breve{H}O_{k,a}.$$

The proofs of the others are shown similarly in i. □

Theorem 3.15. If $a, b \in \mathbb{N}$ and $a \le b$, we obtain

i. $\breve{H}O_{k,2a+b} = O_{k,2a}\breve{H}O_{k,b+2} - \frac{1}{k^4} O_{k,2a-2}\breve{H}O_{k,b}$,

ii. $\breve{H}P_{k,2a+b} = kO_{k,2a}\breve{H}P_{k,b+2} - \frac{1}{k^3} O_{k,2a-2}\breve{H}P_{k,b}$,

iii. $\breve{H}O_{k,2a+b} = O_{k,2a}\breve{H}P_{k,b+1} - \frac{1}{k^2} P_{k,2a-1}\breve{H}O_{k,b}$,

iv. $\breve{H}P_{k,2a+b} = k^2O_{k,2a}\delta^2\breve{H}O_{k,b+1} + \frac{1}{k^2} P_{k,2a-1}\breve{H}P_{k,b}$.

Proof. iv. With the Binet formula, we have

$$k^2O_{k,2a}\delta^2\breve{H}O_{k,b+1} + \frac{1}{k^2} P_{k,2a-1}\breve{H}P_{k,b}$$
$$= k^2 \frac{\alpha^{2a}-\beta^{2a}}{k(\alpha-\beta)}(\alpha-\beta)^2\left(\frac{\overline{\alpha}\alpha^{b+1}-\overline{\beta}\beta^{b+1}}{k(\alpha-\beta)}\right) + \frac{1}{k^2}(\alpha^{2a-1} + \beta^{2a-1})(\overline{\alpha}\alpha^b + \overline{\beta}\beta^b)$$
$$= \overline{\alpha}\alpha^{2a+b}\left(\alpha + \frac{1}{k^2\alpha}\right) + \overline{\beta}\beta^{2a+b}\left(\beta + \frac{1}{k^2\beta}\right)$$

Since, $\alpha\beta = \frac{1}{k^2}$. Thus, we get

$$\breve{H}P_{k,2a+b} = k^2O_{k,2a}\delta^2\breve{H}O_{k,b+1} + \frac{1}{k^2} P_{k,2a-1}\breve{H}P_{k,b}.$$

□

Theorem 3.16. Let $x, y, z \in \mathbb{N}$ and $x, y \ge z$. We obtain

i. $(k^2-4)\breve{H}O_{k,x}^2 - \breve{H}P_{k,x}^2 = \frac{-2\delta}{k^{2x}}(0, -\frac{1}{k^4}, \frac{2}{k^2}, -\frac{1}{k^2})$,

ii. $\breve{H}O_{k,x+y}\breve{H}P_{k,x+z} - \breve{H}O_{k,x+z}\breve{H}P_{k,x+y} = \frac{2}{k^{2x+2z}} O_{k,y-z}(\breve{H}P_{k,0} - 1 + \frac{1}{k^2} + \frac{1}{k^4} + \frac{1}{k^6})$.

Proof. With the Binet formulas, we have

i. $(k^2-4)\breve{H}O_{k,x}^2 - \breve{H}P_{k,x}^2 = (k^2-4)\left(\frac{\overline{\alpha}\alpha^x-\overline{\beta}\beta^x}{k(\alpha-\beta)}\right)\left(\frac{\overline{\alpha}\alpha^x-\overline{\beta}\beta^x}{k(\alpha-\beta)}\right) - (\overline{\alpha}\alpha^x + \overline{\beta}\beta^x)(\overline{\alpha}\alpha^x + \overline{\beta}\beta^x)$

$$= -2\overline{\alpha}\overline{\beta}\alpha^x\beta^x - 2\overline{\beta}\overline{\alpha}\beta^x\alpha^x = -2\alpha^x\beta^x(\overline{\alpha}\overline{\beta} - \overline{\beta}\overline{\alpha}).$$

Thus, we obtain

$$(k^2-4)\breve{H}O_{k,x}^2 - \breve{H}P_{k,x}^2 = \frac{-2\delta}{k^{2x}}(0, -\frac{1}{k^4}, \frac{2}{k^2}, -\frac{1}{k^2}).$$

ii. $\breve{H}O_{k,x+y}\breve{H}P_{k,x+z} - \breve{H}O_{k,x+z}\breve{H}P_{k,x+y}$

$$= \left(\frac{\overline{\alpha}\alpha^{x+y}-\overline{\beta}\beta^{x+y}}{k(\alpha-\beta)}\right)(\overline{\alpha}\alpha^{x+z} + \overline{\beta}\beta^{x+z}) - \left(\frac{\overline{\alpha}\alpha^{x+z}-\overline{\beta}\beta^{x+z}}{k(\alpha-\beta)}\right)(\overline{\alpha}\alpha^{x+y} + \overline{\beta}\beta^{x+y})$$
$$= \frac{\overline{\alpha}\overline{\beta}\alpha^{x+y}\beta^{x+z} - \overline{\beta}\overline{\alpha}\alpha^{x+z}\beta^{x+y} - \overline{\alpha}\overline{\beta}\alpha^{x+z}\beta^{x+y} + \overline{\beta}\overline{\alpha}\alpha^{x+y}\beta^{x+z}}{k(\alpha-\beta)}$$
$$= \frac{\alpha^x\beta^x\alpha^z\beta^z(\alpha^{y-z}-\beta^{y-z})}{k(\alpha-\beta)}(\overline{\alpha}\overline{\beta} + \overline{\beta}\overline{\alpha})$$

Thus, we get

$$\breve{H}O_{k,x+y}\breve{H}P_{k,x+z} - \breve{H}O_{k,x+z}\breve{H}P_{k,x+y} = \frac{2}{k^{2x+2z}} O_{k,y-z}(\breve{H}P_{k,0} - 1 + \frac{1}{k^2} + \frac{1}{k^4} + \frac{1}{k^6}).$$ □

4 Conclusion

In this study, we defined the hyperbolic k-Oresme and k-Oresme-Lucas quaternions. Then, we obtained some properties of these quaternions. Also, we examined the relationships between these quaternions. In addition, we found relations among the hyperbolic k-Oresme and k-Oresme-Lucas quaternions and their conjugates. Furthermore, we calculated the special identities of these quaternions. Moreover, we found the terms of the k-Oresme and k-Oresme-Lucas sequences are associated with their hyperbolic quaternion values. In the future, we can spread a new approach to hyperbolic k-Oresme and k-Oresme-Lucas octonions and sedenions.

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