A note on bivariate bi-periodic Mersenne polynomials

BAHAR KULOĞLU[1], ENZE CUI[2], ENGİN ÖZKAN[3], JAMES F. PETERS[4]

[1]Sivas Science and Technology University,
Department of Engineering Basic Sciences,
Sivas, TÜRKİYE

[2]Computational Intelligence Laboratory,
University of Manitoba, WPG, MB, R3T 5V6, CANADA

[3]Marmara University,
Department of Mathematics,
İstanbul, TÜRKİYE

[4]Computational Intelligence Laboratory,
University of Manitoba, WPG, MB, R3T 5V6, CANADA

Abstract: - We introduce a new generalization $m_n(x, y)$, which we will call Bivariate Bi-periodic Mersenne polynomials depending on whether n is even or odd. We investigated the bivariate and biperiodic forms of Mersenne polynomials, focusing on their unique structural properties, roots, and relationships among coefficients. Key contributions include deriving the generating function and Binet formula, examining the limit behavior of the polynomials, and identifying connections between positive and negative terms. Also, we give the limits of the consecutive terms of the polynomials and some important identities such as Catalan, Cassini and D'Ocagne's identity. We also find the corresponding binomial addition formula.

Key-Words: Mersenne sequence, Mersenne polynomials, Generalized Mersenne sequence, Bi-periodic sequence, Bivariate Bi-periodic sequence

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1 Introduction

As number sequences have played a foundational role in mathematical theory, leading to the discovery of intricate patterns, recursions, and connections across various branches of mathematics. In a similar spirit, the study of Mersenne polynomials and their bivariate and biperiodic forms extends this understanding to higher dimensions and periodic behaviors. Mersenne numbers, originally defined as Mn = 2n − 1, give rise to Mersenne polynomials,
which inherit recursive properties analogous to Fibonacci but with distinct combinatorial features.
For any natural number Some studies on Mersenne numbers by Koshy and Gao [11] have been on the investigation the divisibility properties of these numbers into Catalan numbers. Mersenne sequence has an important place in number theory as it is also involved in computer science because of Mersenne primes. In number theory, Mersenne number of orders n is defined as $2^n - 1$, where n is a non-negative integer. This identity is defined as the Binet formula for the Mersenne sequence.
Mersenne sequence was defined as [7]

$$M_{n+2} = 3M_{n+1} - 2M_n$$

with $M_0 = 0, M_1 = 1$.

The Mersenne sequence has been generalized in many ways where some by preserving recurrence relation and initial condition [7,13,14,17].

Some of these generalizations have been obtained by describing the bi-periodic relation of this sequences and other number sequences [2,5,9,10,12,18-21,23,24,27].

Studies have been carried out on the polynomials of these new number sequences and bivariate polynomials which are the more general form of polynomials [3,4,6,15,16,22,25,26].

In 2002, Catalani [6] worked on the more general form of bivariate polynomials with the matrix approach and focused more on Fibonacci and Lucas numbers. In 2018, Uygun [9] examined Jacobsthal and Jacobsthal-Lucas matrix polynomial sequence and studied the relationship between these polynomials and bivariate polynomials. Working on the more general version of these number sequences in [22,23], she obtained some important properties.

For $n \in \mathbb{N}$ and $a, b \in \mathbb{R}^+$, Edson and Yayenie [9,27] introduced the bi-periodic Fibonacci sequence as follows

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & if\ n\ is\ even \\ bq_{n-1} + q_{n-2}, & if\ n\ is\ odd \end{cases}, \ n \geq 2$$

with $q_0 = 0,\ q_1 = 1$.

The generating function of this sequence is

$$F(x) = \frac{x(1 + ax - x^2)}{1 - (ab + 2)x^2 + 4x^4}$$

also, the Binet's formula of this sequence is

$$q_{nn} = \frac{a^{1-\epsilon(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right)$$

where $\lfloor a \rfloor$is the floor function and $\epsilon(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ is the parity function and α, β are the root of characteristic equation of bi-periodic Fibonacci sequence, $x^2 - abx - ab = 0$.

This paper carries forward earlier work Mersenne polynomials, an extension of the classical Mersenne numbers, have gained attention due to their intriguing algebraic properties and potential applications in number theory and combinatorics. Traditionally, these polynomials are defined by integer sequences that mirror properties of Mersenne numbers and are further generalized to account for complex mathematical behaviors in multidimensional forms. Recently, studies on bivariate and biperiodic forms of Mersenne polynomials have opened new avenues for exploring structural relationships and symmetries within these polynomials, enhancing our understanding of polynomial behavior across different forms [16,17,21].

This study builds upon existing frameworks by exploring bivariate and biperiodic Mersenne polynomials and systematically examining their properties, roots, and coefficients. Our contributions include deriving fundamental expressions such as the generating function, the Binet formula, and limit behavior, and establishing identities that link positive and negative terms. Furthermore, we extend our analysis by proving the Catalan identity specific to these forms and deriving summation formulas that encapsulate these polynomials' recursive nature and growth characteristics. These findings contribute to the broader theoretical foundation of Mersenne polynomials and provide tools for applications in combinatorial identities and recursive sequence analysis.

2 Main Results

This section presents a bi-variate, bi-periodic Mersenne generating function, binet formula and their associated properties.

Definition 2.1. For any $a, b \in \mathbb{R} - \{0\}$ and $x, y \in \mathbb{R}$, for $n \geq 0$ the bivariate bi-periodic Mersenne polynomials is given by

$$m_{n+2}(x, y) = \begin{cases} 3aym_{n+1}(x, y) - 2xm_n(x, y), & if\ n\ is\ even \\ 3bym_{n+1}(x, y) - 2xm_n(x, y), & if\ n\ is\ odd \end{cases} \quad (2.1)$$

with initial conditions $m_0(x, y) = 0,\ m_1(x, y) = 1$. When $a = b = 1$, we have the classic the bivariate Mersenne polynomials [1]. When $a = b = x = y = 1$, we have the Mersenne sequences [7,8].

The first four elements of the bivariate bi-periodic Mersenne polynomials are as follows

$m_0(x, y) = 0,\ m_1(x, y) = 1, m_2(x, y) = 3ay,$
$m_3(x, y) = 9aby^2 - 2x,$
$m_4(x, y) = 27a^2by^3 - 12axy.$

From Definition 2.1, for the bivariate bi-periodic Mersenne polynomials, we get the quadratic equation as follows

$$t^2 - 3abyt + 2abx = 0$$

with roots

$$\alpha = \frac{3aby + \sqrt{9a^2b^2y^2 - 8abx}}{2}$$

and (2.2)

$$\beta = \frac{3aby - \sqrt{9a^2b^2y^2 - 8abx}}{2}.$$

Lemma 2.2. *The bivariate bi-periodic Mersenne polynomials satisfy the following properties*

i. $m_{2n+2}(x, y) = (9aby^2 - 4x)m_{2n}(x, y) - 4x^2 m_{2n-2}(x, y)$
ii. $m_{2n+1}(x, y) = (9aby^2 - 4x)m_{2n-1}(x, y) - 4x^2 m_{2n-3}(x, y).$

Proof.

$i. m_{2n+2}(x, y) = 3aym_{2n+1}(x, y) - 2xm_{2n}(x, y)$

$$= 3ay(3bym_{2n}(x,y) - 2xm_{2n-1}(x,y)) - 2xm_{2n}(x,y)$$
$$= 9aby^2m_{2n} - 6axym_{2n-1} - 2xm_{2n}$$
$$= m_{2n}(9aby^2 - 2x) - 2x(3aym_{2n-1})$$
$$= m_{2n}(9aby^2 - 2x) - 2x(m_{2n} + 2xm_{2n-2})$$
$$= m_{2n}(9aby^2 - 4x) - 4x^2m_{2n-2}$$

ii. $m_{2n+1}(x,y) = 3bym_{2n}(x,y) - 2xm_{2n-1}(x,y)$
$$= 3by(3aym_{2n-1}(x,y) - 2xm_{2n-2}(x,y)) - 2xm_{2n-1}(x,y)$$
$$= 9aby^2m_{2n-1} - 6bxym_{2n-2} - 2xm_{2n-1}$$
$$= m_{2n-1}(9aby^2 - 2x) - 2x(3bym_{2n-2})$$
$$= m_{2n-1}(9aby^2 - 2x) - 2x(m_{2n-1} + 2xm_{2n-3})$$
$$= m_{2n-1}(9aby^2 - 4x) - 4x^2m_{2n-3}$$
□

Lemma 2.3. *There are the following properties.*
- $\alpha + \beta = 3aby^2$, $\alpha\beta = 2abxy^2$
- $(3\alpha - 2x) = \frac{\alpha^2}{aby^2}$, $(3\beta - 2x) = \frac{\beta^2}{aby^2}$
- $(3\alpha - 2x)(3\beta - 2x) = 4x^2$
- $\beta(3\alpha - 2x) = 2\alpha x$, $\alpha(3\beta - 2x) = 2\,\beta x$

Proof. Their proofs can be easily obtained using the definition of α and β.
□

Theorem 2.4. *Let $\mathcal{M}_{(x,y)}(t)$ show the generating function of the bivariate bi-periodic Mersenne polynomials. Then*
$$\mathcal{M}_{(x,y)}(t) = \frac{t(1+3ayt+2xt^2)}{(1-(9aby^2-4x)t^2+4x^2t^4)}.$$

Proof. The most general form of $\mathcal{M}_{(x,y)}(t)$ is as follows
$\mathcal{M}_{(x,y)}(t) = \sum_{n=0}^{\infty} m_n(x,y)t^n = m_0 + m_1t + \cdots + m_kt^k + \cdots$.

By multiplying the last equation by $-3byt$ and $2xt^2$, respectively, we have

$-3byt\mathcal{M}_{(x,y)}(t) = -3by\sum_{n=0}^{\infty} m_n(x,y)t^{n+1} = -3by\sum_{n=1}^{\infty} m_{n-1}(x,y)t^n$,
$2xt^2\,\mathcal{M}_{(x,y)}(t) = 2x\sum_{n=0}^{\infty} m_n(x,y)t^{n+2} = 2x\sum_{n=2}^{\infty} m_{n-2}(x,y)t^n$.
So, from Lemma 2.2, we get

$$(1 - 3byt + 2xt^2)\mathcal{M}_{(x,y)}(t) = m_0(x,y) + \big(m_1(x,y) - 3bym_0(x,y)\big)t + \big(m_2(x,y) - 3bym_1(x,y) + 2xm_0(x,y)\big)t^2 + \big(m_3(x,y) - 3bym_2(x,y) + 2xm_1(x,y)\big)t^3 + \big(m_4(x,y) - 3bym_3(x,y) + 2xm_2(x,y)\big)t^4 + \cdots.$$

$$(1 - 3byt + 2xt^2)\mathcal{M}_{(x,y)}(t) = t + \sum_{n=1}^{\infty}(m_{2n} - 3bym_{2n-1} + 2xm_{2n-2})t^{2n}$$
$$= t + \sum_{n=1}^{\infty}(3aym_{2n-1} - 3bym_{2n-1})t^{2n}$$
$$= t + \sum_{n=1}^{\infty}(3ay - 3by)m_{2n-1}t^{2n}$$
$$= t + (3ay - 3by)t^2 + (3ay - 3by)t\sum_{n=1}^{\infty} m_{2n-1}t^{2n-1}.$$

For simplicity in operations, let's define $\widehat{m}_{(x,y)}(t)$ as follows.
$$\widehat{m}_{(x,y)}(t) = \sum_{n=1}^{\infty} m_{2n-1}t^{2n-1}$$
where
$$m_{2n-1}(x,y) = 3bym_{2n-2}(x,y) - 2xm_{2n-3}(x,y)$$
$$= 3by(3aym_{2n-3}(x,y) - 2xm_{2n-4}(x,y)) - 2xm_{2n-3}(x,y)$$
$$= m_{2n-3}(x,y)(9aby^2 - 2x) - 6bxym_{2n-4}(x,y)$$
$$= (9aby^2 - 4x)m_{2n-3}(x,y) - 4x^2m_{2n-5}(x,y).$$
So, we get
$$\widehat{m}_{(x,y)}(t) = (1 - (9aby^2 - 4x)t^2 + 4x^2t^4)m_{(x,y)}(t)$$
$$= m_1(x,y)t + \big(m_3(x,y) - (9aby^2 - 4x)m_1(x,y)\big)t^3 + (m_5(x,y) - (9aby^2 - 4x)m_3(x,y) + 4x^2m_1(x,y))t^5 + \cdots.$$

From Lemma 2.2, we get

$$(1 - (9aby^2 - 4x)t^2 + 4x^2t^4)m_{(x,y)}(t) = t + 2xt^3 + 0t^5 + 0t^7 + \cdots + 0t^{2n-1} + \cdots$$
$$m_{(x,y)}(t) = \frac{t + 2xt^3}{(1 - (9aby^2 - 4x)t^2 + 4x^2t^4)}.$$

Plugging $\widehat{m}_{(x,y)}(t)$ into $\mathcal{M}_{(x,y)}(t)$, we obtain

$$(1 - 3byt + 2xt^2)\mathcal{M}_{(x,y)}(t) = t + \frac{t(3ay - 3by)(2xt^3 + t)}{(1 - (9aby^2 - 4x)t^2 + 4x^2t^4)}$$

$$\mathcal{M}_{(x,y)}(t) = \frac{t(1 - 9aby^2t^2 + 4xt^2 + 4x^2t^4 + 6axyt^3 + 3ayt - 6bxyt^3 - 3byt)}{(1 - (9aby^2 - 4x)t^2 + 4x^2t^4)(1 - 3byt + 2xt^2)}$$
$$\mathcal{M}_{(x,y)}(t) = \frac{t(1 + 3ayt + 2xt^2)}{(1 - (9aby^2 - 4x)t^2 + 4x^2t^4)}$$

as desired.
□

Theorem 2.5. *The Binet's formula for* $m_m(x,y)$ *is given by*

$$m_m(x,y) = \frac{(ay)^{1-\delta(m)}}{(aby^2)^{\lfloor \frac{m}{2} \rfloor}} \left(\frac{\alpha^m - \beta^m}{\alpha - \beta} \right)$$

where $\delta(m) = m - 2\left\lfloor \frac{m}{2} \right\rfloor$ *is the parity function* [9,27].

Proof. The function can be expressed as follows:

$$\delta(m) = \begin{cases} 0, & if\ m\ is\ even \\ 1, & if\ m\ is\ odd \end{cases}.$$

We can find Binet's formula as follow.

For generating function, we use partial fraction decomposition method, Lemma 2.3 and then we apply Maclaurin's Series expansion.

$$\mathcal{M}_{(x,y)}(t) = \frac{1}{(\alpha - \beta)} \left[\frac{\alpha t + ay}{(1 - (3\alpha - 2x)t^2)} - \frac{\beta t + ay}{(1 - (3\beta - 2x)t^2)} \right]$$

$$= \frac{1}{(\alpha - \beta)} \left[\frac{\alpha t + ay}{-(3\alpha - 2x)\left(t^2 - \frac{1}{(3\alpha - 2x)}\right)} - \frac{\beta t + ay}{-(3\beta - 2x)\left(t^2 - \frac{1}{(3\beta - 2x)}\right)} \right]$$

$$= \frac{1}{(\alpha - \beta)} \left[-\frac{\alpha t + ay}{(3\alpha - 2x)\left(t^2 - \frac{(3\beta - 2x)}{4x^2}\right)} + \frac{\beta t + ay}{(3\beta - 2x)\left(t^2 - \frac{(3\alpha - 2x)}{4x^2}\right)} \right]$$

$$= \frac{1}{(\alpha - \beta)} \left[\frac{\frac{\alpha t}{2x} + ay\frac{3\alpha - 2x}{4x^2}}{\left(t^2 - \frac{(3\alpha - 2x)}{4x^2}\right)} - \frac{\frac{\beta t}{2x} + ay\frac{3\beta - 2x}{4x^2}}{\left(t^2 - \frac{(3\beta - 2x)}{4x^2}\right)} \right]$$

$$= \frac{1}{4x^2(\alpha - \beta)} \left[\frac{2x\alpha t + ay(3\alpha - 2x)}{\left(t^2 - \frac{(3\alpha - 2x)}{4x^2}\right)} - \frac{2x\beta t + ay(3\beta - 2x)}{\left(t^2 - \frac{(3\beta - 2x)}{4x^2}\right)} \right].$$

The Maclaurin series for the function $\frac{A - Bz}{z^2 - C}$ have the form

$$\frac{A - Bz}{z^2 - C} = \sum_{n=0}^{\infty} BC^{-n-1}z^{2n+1} - \sum_{n=0}^{\infty} AC^{-n-1}z^{2n}.$$

So, the generating function $\mathcal{M}_{(x,y)}(t)$ can be expanded as

$$\begin{aligned}
&\mathcal{M}_{(x,y)}(t) \\
&= \frac{1}{4x^2(\alpha - \beta)} \left[\sum_{m=0}^{\infty} (-2x\alpha) \left(\frac{4x^2}{3\alpha - 2x} \right)^{m+1} t^{2m+1} - \sum_{m=0}^{\infty} \frac{ay(3\alpha - 2x)(4x^2)^{m+1}}{(3\alpha - 2x)^{m+1}} t^{2m} \right] \\
&- \frac{1}{4x^2(\alpha - \beta)} \left[\sum_{m=0}^{\infty} (-2x\beta) \left(\frac{4x^2}{3\beta - 2x} \right)^{m+1} t^{2m+1} - \sum_{m=0}^{\infty} \frac{ay(3\beta - 2x)(4x^2)^{m+1}}{(3\beta - 2x)^{m+1}} t^{2m} \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4x^2(\alpha - \beta)} \left[\sum_{m=0}^{\infty} \left((-2x\alpha) \left(\frac{4x^2}{3\alpha - 2x} \right)^{m+1} - (-2x\beta) \left(\frac{4x^2}{3\beta - 2x} \right)^{m+1} \right) t^{2m+1} \right] \\
&+ \frac{1}{4x^2(\alpha - \beta)} \left[\sum_{m=0}^{\infty} \left(\frac{ay(3\beta - 2x)(4x^2)^{m+1}}{(3\beta - 2x)^{m+1}} - \frac{ay(3\alpha - 2x)(4x^2)^{m+1}}{(3\alpha - 2x)^{m+1}} \right) t^{2m} \right]
\end{aligned}$$

$$=\frac{1}{4x^2(\alpha-\beta)}\left[\sum_{m=0}^{\infty}\left(\frac{(-2x\alpha)(4x^2)^{m+1}(3\beta-2x)^{m+1}+(2x\beta)(4x^2)^{m+1}(3\alpha-2x)^{m+1}}{(3\beta-2x)^{m+1}(3\alpha-2x)^{m+1}}\right)t^{2m+1}\right]$$

$$+\frac{1}{4x^2(\alpha-\beta)}\left[\sum_{m=0}^{\infty}\left(\frac{(ay)(4x^2)^{m+1}(3\alpha-2x)^{m+1}(3\beta-2x)-(ay)(4x^2)^{m+1}(3\beta-2x)^{m+1}(3\alpha-2x)}{(3\beta-2x)^{m+1}(3\alpha-2x)^{m+1}}\right)t^{2m}\right]$$

From Lemma 2.3,

$$\mathcal{M}_{(x,y)}(t)=\frac{1}{4x^2(\alpha-\beta)}\left[\sum_{m=0}^{\infty}(4\alpha x^2(3\alpha-2x)^m-4\beta x^2(3\beta-2x)^m)\,t^{2m+1}\right]+\frac{1}{4x^2(\alpha-\beta)}\left[\sum_{m=0}^{\infty}(ay.4x^2(3\alpha-2x)^m-ay.4x^2(3\beta-2x)^m)\,t^{2m}\right]$$

$$=\frac{1}{4x^2(\alpha-\beta)}\left[\sum_{m=0}^{\infty}\left(\frac{4x^2\alpha^{2m+1}}{(aby^2)^m}-\frac{4x^2\beta^{2m+1}}{(aby^2)^m}\right)t^{2m+1}\right]$$

$$+\frac{1}{4x^2(\alpha-\beta)}\left[\sum_{m=0}^{\infty}\left(\frac{ay.4x^2\alpha^{2m}}{(aby^2)^m}-\frac{ay.4x^2\beta^{2m}}{(aby^2)^m}\right)t^{2m}\right]$$

$$=\frac{1}{\alpha-\beta}\left[\sum_{m=0}^{\infty}\left(\frac{1}{aby^2}\right)^m(\alpha^{2m+1}-\beta^{2m+1})t^{2m+1}\right]$$

$$+\frac{1}{\alpha-\beta}\left[ay\sum_{m=0}^{\infty}\left(\frac{1}{aby^2}\right)^m(\alpha^{2m}-\beta^{2m})t^{2m}\right].$$

From the parity function, the expansion can be considered into the following form

$$\mathcal{M}_{(x,y)}(t)=\sum_{m=0}^{\infty}\frac{(ay)^{1-\delta(m)}}{(aby^2)^{\left\lfloor\frac{m}{2}\right\rfloor}}\left(\frac{\alpha^m-\beta^m}{\alpha-\beta}\right)t^m.$$

hence by comparing the above with $\mathcal{M}_{(x,y)}(t)=\sum_{m=0}^{\infty}m_m(x,y)t^m$, it follows that

$$m_m(x,y)=\frac{(ay)^{1-\delta(m)}}{(aby^2)^{\left\lfloor\frac{m}{2}\right\rfloor}}\left(\frac{\alpha^m-\beta^m}{\alpha-\beta}\right)$$

as desired.

Theorem 2.6. *The limit of every two consecutive terms of the polynomial is as follows*

$$\lim_{n\to\infty}\frac{m_{2m+1}(x,y)}{m_{2m}(x,y)}=\frac{\alpha}{ay}$$

and

$$\lim_{n\to\infty}\frac{m_{2m}(x,y)}{m_{2m-1}(x,y)}=\frac{\alpha}{by}.$$

Proof. Considering that $|\beta|<\alpha$ and properties of the limit, it is obtained

$$\lim_{n\to\infty}\frac{m_{2m+1}(x,y)}{m_{2m}(x,y)}=\frac{\frac{(ay)^{1-\delta(2m+1)}}{(aby^2)^{\left\lfloor\frac{2m+1}{2}\right\rfloor}}\left(\frac{\alpha^{2m+1}-\beta^{2m+1}}{\alpha-\beta}\right)}{\frac{(ay)^{1-\delta(2m)}}{(aby^2)^{\left\lfloor\frac{2m}{2}\right\rfloor}}\left(\frac{\alpha^{2m}-\beta^{2m}}{\alpha-\beta}\right)}$$

$$=\frac{\alpha^{2m+1}-\beta^{2m+1}}{ay(\alpha^{2m}-\beta^{2m})}=\frac{\alpha}{ay}.$$

Similarly,

$$\lim_{n\to\infty}\frac{m_{2m}(x,y)}{m_{2m-1}(x,y)}=\frac{\frac{(ay)^{1-\delta(2m)}}{(aby^2)^{\left\lfloor\frac{2m}{2}\right\rfloor}}\left(\frac{\alpha^{2m}-\beta^{2m}}{\alpha-\beta}\right)}{\frac{(ay)^{1-\delta(2m-1)}}{(aby^2)^{\left\lfloor\frac{2m-1}{2}\right\rfloor}}\left(\frac{\alpha^{2m-1}-\beta^{2m-1}}{\alpha-\beta}\right)}$$

$$=\frac{\frac{ay}{(aby^2)^m}\left(\frac{\alpha^{2m}-\beta^{2m}}{\alpha-\beta}\right)}{\frac{1}{(aby^2)^{m-1}}\left(\frac{\alpha^{2m-1}-\beta^{2m-1}}{\alpha-\beta}\right)}=\frac{\alpha}{by}$$

as desired. We can conclude that the bivariate bi-periodic Mersenne polynomial does not converge.

□

Theorem 2.7. *There is a relationship between positive terms and their corresponding negative terms in the polynomials is*

$$m_{-n}(x,y)=-m_n(x,y)(2x)^{-n}.$$

Proof. From Binet's formula,

$$m_{-n}(x,y)=\frac{(ay)^{1-\delta(-n)}}{(aby^2)^{\left\lfloor\frac{-n}{2}\right\rfloor}}\left(\frac{\alpha^{-n}-\beta^{-n}}{\alpha-\beta}\right)$$

$$=\frac{(ay)^{1-\delta(n)}}{(aby^2)^{\left\lfloor\frac{-n}{2}\right\rfloor}}\left(\frac{\beta^n-\alpha^n}{(\alpha-\beta)(2abxy^2)^n}\right)$$

$$=\frac{(ay)^{1-\delta(n)}}{(aby^2)^{\left\lfloor\frac{n}{2}\right\rfloor}}\left(\frac{\beta^n-\alpha^n}{(\alpha-\beta)(2x)^n}\right)=-m_n(x,y)(2x)^{-n}.$$

□

Theorem 2.8. (Catalan Identity)

Let n and r *any two nonnegative integers such that* $n\geq r$, *we have*

$$a^{\delta(n-r)}b^{1-\delta(n-r)}m_{n-r}(x,y)m_{n+r}(x,y)-a^{\delta(n)}b^{1-\delta(n)}m_n^2(x,y)=-(2x)^{n-r}a^{\delta(r)}b^{1-\delta(r)}m_r^2(x,y).$$

Proof. From Binet's formula, we obtain

$$a^{\delta(n-r)}b^{1-\delta(n-r)}m_{n-r}(x,y)m_{n+r}(x,y)$$
$$=a^{\delta(n-r)}b^{1-\delta(n-r)}\frac{(ay)^{1-\delta(n-r)}}{(aby^2)^{\left\lfloor\frac{n-r}{2}\right\rfloor}}\left(\frac{\alpha^{n-r}-\beta^{n-r}}{\alpha-\beta}\right)\frac{(ay)^{1-\delta(n+r)}}{(aby^2)^{\left\lfloor\frac{n+r}{2}\right\rfloor}}\left(\frac{\alpha^{n+r}-\beta^{n+r}}{\alpha-\beta}\right)$$
$$=\frac{(ay)^{2-\delta(n-r)-\delta(n+r)}a^{\delta(n-r)}b^{1-\delta(n-r)}}{(aby^2)^{n-\delta(n-r)}}\left(\frac{\alpha^{n-r}-\beta^{n-r}}{\alpha-\beta}\right)\left(\frac{\alpha^{n+r}-\beta^{n+r}}{\alpha-\beta}\right)$$

$$=\frac{a}{(aby^2)^{n-1}}\left(\frac{\alpha^{2n}+\beta^{2n}-(\alpha\beta)^{n-r}(\alpha^{2r}+\beta^{2r})}{(\alpha-\beta)^2}\right).$$

From that $\delta(n)=n-2\left\lfloor\frac{n}{2}\right\rfloor$, we get

$$a^{\delta(n)}b^{1-\delta(n)}m_n^2(x,y)$$
$$=a^{\delta(n)}b^{1-\delta(n)}\frac{(ay)^{2-2\delta(n)}}{(aby^2)^{2\left\lfloor\frac{n}{2}\right\rfloor}}\left(\frac{\alpha^{2n}+\beta^{2n}-2(\alpha\beta)^n}{(\alpha-\beta)^2}\right)$$

$$=\frac{a^{2-\delta(n)}(y^2)^{1-\delta(n)}b^{1-\delta(n)}}{a^{n-\delta(n)}(y^2)^{n-\delta(n)}b^{n-\delta(n)}}\left(\frac{\alpha^{2n}+\beta^{2n}-2(\alpha\beta)^n}{(\alpha-\beta)^2}\right)$$

$$=\frac{a}{a^{n-1}b^{n-1}(y^2)^{n-1}}\left(\frac{\alpha^{2n}+\beta^{2n}-2(\alpha\beta)^n}{(\alpha-\beta)^2}\right)=\frac{a}{(aby^2)^{n-1}}\left(\frac{\alpha^{2n}+\beta^{2n}-2(\alpha\beta)^n}{(\alpha-\beta)^2}\right).$$

$$=\frac{a}{(aby^2)^{n-1}}\left((\alpha\beta)^{n-r}\frac{-\alpha^{2r}-\beta^{2r}+2(\alpha\beta)^r}{(\alpha-\beta)^2}\right)$$

$$=\frac{-a}{(aby^2)^{n-1}}\left((\alpha\beta)^{n-r}\left(\frac{\alpha^r-\beta^r}{\alpha-\beta}\right)^2\right)$$

$$=\frac{-a}{(aby^2)^{n-1}}\left((2abxy^2)^{n-r}\frac{(aby^2)^{2\left\lfloor\frac{n}{2}\right\rfloor}}{(ay)^{2-2\delta(r)}}m_r^2(x,y)\right)$$

$$=\frac{-(-2)^{n-r}a(ab)^{n-r}(xy^2)^{n-r}(ab)^{r-\delta(r)}(y^2)^{r-\delta(r)}}{(ab)^{n-1}(y^2)^{n-1}(a)^{2-2\delta(r)}(y^2)^{1-\delta(r)}}m_r^2(x,y)$$

$$=-(2x)^{n-r}a^{\delta(r)}b^{1-\delta(r)}m_r^2(x,y).$$

□

Theorem 2.9. (Cassini Identity)

For $n \in \mathbb{Z}^+$, we get

$$a^{\delta(n-1)}b^{1-\delta(n-1)}m_{n-1}(x,y)m_{n+1}(x,y) - a^{\delta(n)}b^{1-\delta(n)}m_n^2(x,y) = -(2x)^{n-1}a.$$

Proof. The proof is seen easily by choosing $r = 1$ in Catalan identity.

□

Theorem 2.10. (d'Ocagne Identity)

For $r, s \in \mathbb{Z}^+$ and $r \geq s$, we have

$$a^{\delta(rs+r)}b^{\delta(rs+s)}m_r(x,y)m_{s+1}(x,y) - a^{\delta(rs+s)}b^{\delta(rs+r)}m_{r+1}(x,y)m_s(x,y) = (2x)^s a^{\delta(r-s)}m_{r-s}(x,y).$$

Proof. Let's consider the following equations.

$$\delta(r) + \delta(s+1) - 2\delta(rs+r) = \delta(r+1) + \delta(s) - 2\delta(rs+s) = 1 - \delta(r-s)$$
(2.3)

$$\delta(r-s) = \delta(rs+r) + \delta(rs+s)$$
(2.4)

$$\frac{r-s-\delta(r-s)}{2} = \left\lfloor\frac{r}{2}\right\rfloor + \left\lfloor\frac{s+1}{2}\right\rfloor - \delta(rs+r) - s$$
(2.5)

$$\frac{r-s-\delta(r-s)}{2} = \left\lfloor\frac{r+1}{2}\right\rfloor + \left\lfloor\frac{s}{2}\right\rfloor - \delta(rs+r) - s$$
(2.6)

$$\frac{r-s-\delta(r-s)}{2} = \left\lfloor\frac{r-s}{2}\right\rfloor$$
(2.7)

By using the extended Binet's formula (2.3), (2.4), (2.5), (2.6), it is obtained

$$€ = a^{\delta(rs+r)}b^{\delta(rs+s)}m_r(x,y)m_{s+1}(x,y)$$
$$= \frac{a(a)^{\delta(rs+r)+1-\delta(r)-\delta(s+1)}b^{\delta(rs+s)}y^{2-\delta(r)-\delta(s+1)}}{(aby^2)^{\left\lfloor\frac{r}{2}\right\rfloor+\left\lfloor\frac{s+1}{2}\right\rfloor}}\left(\frac{\alpha^r-\beta^r}{\alpha-\beta}\right)\left(\frac{\alpha^{s+1}-\beta^{s+1}}{\alpha-\beta}\right)$$

$$= \frac{a(a)^{\delta(r-s)-\delta(rs+r)}b^{\delta(rs+s)}y^{2-\delta(r)-\delta(s+1)}}{(aby^2)^{\frac{r-s-\delta(r-s)}{2}+\delta(rs+s)+s}}\left(\frac{\alpha^r-\beta^r}{\alpha-\beta}\right)\left(\frac{\alpha^{s+1}-\beta^{s+1}}{\alpha-\beta}\right)$$

$$= \frac{a(a)^{\delta(rs+s)}b^{\delta(rs+s)}y^{2-\delta(r)-\delta(s+1)}}{(aby^2)^{\frac{r-s-\delta(r-s)}{2}+\delta(rs+s)+s}}\left(\frac{\alpha^r-\beta^r}{\alpha-\beta}\right)\left(\frac{\alpha^{s+1}-\beta^{s+1}}{\alpha-\beta}\right)$$

$$= \frac{a(ab)^{-s}y^{1-r-s}}{(ab)^{\frac{r-s-\delta(r-s)}{2}}}\left(\frac{\alpha^r-\beta^r}{\alpha-\beta}\right)\left(\frac{\alpha^{s+1}-\beta^{s+1}}{\alpha-\beta}\right)$$

on the other hand, similarly,

$$¥ = a^{\delta(rs+s)}b^{\delta(rs+r)}m_{r+1}(x,y)m_s(x,y)$$

$$= \frac{a(ab)^{-s}y^{1-r-s}}{(ab)^{\frac{r-s-\delta(r-s)}{2}}}\left(\frac{\alpha^r-\beta^r}{\alpha-\beta}\right)\left(\frac{\alpha^{s+1}-\beta^{s+1}}{\alpha-\beta}\right)$$

from (2.7), we obtain

$$€ - ¥$$
$$= \frac{a(ab)^{-s}y^{1-r-s}}{(ab)^{\frac{r-s-\delta(r-s)}{2}}}\left(\frac{(\alpha\beta)^s[-\beta\alpha^{r-s} - \alpha\beta^{r-s} + \alpha^{r-s+1} + \beta^{r-s+1}]}{(\alpha-\beta)^2}\right)$$

$$= \frac{(2x)^s ay}{(ab)^{\left\lfloor\frac{r-s}{2}\right\rfloor}y^{r-s}}\left(\frac{\alpha^{r-s}-\beta^{r-s}}{\alpha-\beta}\right)$$

$$= \frac{(2x)^s m_{r-s}(x,y)a^{\delta(r-s)}y^{r-s}}{y^{r-s}}\left(\frac{\alpha^{r-s}-\beta^{r-s}}{\alpha-\beta}\right)$$

$$= (2x)^s a^{\delta(r-s)}m_{r-s}(x,y).$$

□

Theorem 2.11. *For any numbers $n \in \mathbb{Z}^+$, we have*

$$\sum_{m=0}^{n} \binom{n}{m} 3^m (-2x)^{n-m} (aby^2)^{\left\lfloor \frac{m}{2} \right\rfloor} (ay)^{\delta(m)} m_m(x,y) = m_{2n}(x,y)$$

$$\sum_{m=0}^{n} \binom{n}{m} 3^m (-2x)^{n-m} (aby^2)^{\left\lfloor \frac{m+1}{2} \right\rfloor} (ay)^{\delta(m+1)} m_{m+1}(x,y) = m_{2n+1}(x,y).$$

Proof. By using Binet's formula, we have

$$(ay)\frac{(3\alpha)^m - (3\beta)^m}{\alpha - \beta} = 3^m (aby^2)^{\left\lfloor \frac{m}{2} \right\rfloor} (ay)^{\delta(m)} m_m(x,y)$$

from above equation and binomial expansion, we get

$$\sum_{m=0}^{n} \binom{n}{m} 3^m (-2x)^{n-m} (aby^2)^{\left\lfloor \frac{m}{2} \right\rfloor} (ay)^{\delta(m)} m_m(x,y)$$

$$= \sum_{m=0}^{n} \binom{n}{m} (-2x)^{n-m} (ay) \frac{(3\alpha)^m - (3\beta)^m}{\alpha - \beta}$$

$$= \frac{ay}{\alpha - \beta}\left(\sum_{m=0}^{n} \binom{n}{m} (3\alpha)^m (-2x)^{n-m} - \sum_{m=0}^{n} \binom{n}{m} (3\beta)^m (-2x)^{n-m}\right)$$

$$= \frac{(ay)}{\alpha - \beta}\left((3\alpha - 2x)^n - (3\beta - 2x)^n\right)$$

$$= \frac{ay}{\alpha - \beta}\left(\left(\frac{\alpha^2}{aby^2}\right)^n - \left(\frac{\beta^2}{aby^2}\right)^n\right) = \frac{ay}{(aby^2)^n}\frac{(\alpha)^{2n} - (\beta)^{2n}}{\alpha - \beta} = m_{2n}(x,y)$$

similarly,

$$\sum_{m=0}^{n} \binom{n}{m} 3^m (-2x)^{n-m} (aby^2)^{\left\lfloor \frac{m+1}{2} \right\rfloor} (ay)^{\delta(m+1)} m_{m+1}(x,y) =$$

$$= \sum_{m=0}^{n} \binom{n}{m} (-2x)^{n-m} (ay) \frac{\alpha(3\alpha)^m - \beta(3\beta)^m}{\alpha - \beta}$$

$$= \frac{ay}{\alpha - \beta}\left(\alpha\sum_{m=0}^{n} \binom{n}{m} (3\alpha)^m (-2x)^{n-m} - \beta\sum_{m=0}^{n} \binom{n}{m} (3\beta)^m (-2x)^{n-m}\right)$$

$$= \frac{(ay)}{\alpha - \beta}\left(\alpha(3\alpha - 2x)^n - \beta(3\beta - 2x)^n\right)$$

$$= \frac{ay}{\alpha - \beta}\left(\alpha\left(\frac{\alpha^2}{aby^2}\right)^n - \beta\left(\frac{\beta^2}{aby^2}\right)^n\right) = \frac{ay}{(aby^2)^n}\frac{(\alpha)^{2n+1} - (\beta)^{2n+1}}{\alpha - \beta} = m_{2n+1}(x,y).$$

□

4 Conclusion

The exploration of bivariate and biperiodic Mersenne polynomials reveals deep connections among their roots, coefficients, and unique properties, contributing significantly to the current understanding of these polynomials. When compared to earlier studies, this work advances the field by deriving the generating function, Binet formula, and summation formulas, which offer fresh perspectives and insights. The identification of links to Catalan identities and the behavior of positive and negative terms further strengthens the relevance and impact of these findings. This research not only adds to the theoretical framework of Mersenne polynomials but also opens up new avenues for future investigations.

Expanding this study to other number sequences and exploring the relationships among them could yield valuable insights, enhancing both the theoretical and practical applications of these polynomials in various mathematical domains.

Appendix

Marin Mersenne (1588-1648)  observed conjectured incorrectly that the numbers $2^{p-1}, p$ a prime, were prime for the numbers 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, and 257 and were composite for all other positive integers $p < 257$. However, Mersenne overlooked the primes $2^{61}-1, 2^{89}-1, 2^{107}-1$. The correct list is

$$\mathbf{2}, \mathbf{3}, \mathbf{5}, 7, \mathbf{13}, 17, 19, 31, 61, \mathbf{89}, 107, 127$$

for the Mersenne primes

$$3, 7, 31, 127, 8191, 131071, 524287, \overbrace{2147483647}^{\boldsymbol{M_{31} = 2^{31}-1}},$$

$$\overbrace{147573952589676412927}^{\boldsymbol{M_{67} = 2^{67}-1}}, \ and$$

$$\overbrace{170141183460469231731687303715884105727}^{\boldsymbol{M_{127} = 2^{127}-1}.}$$

Some studies on Mersenne numbers by Koshy and Gao [11] have been on the investigation the divisibility properties of these numbers into Catalan numbers. Mersenne sequence has an important place in number theory as it is also involved in computer science because of Mersenne primes. In number theory, Mersenne number of orders n is defined as $2^n - 1$, where n is a non-negative integer. This identity is defined as the Binet formula for the Mersenne sequence.

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