Some Binomial Sums of k-Pell, k-Pell-Lucas and Modified k-Pell Numbers

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Abstract: The main goal of this paper is to find some new identities containing k-Pell and k-Pell-Lucas numbers. In addition, we use these identities to prove binomial properties of k-Pell, k-Pell-Lucas, and modified k-Pell numbers. It is possible to prove some new properties of k-Pell, k-Pell-Lucas and modified k-Pell numbers based on the obtained results. The generated functions can also be expressed in closed form using them.

Key–Words: Pell Numbers, Pell-Lucas Numbers, k-Pell Numbers, k-Pell-Lucas Numbers.

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1 Introduction

The Fibonacci sequence is one of the most renowned and exceptional integer sequence in number theory and has been regularly studied in different branches of mathematics. There is the Pell number, which is as crucial as the Fibonacci number. Like the Fibonacci number, the Pell number can also be generalized by changing recurrence relations, initial conditions or the two of them. One particular example of Pell number is k-Pell and k-Pell-Lucas numbers. In last few years, different authors studied Pell and Pell-Lucas numbers, see [14, 11, 10, 2, 1, 13]

In recent years, many researchers have engaged in the study of the generalizations of specific sequences of positive integers. In particular, the study of the k-Pell sequence, the k-Pell-Lucas sequence, and the modified k-Pell sequence, see [4, 7, 9, 5, 8, 15].

The purpose of this paper is to examine some properties of the k-Pell, k-Pell-Lucas, and modified k-Pell numbers. All three k-Pell numbers, k-Pell-Lucas and modified k-Pell numbers are generalizations of the Pell numbers, which are used to calculate area of cycloid. Binomial properties of these numbers help us to understand their behaviour and applications.

Definition 1.1. (**Catarino** [4]). *The k-Pell numbers satisfy the recurrence relation* $\rho_{k,n+1} = 2\rho_{k,n} + k\rho_{k,n-1}$, *for* $n \geq 1$ *with* $\rho_{k,0} = 0$ *and* $\rho_{k,1} = 1$.

Definition 1.2. (**Catarino** [4]). *The k-Pell-Lucas numbers satisfy the recurrence relation* $\varrho_{k,n+1} = 2\varrho_{k,n} + k\varrho_{k,n-1}$, *for* $n \geq 1$ *with* $\varrho_{k,0} = 2$ *and* $\varrho_{k,1} = 2$.

Definition 1.3. (**Catarino** [6]). *The modified k-Pell numbers satisfy the recurrence relation* $\xi_{k,n+1} = 2\xi_{k,n} + k\xi_{k,n-1}$, *for* $n \geq 1$ *with* $\xi_{k,0} = 1$ *and* $\xi_{k,1} = 1$.

The Binet formula of $\rho_{k,n}$, $\varrho_{k,n}$ and $\xi_{k,n}$ is given by (Catarino [6])

$$\rho_{k,n} = \frac{\zeta_1^n - \zeta_2^n}{\zeta_1 - \zeta_2}, \tag{1}$$

$$\varrho_{k,n} = \zeta_1^n + \zeta_2^n, \tag{2}$$

$$\xi_{k,n} = \frac{\zeta_1^n - \zeta_2^n}{2}. \tag{3}$$

The characteristic roots ζ_1 and ζ_2 appeared in (1), (2) and (3) satisfy the following relations (Catarino [6]):

$$\zeta_1 = 1 + \sqrt{1+k}, \tag{4}$$

$$\zeta_2 = 1 - \sqrt{1+k}, \tag{5}$$

$$\zeta_1 - \zeta_2 = 2\sqrt{1+k} = 2\sqrt{\delta}, \tag{6}$$

$$\zeta_1 + \zeta_2 = 2, \tag{7}$$

$$\zeta_1\zeta_2 = -k, \tag{8}$$

$$\zeta_1^2 = 2\zeta_1 + k \tag{9}$$

$$\zeta_2^2 = 2\zeta_2 + k \tag{10}$$

For $k = 1$, the k-Pell number converts into a well-known Pell number. These numbers are the source of many fascinating properties. In the last few years, many authors devoted in the studies of these numbers, see [4, 7, 9, 5, 8, 15, 6, 12] and the references cited in there. Some of these identities are listed below.

Lemma 1.1. (**Catarino** [6]). *Let* $n, m \in \mathbb{Z}^+$. *Then*

(1) $\rho_{k,n+1} + k\rho_{k,n-1} = \varrho_{k,n},$ (11)

(2) $\rho_{k,n+1} - \rho_{k,n} = \xi_{k,n},$ (12)

(3) $\varrho_{k,n} + k\varrho_{k,n-1} = 2\rho_{k,n}(1+k),$ (13)

(4) $\rho_{k,n-r}\rho_{k,n+r} - \rho_{k,n}^2 = -(-k)^{n-r},$ (14)

(5) $\rho_{k,n-1}\rho_{k,n+1} - \rho_{k,n}^2 = -(-k)^{n-1},$ (15)

(6) $\varrho_{k,n-1}\varrho_{k,n+1} - \varrho_{k,n}^2 = 4(-k)^{n-1}(1+k).$ (16)

Carlitz [3] in 1970 formulated distinct Fibonacci and Lucas identities. In the year 1997, Zhang [16] established varied identities for second-order integer sequences. Motivated necessarily by the above-cited works, we focus on investigating various binomial sums for the numbers $\rho_{k,n}$, $\varrho_{k,n}$ and $\xi_{k,n}$.

2 Binomial Identities Involving $\rho_{k,n}$, $\varrho_{k,n}$ and $\xi_{k,n}$

We aim here to obtain some essential identities involving $\rho_{k,n}$ and $\varrho_{k,n}$ numbers. In this section, we formulate some binomial sums for $\rho_{k,n}$, $\varrho_{k,n}$ and $\xi_{k,n}$. First, we prove the Lemma 2.1 which plays a crucial role in the proofs of the Theorems 2.1 to 2.5.

Lemma 2.1. *Let* $u = \zeta_1$ *or* ζ_2. *Then*

(a) $u^n = u\rho_{k,n} + k\rho_{k,n-1},$ (17)

(b) $u^{2n} = u^n\varrho_{k,n} - (-k)^n,$ (18)

(c) $u^{tn} = u^n\frac{\rho_{k,tn}}{\rho_{k,n}} - (-k)^n\frac{\rho_{k,(t-1)n}}{\rho_{k,n}},$ (19)

(d) $u^{sn}\rho_{k,rn} - u^{rn}\rho_{k,sn} = (-k)^{sn}\rho_{k,(r-s)n}.$ (20)

Proof. We apply P. M. I. on n to prove (a). For $n = 2$, we have from (9) and (10)

$$\zeta_1^2 = \zeta_1\rho_{k,2} + k\rho_{k,1},$$
$$\zeta_2^2 = \zeta_2\rho_{k,2} + k\rho_{k,1}.$$

Now consider that the result is true for n. Thus, we have

$$\zeta_1^n = \zeta_1\rho_{k,n} + k\rho_{k,n-1}, \quad (21)$$
$$\zeta_2^n = \zeta_2\rho_{k,n} + k\rho_{k,n-1}. \quad (22)$$

Moreover, by applying (9) and (21), we achieve

$$\begin{aligned}\zeta_1^{n+1} &= \zeta_1{\zeta_1}^n \\ &= \zeta_1(\zeta_1\rho_{k,n} + k\rho_{k,n-1}) \\ &= {\zeta_1}^2\rho_{k,n} + k\zeta_1\rho_{k,n-1} \\ &= (2\zeta_1 + k)\rho_{k,n} + k\zeta_1\rho_{k,n-1} \\ &= (2\rho_{k,n} + k\rho_{k,n-1})\zeta_1 + k\rho_{k,n} \\ &= \rho_{k,n+1}\zeta_1 + k\rho_{k,n}.\end{aligned}$$

Similarly, we can prove that

$$\zeta_2^{n+1} = \zeta_2\rho_{k,n+1} + k\rho_{k,n}.$$

Thus the proof of (a).

(b) By using the result (a), we can write

$$\begin{aligned}u^{2n} &= \rho_{k,n}u^{n+1} + ku^n\rho_{k,n-1} \\ &= \rho_{k,n}(u\rho_{k,n+1} + k\rho_{k,n}) + ku^n\rho_{k,n-1} \\ &= u\rho_{k,n}\rho_{k,n+1} + k\rho_{k,n-1}u^n + k\rho_{k,n}^2 \\ &= (u^n - k\rho_{k,n-1})\rho_{k,n+1} + k\rho_{k,n-1}u^n + k\rho_{k,n}^2 \\ &= u^n(\rho_{k,n+1} + k\rho_{k,n-1}) + k(\rho_{k,n}^2 - \rho_{k,n+1}\rho_{k,n-1}).\end{aligned}$$

At last, by applying (11) and (15), we achieve

$$u^{2n} = \varrho_{k,n}u^n - (-k)^n.$$

(c) Let $u = \zeta_1$. By employing the Binet formula of $\rho_{k,n}$, we can write

$$\begin{aligned}&{\zeta_1}^n\frac{\rho_{k,tn}}{\rho_{k,n}} - (-k)^n\frac{\rho_{k,(t-1)n}}{\rho_{k,n}} \\ &= \frac{1}{\rho_{k,n}}\left\{\left(\frac{{\zeta_1}^{tn} - {\zeta_2}^{tn}}{\zeta_1 - \zeta_2}\right){\zeta_1}^n - (\zeta_1\zeta_2)^n\left(\frac{{\zeta_1}^{(t-1)n} - {\zeta_2}^{(t-1)n}}{\zeta_1 - \zeta_2}\right)\right\} \\ &= \frac{1}{\rho_{k,n}}\left\{\frac{\zeta_1^{tn}\zeta_1^n - \zeta_2^{tn}\zeta_1^n - \zeta_2^n\zeta_1^{tn} + \zeta_1^n\zeta_2^{tn}}{\zeta_1 - \zeta_2}\right\} \\ &= \frac{1}{\rho_{k,n}}\left\{\frac{\zeta_1^{tn}(\zeta_1^n - \zeta_2^n)}{\zeta_1 - \zeta_2}\right\} \\ &= \frac{1}{\rho_{k,n}}(\zeta_1^{tn}\rho_{k,n}) \\ &= \zeta_1^{tn}.\end{aligned}$$

Similarly, if $u = \zeta_2$ then, we get the desired result. The proof of (d) is analogous to (c). Hence, we discard the proof. □

Theorem 2.1. *Let $n, r, s, t \in \mathbb{Z}^+$ with $t \geq 1$. Then prove that*

(1) $\rho_{k,n+t} = \rho_{k,n}\rho_{k,t+1} + k\rho_{k,n-1}\rho_{k,t}$,
(2) $\varrho_{k,n+t} = \rho_{k,n}\varrho_{k,t+1} + k\rho_{k,n-1}\varrho_{k,t}$
(3) $\xi_{k,n+t} = \rho_{k,n}\xi_{k,t+1} + k\rho_{k,n-1}\xi_{k,t}$
(4) $\rho_{k,2n+t} = \varrho_{k,n}\rho_{k,n+t} - (-k)^n\rho_{k,t}$,
(5) $\varrho_{k,2n+t} = \varrho_{k,n}\varrho_{k,n+t} - (-k)^n\varrho_{k,t}$,
(6) $\xi_{k,2n+t} = \varrho_{k,n}\xi_{k,n+t} - (-k)^n\xi_{k,t}$,
(7) $\rho_{k,sn+t}\rho_{k,n} = \rho_{k,sn}\rho_{k,n+t} - (-k)^n\rho_{k,(s-1)n}\rho_{k,t}$,
(8) $\varrho_{k,sn+t}\rho_{k,n} = \rho_{k,sn}\varrho_{k,n+t} - (-k)^n\rho_{k,(s-1)n}\varrho_{k,t}$,
(9) $\xi_{k,sn+t}\rho_{k,n} = \rho_{k,sn}\xi_{k,n+t} - (-k)^n\rho_{k,(s-1)n}\xi_{k,t}$,
(10) $\rho_{k,sn+t}\rho_{k,rn} - \rho_{k,rn+t}\rho_{k,sn} = (-k)^{sn}\rho_{k,t}\rho_{k,(r-s)n}$,
(11) $\varrho_{k,sn+t}\rho_{k,rn} - \varrho_{k,rn+t}\rho_{k,sn} = (-k)^{sn}\varrho_{k,t}\rho_{k,(r-s)n}$,
(12) $\xi_{k,sn+t}\rho_{k,rn} - \xi_{k,rn+t}\rho_{k,sn} = (-k)^{sn}\xi_{k,t}\rho_{k,(r-s)n}$.

Proof. The proofs of (2) to (12) are analogous to (1). Hence, we prove the result (1). Applying the Lemma 2.1(a), we can write

$$\zeta_1^n = \rho_{k,n}\zeta_1 + k\rho_{k,n-1}, \quad (23)$$
$$\zeta_2^n = \rho_{k,n}\zeta_2 + k\rho_{k,n-1}. \quad (24)$$

Now, by multiplying (23) by $\frac{\zeta_1^t}{\zeta_1-\zeta_2}$, (24) by $\frac{\zeta_2^t}{\zeta_1-\zeta_2}$ and subtracting, we achieve

$$\frac{\zeta_1^{n+t}-\zeta_2^{n+t}}{\zeta_1-\zeta_2} = \rho_{k,n}\left(\frac{\zeta_1^{t+1}-\zeta_2^{t+1}}{\zeta_1-\zeta_2}\right) + k\rho_{k,n-1}\left(\frac{\zeta_1^t-\zeta_2^t}{\zeta_1-\zeta_2}\right).$$

In conclusion, by employing the Binet formula of $\rho_{k,n}$, we get

$$\rho_{k,n+t} = \rho_{k,n}\rho_{k,t+1} + k\rho_{k,n-1}\rho_{k,t}.$$

Thus the proof of result (1).

□

Theorem 2.2. *Let $n, r, s, t \in \mathbb{Z}^+$ with $t \geq 1$. Then*

(1) $\rho_{k,rn+t} = \sum_{i=0}^{n}\binom{n}{i}k^{n-i}\rho_{k,r}^i\rho_{k,r-1}^{n-i}\rho_{k,i+t}$,

(2) $\varrho_{k,rn+t} = \sum_{i=0}^{n}\binom{n}{i}k^{n-i}\rho_{k,r}^i\rho_{k,r-1}^{n-i}\varrho_{k,i+t}$,

(3) $\xi_{k,rn+t} = \sum_{i=0}^{n}\binom{n}{i}k^{n-i}\rho_{k,r}^i\rho_{k,r-1}^{n-i}\xi_{k,i+t}$,

(4) $\rho_{k,n+t}\rho_{k,r}^n = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}k^{n-i}\rho_{k,r-1}^{n-i}\rho_{k,ri+t}$,

(5) $\varrho_{k,n+t}\rho_{k,r}^n = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}k^{n-i}\rho_{k,r-1}^{n-i}\varrho_{k,ri+t}$,

(6) $\xi_{k,n+t}\rho_{k,r}^n = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}k^{n-i}\rho_{k,r-1}^{n-i}\xi_{k,ri+t}$,

(7) $\rho_{k,t}\rho_{k,r-1}^n = \frac{1}{k^n}\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\rho_{k,r}^{n-i}\rho_{n+(r-1)i+t}$,

(8) $\varrho_{k,t}\rho_{k,r-1}^n = \frac{1}{k^n}\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\rho_{k,r}^{n-i}\varrho_{n+(r-1)i+t}$,

(9) $\xi_{k,t}\rho_{k,r-1}^n = \frac{1}{k^n}\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\rho_{k,r}^{n-i}\xi_{n+(r-1)i+t}$.

Proof. Starting with the Lemma 2.1(a), we have

$$\zeta_1^r = \rho_{k,r}\zeta_1 + k\rho_{k,r-1},$$
$$\zeta_2^r = \rho_{k,r}\zeta_2 + k\rho_{k,r-1}.$$

By making use of the binomial theorem, we get

$$\zeta_1^{rn} = \sum_{i=0}^{n}\binom{n}{i}k^{(n-i)}\rho_{k,r}^i\rho_{k,r-1}^{n-i}\zeta_1^i, \quad (25)$$
$$\zeta_2^{rn} = \sum_{i=0}^{n}\binom{n}{i}k^{(n-i)}\rho_{k,r}^i\rho_{k,r-1}^{n-i}\zeta_2^i. \quad (26)$$

Moreover, by multiplying an equation (25) by $\frac{\zeta_1^t}{\zeta_1-\zeta_2}$, equation (26) by $\frac{\zeta_2^t}{\zeta_1-\zeta_2}$ and subtracting, we obtain

$$\frac{\zeta_1^{rn+t}-\zeta_2^{rn+t}}{\zeta_1-\zeta_2} = \sum_{i=0}^{n}\binom{n}{i}k^{(n-i)}\rho_{k,r}^i\rho_{k,r-1}^{n-i}(\frac{\zeta_1^{i+t}-\zeta_2^{i+t}}{\zeta_1-\zeta_2}).$$

Finally, using the Binet formula of $\rho_{k,n}$, we get the required result

$$\rho_{k,rn+t} = \sum_{i=0}^{n}\binom{n}{i}k^{(n-i)}\rho_{k,r}^i\rho_{k,r-1}^{n-i}\rho_{k,i+t}.$$

Again, by multiplying equation (25) by ζ_1^t, (26) by ζ_2^t and adding, we achieve

$$\zeta_1^{rn+t}+\zeta_2^{rn+t}=\sum_{i=0}^{n}\binom{n}{i}k^{(n-i)}\rho_{k,r}^{i}\rho_{k,r-1}^{n-i}(\zeta_1^{i+t}+\zeta_2^{i+t}).$$

Furthermore, by applying the Binet formula of $\varrho_{k,n}$, we get

$$\varrho_{k,rn+t}=\sum_{i=0}^{n}\binom{n}{i}k^{(n-i)}\rho_{k,r}^{i}\rho_{k,r-1}^{n-i}\varrho_{k,i+t}.$$

Thus, the result (2).
The proof of (3)-(9) is similar to (1) and (2). Hence, we discard the proof. □

Theorem 2.3. *If $n,r,s,t\in\mathbb{Z}^+$ with $t\geq 1$ then*

(1) $\rho_{k,2rn+t}=\sum_{i=0}^{n}\binom{n}{i}(-1)^{(r+1)(n-i)}k^{r(n-i)}\varrho_{k,r}^{i}\rho_{k,ri+t}$,

(2) $\varrho_{k,2rn+t}=\sum_{i=0}^{n}\binom{n}{i}(-1)^{(r+1)(n-i)}k^{r(n-i)}\varrho_{k,r}^{i}\varrho_{k,ri+t}$,

(3) $\xi_{k,2rn+t}=\sum_{i=0}^{n}\binom{n}{i}(-1)^{(r+1)(n-i)}k^{r(n-i)}\varrho_{k,r}^{i}\xi_{k,ri+t}$,

(4) $\rho_{k,rn+t}\varrho_{k,r}^{n}=\sum_{i=0}^{n}\binom{n}{i}(-k)^{r(n-i)}\rho_{k,2ri+t}$,

(5) $\varrho_{k,rn+t}\varrho_{k,r}^{n}=\sum_{i=0}^{n}\binom{n}{i}(-k)^{r(n-i)}\varrho_{k,2ri+t}$,

(6) $\xi_{k,rn+t}\varrho_{k,r}^{n}=\sum_{i=0}^{n}\binom{n}{i}(-k)^{r(n-i)}\xi_{k,2ri+t}$,

(7) $\rho_{k,t}(-k)^{rn}=\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\varrho_{k,r}^{i}\rho_{k,(2n-i)r+t}$,

(8) $\varrho_{k,t}(-k)^{rn}=\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\varrho_{k,r}^{i}\varrho_{k,(2n-i)r+t}$,

(9) $\xi_{k,t}(-k)^{rn}=\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\varrho_{k,r}^{i}\xi_{k,(2n-i)r+t}$.

Proof. By using the Lemma 2.1(b), we rewrite

$$\zeta_1^{2r}=\zeta_1^{r}\varrho_{k,r}-(-k)^{r},$$
$$\zeta_2^{2r}=\zeta_2^{r}\varrho_{k,r}-(-k)^{r}.$$

By making use of the binomial theorem, we get

$$\zeta_1^{2rn}=\sum_{i=0}^{n}\binom{n}{i}(-1)^{(r+1)(n-i)}\varrho_{k,r}^{i}\zeta_1^{ri}(k)^{r(n-i)},\quad(27)$$
$$\zeta_2^{2rn}=\sum_{i=0}^{n}\binom{n}{i}(-1)^{(r+1)(n-i)}\varrho_{k,r}^{i}\zeta_2^{ri}(k)^{r(n-i)}.\quad(28)$$

Multiplying an equation (27) by $\dfrac{\zeta_1^t}{\zeta_1-\zeta_2}$, equation (28) by $\dfrac{\zeta_2^t}{\zeta_1-\zeta_2}$ and subtracting, we obtain

$$\frac{\zeta_1^{2rn+t}-\zeta_2^{2rn+t}}{\zeta_1-\zeta_2}=\sum_{i=0}^{n}\binom{n}{i}\varrho_{k,r}^{i}(-1)^{(n-i)(r+1)}k^{r(n-i)}\left(\frac{\zeta_1^{ri+t}-\zeta_2^{ri+t}}{\zeta_1-\zeta_2}\right).$$

Now, using the Binet formula of $\rho_{k,n}$, we achieve

$$\rho_{k,2rn+t}=\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)(r+1)}k^{r(n-i)}\varrho_{k,r}^{i}\rho_{k,ri+t}.$$

Thus the result (1).

Moreover, by multiplying equation (27) by ζ_1^t, (28) by ζ_2^t and adding, we get

$$\zeta_1^{2rn+t}+\zeta_2^{2rn+t}=\sum_{i=0}^{n}\binom{n}{i}\varrho_{k,r}^{i}(-1)^{(n-i)(r+1)}k^{r(n-i)}(\zeta_1^{ri+t}+\zeta_2^{ri+t}).$$

Again, using the Binet formula of $\varrho_{k,n}$, we obtain the desired result

$$\varrho_{k,2rn+t}=\sum_{i=0}^{n}\binom{n}{i}\varrho_{k,r}^{i}(-1)^{(n-i)(r+1)}2^{r(n-i)}\varrho_{k,ri+t}.$$

Hence the proof of result (2).

The proofs of (3)-(6) are analogous to (1) and (2). Hence, we omit the proof. □

Theorem 2.4. *Let $n, r, s, t, l \in \mathbb{Z}^+$ with $t \geq 1$. Then*

(1) $\rho_{k,trn+l}\rho_{k,r}^{n} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(r+1)(n-i)}k^{r(n-i)} \rho_{k,tr}^{i}\rho_{k,(t-1)r}^{(n-i)}\rho_{k,ri+l}$,

(2) $\varrho_{k,trn+l}\rho_{k,r}^{n} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(r+1)(n-i)}k^{r(n-i)}\rho_{k,tr}^{i} \rho_{k,(t-1)r}^{(n-i)}\varrho_{k,ri+l}$,

(3) $\xi_{k,trn+l}\rho_{k,r}^{n} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(r+1)(n-i)}k^{r(n-i)}\rho_{k,tr}^{i} \rho_{k,(t-1)r}^{(n-i)}\xi_{k,ri+l}$,

(4) $\rho_{k,rn+l}\rho_{k,tr}^{n} = \sum_{i=0}^{n}\binom{n}{i}(-k)^{r(n-i)}\rho_{k,r}^{i} \rho_{k,(t-1)r}^{(n-i)}\rho_{k,tri+l}$,

(5) $\varrho_{k,rn+l}\rho_{k,tr}^{n} = \sum_{i=0}^{n}\binom{n}{i}(-k)^{r(n-i)}\rho_{k,r}^{i} \rho_{k,(t-1)r}^{(n-i)}\varrho_{k,tri+l}$,

(6) $\xi_{k,rn+l}\rho_{k,tr}^{n} = \sum_{i=0}^{n}\binom{n}{i}(-k)^{r(n-i)}\rho_{k,r}^{i} \rho_{k,(t-1)r}^{(n-i)}\xi_{k,tri+l}$,

(7) $(-k)^{rn}\rho_{k,l}\rho_{k,(t-1)r}^{n} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}\rho_{k,tr}^{i} \rho_{k,r}^{(n-i)}\rho_{k,ri+l}$,

(8) $(-k)^{rn}\varrho_{k,l}\rho_{k,(t-1)r}^{n} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}\rho_{k,tr}^{i} \rho_{k,r}^{(n-i)}\varrho_{k,ri+l}$,

(9) $(-k)^{rn}\xi_{k,l}\rho_{k,(t-1)r}^{n} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}\rho_{k,tr}^{i} \rho_{k,r}^{(n-i)}\xi_{k,ri+l}$

Proof. By making use of the Lemma 2.1(c), we have

$$\zeta_1^{tr} = \zeta_1^{r}\frac{\rho_{k,tr}}{\rho_{k,r}} - (-k)^{r}\frac{\rho_{k,(t-1)r}}{\rho_{k,r}},$$
$$\zeta_2^{tr} = \zeta_2^{r}\frac{\rho_{k,tr}}{\rho_{k,r}} - (-k)^{r}\frac{\rho_{k,(t-1)r}}{\rho_{k,r}}.$$

Thanks to the binomial theorem. By making use of it, we get

$$\zeta_1^{trn} = \frac{1}{\rho_{k,r}^{n}}\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)(r+1)}k^{r(n-i)}\rho_{k,tr}^{i}\zeta_1^{ri}\rho_{k,(t-1)r}^{(n-i)}, \tag{29}$$

$$\zeta_2^{trn} = \frac{1}{\rho_{k,r}^{n}}\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)(r+1)}k^{r(n-i)}\rho_{k,tr}^{i}\zeta_2^{ri}\rho_{k,(t-1)r}^{(n-i)}. \tag{30}$$

Now, by multiplying an equation (29) by $\frac{\zeta_1^{l}}{\zeta_1-\zeta_2}$, equation (30) by $\frac{\zeta_2^{l}}{\zeta_1-\zeta_2}$ and subtracting, we obtain

$$\frac{\zeta_1^{trn+l}-\zeta_2^{trn+l}}{\zeta_1-\zeta_2}\rho_{k,r}^{n} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)(r+1)}k^{r(n-i)}\rho_{k,tr}^{i}\rho_{k,(t-1)r}^{n-i}\left(\frac{\zeta_1^{ri+l}-\zeta_2^{ri+l}}{\zeta_1-\zeta_2}\right).$$

Furthermore, by applying the Binet formula of $\rho_{k,n}$, we achieve the required result

$$\rho_{k,trn+l}\rho_{k,r}^{n} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)(r+1)}k^{r(n-i)}\rho_{k,r}^{i}\rho_{k,(t-1)r}^{n-i}\rho_{k,ri+l}.$$

Thus the result (1).
Again, multiplying equation (29) by ζ_1^{l}, (30) by ζ_2^{l} and adding, we obtain

$$\zeta_1^{trn+l}+\zeta_2^{trn+l} = \frac{1}{\rho_{k,r}^{n}}\sum_{i=0}^{n}\binom{n}{i}\rho_{k,tr}^{i}\rho_{k,(t-1)r}^{n-i} (-1)^{(n-i)(r+1)}2^{r(n-i)}\left(\zeta_1^{ri+l}+\zeta_2^{ri+l}\right).$$

Finally, by employing the Binet formula of $\varrho_{k,n}$, we get the desired result

$$\varrho_{k,trn+l}\rho_{k,r}^{n} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)(r+1)}k^{r(n-i)}\rho_{k,tr}^{i}\rho_{k,(t-1)r}^{n-i}\varrho_{k,ri+l}.$$

This ends the proof of the result (2).
The proof of (3)-(6) is identical to (1) and (2). Hence, we omit the proof. □

Theorem 2.5. *If $n, r, s, t \in \mathbb{Z}^+$ with $t \geq 1$ then the* *following identities hold*

(1) $(-k)^{smn}\rho_{k,t}\rho^n_{k,(r-s)m} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}\rho^i_{k,rm}$
$\rho^{(n-i)}_{k,sm}\rho_{k,smi+rm(n-i)+t}$,

(2) $(-k)^{smn}\varrho_{k,t}\rho^n_{k,(r-s)m} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}\rho^i_{k,rm}$
$\rho^{(n-i)}_{k,sm}\varrho_{k,smi+rm(n-i)+t}$,

(3) $(-k)^{smn}\xi_{k,t}\rho^n_{k,(r-s)m} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}\rho^i_{k,rm}$
$\rho^{(n-i)}_{k,sm}\xi_{k,smi+rm(n-i)+t}$,

(4) $\rho^n_{k,rm}\rho_{k,smn+t} = \sum_{i=0}^{n}\binom{n}{i}(-k)^{sm(n-i)}\rho^i_{k,sm}$
$\rho^{(n-i)}_{k,(r-s)m}\rho_{k,rmi+t}$,

(5) $\rho^n_{k,rm}\varrho_{k,smn+t} = \sum_{i=0}^{n}\binom{n}{i}(-k)^{sm(n-i)}\rho^i_{k,sm}$
$\rho^{(n-i)}_{k,(r-s)m}\varrho_{k,rmi+t}$,

(6) $\rho^n_{k,rm}\xi_{k,smn+t} = \sum_{i=0}^{n}\binom{n}{i}(-k)^{sm(n-i)}\rho^i_{k,sm}$
$\rho^{(n-i)}_{k,(r-s)m}\xi_{k,rmi+t}$,

(7) $\rho^n_{k,sm}\rho_{k,rmn+t} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(sm+1)(n-i)}$
$k^{sm(n-i)}\rho^i_{k,rm}\rho^{(n-i)}_{k,(r-s)m}\rho_{k,smi+t}$,

(8) $\rho^n_{k,sm}\varrho_{k,rmn+t} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(sm+1)(n-i)}$
$k^{sm(n-i)}\rho^i_{k,rm}\rho^{(n-i)}_{k,(r-s)m}\varrho_{k,smi+t}$,

(9) $\rho^n_{k,sm}\xi_{k,rmn+t} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(sm+1)(n-i)}$
$k^{sm(n-i)}\rho^i_{k,rm}\rho^{(n-i)}_{k,(r-s)m}\xi_{k,smi+t}$

Proof. Applying the Lemma 2.1(d), we can write

$$\zeta_1^{sm}\rho_{k,rm} - \zeta_1^{rm}\rho_{k,sm} = (-k)^{sm}\rho_{k,(r-s)m},$$
$$\zeta_2^{sm}\rho_{k,rm} - \zeta_2^{rm}\rho_{k,sm} = (-k)^{sm}\rho_{k,(r-s)m}.$$

Thanks to the binomial theorem. By applying it, we get

$$(-k)^{smn}\rho^n_{k,(r-s)m} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\rho^i_{k,rm}\rho^{n-i}_{k,sm}\zeta_1^{smi+rm(n-i)}, \tag{31}$$

$$(-k)^{smn}\rho^n_{k,(r-s)m} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\rho^i_{k,rm}\rho^{n-i}_{k,sm}\zeta_2^{smi+rm(n-i)}. \tag{32}$$

Further, multiplying an equation (31) by $\frac{\zeta_1^t}{\zeta_1-\zeta_2}$, equation (32) by $\frac{\zeta_2^t}{\zeta_1-\zeta_2}$ and subtracting, we obtain

$$(-k)^{smn}\rho^n_{k,(r-s)m}\left(\frac{\zeta_1^t-\zeta_2^t}{\zeta_1-\zeta_2}\right)$$

$$= \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\rho^i_{k,rm}\rho^{n-i}_{k,sm}\left(\frac{\zeta_1^{smi+rm(n-i)+t}-\zeta_2^{smi+rm(n-i)+t}}{\zeta_1-\zeta_2}\right).$$

Finally, applying the Binet formula of $\rho_{k,n}$, we achieve the required result

$$(-k)^{smn}\rho^n_{k,(r-s)m}\rho_{k,t} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\rho^i_{k,rm}\rho^{n-i}_{k,sm}$$
$$\rho_{k,smi+rm(n-i)+t}.$$

Again, by multiplying equation (31) by ζ_1^t, (32) by ζ_2^t and adding, we get

$$(-2)^{smn}\rho^n_{k,(r-s)m}\varrho_{k,t} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\rho^i_{k,rm}\rho^{n-i}_{k,sm}$$
$$\varrho_{k,smi+rm(n-i)+t}$$

Thus the result (2).
The proof of (3) - (6) is analogous to the proof of (1) and (2). Hence, we omit the proof. □

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