

Higher Order Sliding Mode Control: A Control Lyapunov Function Based Approach

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Abstract: The paper presents a new method for higher order sliding mode control using control Lyapunov function for chain of integrator system with nonlinear uncertainties. The stability proof of the suggested scheme is analyzed in terms of two Lyapunov functions using appropriate switching function. Using these Lyapunov functions and switching scheme it is proved that, output and its higher order derivatives converge to origin in finite time. Simulation results illustrate the efficacy of the method.

Key-Words: Higher order sliding mode control, Control Lyapunov function, Finite time stability

1 Introduction

Sliding mode control (SMC) has many attractive features such as invariance to matched uncertainties, simplicity in design, robustness against perturbations and some others [1]-[3]. The characteristic feature of continuous-time SMC system is that sliding mode occurs on a prescribed manifold (sliding surface), where switching control is employed to maintain the states on the surface [4]. Although, high-frequency switching is theoretically desirable from the robustness point of view, it is usually hard to achieve in practice because of physical constraints, such as processor computational speed, A/D and D/A conversions delays, actuator bandwidth, etc [3]. Also this results in high frequency oscillations, called chattering. Moreover for the classical SMC to be applied, the relative degree of the control input with respect to the output should be equal to one. To overcome these difficulties a new area called “higher order sliding mode” was looked into. Its main idea is to reduce to zero, not only the sliding function, but also its high order derivatives. Stress was given to finite time stabilization of states. Several second order sliding mode algorithms are described in [4, 9, 11]. In 2001, the first arbitrary order sliding mode controller was proposed [12] by tuning only one gain parameter. Such controller allowed solving the finite-time output stabilization and exact disturbance compensation problem for an output with an arbitrary relative degree. There, its finite time convergence is proved by means of geometrical (point-to-point transformation) method. However, the

convergence rate is not arbitrarily selected. One of the main problems of algorithms [12]-[14] is parameter adjustment. Therefore convergence rate is not in the designer’s hand. Another proposal of arbitrary higher order sliding mode is reported in [16]. It offers several advantages such as practical applicability and constructive approach. However, this approach suffers from a major drawback that the system trajectories reach to only an arbitrary small neighborhood of the origin in finite time. Similar type of approach is used in [18]. Based on the information of initial and final values for each state variable for the control input the higher order controller is designed. In [17], the controller is based on integral sliding mode, but it directly depends on the initial conditions of the system and complex off-line computations are needed before starting the control action. In 2007, a new type of arbitrary-order controller [15], which is r^{th} -sliding homogeneous, controller was proposed. Considering all the above mentioned drawbacks, in 2009 [19] a new proposal of higher order sliding mode came into existence, which was based on combined approach of geometrical homogeneity based linear controller [21] and classical sliding mode technique. Since this controller is based on geometrical homogeneity principle, it is again not possible to calculate exact time of convergence. Recently for calculation of exact time of convergence, the geometrical proof of second order sliding mode is replaced by (a) Moreno et.al., [24] using standard Lyapunov equation and LMI to adjust gain, and (b) by solving partial differential equation to derive a Lyapunov function by Polyakov et.al [23]. To

the best of authors knowledge, only few arbitrary order sliding mode control [20] exists in literature which is fully based on the Lyapunov approach, which was an extension of [5]-[8]. In [5]-[8] finite time stabilization of chain of integrator without uncertainty based on controllability function method is reported. But it is limited to a disturbance free environment. In the present paper a totally different methodology based on controllability function is presented, for chain of integrator system with nonlinear uncertainty.

The main aim of this paper is to propose a new controller, which is fully based on Lyapunov approach, so that higher order sliding mode establishment is guaranteed in exact finite time and which also removes all the above mentioned drawbacks. Here, the problem of higher order sliding mode is formulated in terms of input-output terms using successive derivatives of sliding variable [10], and then the controller is designed for the finite time stabilization of integrator chain with nonlinear uncertainties. These are similar to uncertain linear system with bounded non structured parametric uncertainties. Therefore, the problem can be viewed as a lower or dimension manifold that contains a fully linear system coupled with a nonlinear uncertain integrator. A control Lyapunov function is used based on controllability function method [5]-[8] for designing the lower dimensional switching manifold and discontinuous control (based on classical sliding mode) in order to ensure the robustness with respect to the uncertainties.

The organization of the paper is as follows. In the Section II, the concept of higher-order sliding modes is introduced. In Section III, a brief review on controllability function method for finite time stabilization is presented. Formulation of higher order sliding mode control using controllability function method is presented in Section IV. In Section V, the first, second and third order sliding mode algorithms are explained systematically. Numerical examples are presented in Section VI to illustrate the methods followed by the concluding Section VII.

2 Concept of Higher-Order Sliding Modes

First, let us briefly introduce the higher-order SMC systems. Consider a smooth dynamic affine system

$$\dot{x} = f(x) + g(x)\nu \quad (1)$$

where $x \in \mathbb{R}^n$ is the system state, $y = \sigma(x, t)$ is the system output, $\nu \in \mathbb{R}$ is the scalar control and $f(x)$ and $g(x)$ are some smooth functions. Higher-order

sliding manifold is given as follow

$$s_k = \left\{ x : \frac{d^k}{dt^k} \sigma(x) = 0, \dots, k = 0, 1, \dots, r-1 \right\}, \quad (2)$$

is a nonempty set and consists locally of Filippov trajectories, where σ is a smooth function (this is considered as the sliding variable). The trajectories of this provide the successive time derivative of σ . The motion on set (2) is called r^{th} -order sliding mode [10], which gives the dynamic smoothness degree in some vicinity of the sliding mode.

The relative degree r of the system is assumed to be constant and known. In other words, for the first time the control explicitly appears in the r^{th} total time derivative of σ . $\sigma^r = h(t, x) + l(t, x)\nu$, where

$$\begin{aligned} h(t, x) &= \sigma^r|_{\nu=0}, \\ l(t, x) &= \frac{\partial}{\partial \nu} \sigma^r \neq 0, \quad 0 < K_m \leq \frac{\partial}{\partial \nu} \sigma^r \leq K_M, \\ |\sigma^r|_{\nu=0} &\leq C_0, \quad k_m, k_M, C_0 \in \mathbb{R}^+. \end{aligned}$$

For finite time stabilization of a linear system with uncertainties at origin, ν takes the form given by $\nu = \varphi(\sigma, \dot{\sigma}, \dots, \sigma^{r-1})$. Based on this proposal, several sliding mode controllers (the sub-optimal controller, twisting controller, the terminal sliding mode controller and super twisting controller) were proposed in continuous time. Moreover a general output based controller for r^{th} relative degree system has been also developed.

3 A Brief Review On Controllability Function Method for finite time stabilization [6], [7]

Consider the controllable linear system

$$\dot{x} = Ax + Bu, \quad (3)$$

where $A \in \mathbb{R}^{p \times p}$ and $B \in \mathbb{R}^p$ are defined as

$$A = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B^T = (0 \ 0 \ \dots \ 1). \quad (4)$$

Let $N(\Theta) = N^T(\Theta)$ be a symmetric positive definite matrix, which represents the solution of the following differential equation: [7]

$$\frac{dX}{d\Theta} = \frac{1}{\beta} \Theta^{\frac{1}{\alpha}-1} \left[-AX - XA^T - \beta \Theta^{-\frac{1}{\alpha}} X + BB^T \right], \quad (5)$$

where

$$[N(\Theta)]_{ij} = (-1)^{i+j} \left(\frac{\alpha}{\beta} \Theta^{\frac{1}{\alpha}} \right)^{2p-i-j+1} (2p-i-j)! \times [(p-i)!(p-j)! \times [(\alpha+1)\dots(\alpha+2p-i-j+1)]]^{-1}$$

$$i, j = 1, \dots, p, \quad (6)$$

and $\alpha > 1, \beta > 0$ are constants and $\Theta(x)$ is the unique positive solution of the equation

$$\Theta^{1+\frac{1}{\alpha}(2p-1)} = \sum_{i,j=1}^p [F(\alpha, \beta)]_{ij}^{-1} \Theta^{\frac{1}{p}(i+j-2)} x_i x_j,$$

$$x \neq 0 \text{ and } \Theta(0) = 0. \quad (7)$$

In (7) $[F(\alpha, \beta)]_{ij}^{-1}$ are the elements of $[F(\alpha, \beta)]^{-1}$ the inverse matrix of $[F(\alpha, \beta)]$ where elements are given by

$$[F(\alpha, \beta)]_{ij} = (-1)^{i+j} \left(\frac{\alpha}{\beta} \right)^{2p-i-j+1} (2p-i-j)! \times [(p-i)!(p-j)! \times [(\alpha+1)\dots(\alpha+2p-i-j+1)]]^{-1}$$

$$i, j = 1, \dots, p. \quad (8)$$

Note- Consider the nonlinear system

$$\dot{x} = f(x, u). \quad (9)$$

Assume that the ancillary function $\Theta = \Theta(x)$ (controllability function) exists. The function satisfies the following conditions [8]:-

- $\Theta(x) > 0$ for $x \neq 0, \Theta(0) = 0$.
- $\Theta(x)$ is continuous everywhere and continuously differentiable everywhere except for the origin.
- There exists a number $c > 0$, such that the set

$$Q = \{x \in \mathbb{R}^n : \Theta(x) \leq c\},$$

is bounded.

Suppose also that there exists the control $u = u(x, \Theta(x))$, such that the differential equation

$$\frac{\partial \Theta}{\partial x} f(x, u) \leq -\beta \Theta^{1-\frac{1}{\alpha}}(x), \quad \alpha \geq 1, \beta > 0, \quad (10)$$

holds. This means that motion takes place in the negative direction of the function $\Theta(x)$ i.e., with lesser magnitude of $\Theta(x)$ and finally reaches the origin in finite time.

It is shown in [7] that the control obtained using controllability function method stabilize (3) in finite time. This is briefly discussed in the following theorem [7].

Theorem 1 [7] *The feedback control law*

$$u(x) = -\frac{1}{2} \sum_{j=1}^p [F(\alpha, \beta)]_{pj}^{-1} \times \Theta^{\frac{1}{\alpha}(-p+j-1)} x_j, \quad (11)$$

where $[F(\alpha, \beta)]_{pj}^{-1}, j = 1, \dots, p$ are the elements of the last line of $[F(\alpha, \beta)]^{-1}$ and Θ is defined as in (7). Then all solution $x(t, t_0)$ of the closed loop system after applying (11) satisfies

$$\lim_{t \rightarrow T} x(t) = 0 \text{ where } T = \frac{\alpha}{\beta} \Theta^{\frac{1}{\alpha}}(x_0), \quad (12)$$

Thus system(3) globally exponentially stabilizes at the origin in finite time T , where feedback control satisfies $|u(x)| \leq \eta_0, \forall x \in \mathbb{R}^p$ and η_0 is positive constant.

Proof:-

In control law (11) $[F(\alpha, \beta)]_{pj}^{-1} \times \Theta^{\frac{1}{\alpha}(-p+j-1)}$ are the element of the last line of $N^{-1}(\Theta)$, that is

$$\left[[F(\alpha, \beta)]_{p1}^{-1} \Theta^{-\frac{p}{\alpha}}, \dots, [F(\alpha, \beta)]_{pp}^{-1} \Theta^{-\frac{1}{\alpha}} \right]$$

$$= B^T N^{-1}(\Theta). \quad (13)$$

Closed loop system (3) becomes after substituting (11)

$$\dot{x} = \left(A - \frac{1}{2} B B^T N^{-1}(\Theta) \right) x. \quad (14)$$

Remark 2 *Without loss of generality, one can suppose that $x(t)$ is defined in the interval $[0, T[$. This implies that $\Theta \neq 0$ and the matrices $N(\Theta)$ and $N^{-1}(\Theta)$ exist.*

Let the Lyapunov candidate function be defined as follows

$$\Theta(x) := x^T N^{-1}(\Theta) x. \quad (15)$$

$\Theta(x)$ is a quadratic positive definite function of class C^1 . Introducing the following function,

$$\Xi(\Theta, x) = \Theta(x) - x^T N^{-1}(\Theta) x. \quad (16)$$

From the equation (15) and (16), it follows that

$$d\Xi = \Xi'_{\Theta} \frac{\partial \Theta}{\partial x} + \Xi'_x = 0, \quad (17)$$

where

$$\Xi'_{\Theta} = \frac{\partial \Xi}{\partial \Theta}(\Theta, x) = 1 - x^T \frac{d}{d\Theta} N^{-1}(\Theta) x$$

$$= x^T \left(\frac{1}{\Theta} N^{-1}(\Theta) - \frac{d}{d\Theta} N^{-1}(\Theta) \right) x \quad (18)$$

$$\Xi'_x = \frac{\partial \Xi}{\partial x}(\Theta, x) = -N^{-1}(\Theta) x. \quad (19)$$

Hence

$$\frac{\partial \Theta}{\partial x} = -\frac{\Xi'_x}{\Xi'_\Theta}. \quad (20)$$

Taking the first time derivative of Lyapunov function (15), we get

$$\begin{aligned} \dot{\Theta}(x) &= \left\langle \frac{\partial \Theta}{\partial x}, \dot{x} \right\rangle = \left\langle \frac{\Xi'_x}{\Xi'_\Theta} \left(A - \frac{1}{2} BB^T N^{-1} \Theta \right) x \right\rangle \\ &= \frac{1}{\Xi'_\Theta} x^T [A^T N^{-1}(\Theta) + N^{-1}(\Theta)A] x \\ &\quad - \frac{1}{\Xi'_\Theta} x^T [N^{-1}(\Theta)BB^T N^{-1}(\Theta)] x. \end{aligned} \quad (21)$$

Note- $\langle \cdot, \cdot \rangle$ denotes the inner product.

From (5) and (21)

$$\begin{aligned} &A^T N^{-1}(\Theta) + N^{-1}(\Theta)A - N^{-1}(\Theta)BB^T N^{-1}(\Theta) \\ &= \beta \Theta^{1-\frac{1}{\alpha}} \left[\frac{d}{d\Theta} N^{-1}(\Theta) - \frac{1}{\Theta} N^{-1}(\Theta) \right]. \end{aligned} \quad (22)$$

Hence

$$\begin{aligned} &x^T [A^T N^{-1}(\Theta) + N^{-1}(\Theta)A - N^{-1}(\Theta)BB^T N^{-1}(\Theta)] x \\ &= -\beta \Theta^{1-\frac{1}{\alpha}} \Xi'_\Theta. \end{aligned} \quad (23)$$

From (21) and (23)

$$\dot{\Theta}(x) = -\beta \Theta^{1-\frac{1}{\alpha}}(x). \quad (24)$$

which is negative definite because $\alpha > 1$ and $\beta > 0$. After integration of Eqn.(24), one can get

$$\Theta = \left(\frac{1}{\alpha} (-\beta t + W) \right)^\alpha. \quad (25)$$

where W is an integration constant. For nonzero initial condition $\Theta(x_0) \neq 0$, hence

$$W = \alpha \Theta^{\frac{1}{\alpha}}(x_0) \quad (26)$$

Now putting $\Theta = 0$, one can get

$$T = \frac{\alpha}{\beta} \Theta^{\frac{1}{\alpha}}(x_0)$$

This implies that the state $x(t) \in [0, T[$ satisfies $x(T) = 0$. Thus, one can conclude that $x(t)$ converges to zero in finite time. Apart from that we have to also prove that $x(t) = 0$ for all $t \geq T$.

From the **Lasalle Invariance Principle**, it obvious that the largest invariant set contained in $I = \{x \in \mathbb{R}^p, \dot{\Theta}(x)\} = 0$ is the manifold $\Theta(x) = 0$. Now from definition of Θ , $\Theta(x) = 0 \Rightarrow x = 0$. This implies that the origin is the largest invariant set contained in I . Since $x(T) = 0$, one can say that that state $x(t) = 0$ remains at zero for all $t \geq T$. Only $x(t) = 0$ is the largest invariant set of \mathbb{R}^p , this implies that all trajectory converges towards origin in finite time T . \square

4 Higher Order Sliding Mode Control Using Controllability Function Method

Consider the nonlinear system (1) with a relative degree r with respect to σ . The r^{th} order sliding mode control with respect to σ is equivalent to the finite time stabilization to zero of integrator chain with nonlinear uncertainties [10]

$$\begin{aligned} \dot{z}_1 &= A_{11}z_1 + A_{12}z_2 \\ \dot{z}_2 &= \varphi + \gamma\nu, \end{aligned} \quad (27)$$

where $z_1 = [\sigma \dots \sigma^{(r-2)}]^T$, $z_2 = \sigma^{(r-1)}$, $\varphi = L_f^r \sigma$, $\gamma = L_g L_f^{r-1} \sigma$ and A_{11}, A_{12} defined by $A_{11} = A$, $A_{12} = B$. Assume that $\varphi = \hat{\varphi} + \Delta\varphi$ and $\gamma = \hat{\gamma} + \Delta\gamma$ are divided into nominal part i.e. $\hat{\varphi}$ and $\hat{\gamma}$, known a priori and uncertain bounded functions $\Delta\varphi$ and $\Delta\gamma$. Also uncertain bounded function satisfies the following inequalities

$$|\Delta\varphi - \Delta\gamma\hat{\gamma}^{-1}\hat{\varphi}| \leq \hat{\alpha}, \quad |\Delta\gamma\hat{\gamma}^{-1}| \leq 1 - \hat{\beta} \quad (28)$$

where $\hat{\gamma}$ is non-singular and there are a priori known constant $\hat{\alpha}$, along with priori known constant $0 \leq \hat{\beta} \leq 1$.

Let the system (27) can be stabilized in finite time by the control

$$\nu = \hat{\gamma}^{-1} [-\hat{\varphi} + \hat{u}] \quad (29)$$

(27) can also be written as

$$\begin{aligned} \dot{z}_1 &= A_{11}z_1 + A_{12}z_2 \\ \dot{z}_2 &= (\Delta\varphi - \Delta\gamma\hat{\gamma}^{-1}\hat{\varphi}) + (1 + \Delta\gamma\hat{\gamma}^{-1})\hat{u}, \end{aligned} \quad (30)$$

Our main aim to design \hat{u} so that the transformed system (30) is finite time stable in spite of any matched uncertainty.

Theorem 3 The control input \hat{u} which is defined as

$$\hat{u} = u - K \text{sign}(s), \quad (31)$$

and corresponding

$$\nu = \hat{\gamma}^{-1} [-\hat{\varphi} + u - K \text{sign}(s)] \quad (32)$$

with

$$K \geq \frac{\hat{\alpha} + \hat{\eta} + (1 - \hat{\beta})|u|}{2 - \hat{\beta}}, \quad (33)$$

and $\hat{\eta} > 0$ and $s \in \mathbb{R}$ be a proposed sliding surface as

$$s = \sigma^{(r-1)} - \sigma^{(r-1)}(t_0) - \int_{t_0}^t u dt, \quad (34)$$

where $s(x(t_0)) = 0$ at initial time $t = t_0$, (so the system always starts at the sliding manifold), leads to the establishment of r^{th} -order sliding mode with respect to σ by attracting each trajectory in finite time.

proof:-

Taking the Lyapunov candidate function as

$$V = \frac{1}{2}s^2 \quad (35)$$

The time derivative of candidate Lyapunov function is given as:

$$\dot{V} = s\dot{s} \quad (36)$$

Using Eqn.(30), (34) and (31)

$$\begin{aligned} \dot{V} &= s [(\Delta\varphi - \Delta\gamma\hat{\gamma}^{-1}\hat{\varphi}) + (1 + \Delta\gamma\hat{\gamma}^{-1})\hat{u} - u] \\ &= s [(\Delta\varphi - \Delta\gamma\hat{\gamma}^{-1}\hat{\varphi}) + (1 + \Delta\gamma\hat{\gamma}^{-1})u - u] \\ &\quad - s [(1 + \Delta\gamma\hat{\gamma}^{-1})(K\text{sign}(s))] \\ &\leq s [\hat{\alpha} + (1 - \hat{\beta})u - (2 - \hat{\beta})K\text{sign}(s)] \end{aligned} \quad (37)$$

For finite time stabilization to zero of vector $z = [z_1^T \ z_2^T]^T = [\sigma \ \dot{\sigma} \ \dots \ \sigma^{r-1}]^T$ we select the gain K as (33), such that the η -reachability condition of sliding mode is satisfied. One gets

$$s\dot{s} \leq -\hat{\eta}|s| \quad (38)$$

It means that if inequality and (33) is satisfied, then (38) conform the finite time reachability to s . During sliding equivalent control is given by $s = \dot{s} = 0$.

$$\begin{aligned} (\Delta\varphi - \Delta\gamma\hat{\gamma}^{-1}\hat{\varphi}) + (1 + \Delta\gamma\hat{\gamma}^{-1})\hat{u} - u &= 0 \\ \Rightarrow (\Delta\varphi - \Delta\gamma\hat{\gamma}^{-1}\hat{\varphi}) + (1 + \Delta\gamma\hat{\gamma}^{-1})\hat{u} &= u \end{aligned} \quad (39)$$

Substituting (39) into (30), the closed loop dynamics becomes

$$\begin{aligned} \dot{z}_1 &= A_{11}z_1 + A_{12}z_2 \\ \dot{z}_2 &= u, \end{aligned} \quad (40)$$

The above equation becomes similar to chain of integrators free from any uncertainty. Hence we can easily design the controllability function based controller for the (40).
□

Remark 4 For feasibility of control law (32) (so that uncertain chain of integrators is finite time stable), switching variable (34) must be continuous. The continuity of switching variable is always guaranteed because third term of switching variable s taken after integration of control u (generated for the chain of integrators without any uncertainty using controllability function method, although some particular value of α and β control u may be discontinuous).

Remark 5 Whenever one can try to find the analytical solution of (7), it is not always simple but fortunately, it can be solved numerically. In fact for given value of α and β Eqn.(7) can be expressed as a polynomial in Θ , for which one can easily find its numerical value. But in some cases for example if the order of the system is one, two or three one can obtain the explicit expression of Θ for particular value of α and β .

Remark 6 In this paper for resolution of solution of Eqn.(7) for Θ and finding the control ν , polynomial algorithm of Matlab is used everywhere.

5 Illustrations By First(reaching law), Second And Third Order Sliding Mode

The first order sliding mode algorithm(reaching law) is,

$$\dot{\sigma}_1 = \hat{\varphi} + \Delta\varphi + (\hat{\gamma} + \Delta\gamma)\nu, \quad (41)$$

where σ_1 is the output(sliding variable). Using Eqns.(6), (7) and (11) we get

$$\begin{aligned} N(\Theta) &= \frac{\alpha\Theta^{\frac{1}{\alpha}}}{\beta(\alpha+1)}, \quad N^{-1}(\Theta) = \frac{\beta(\alpha+1)}{\alpha\Theta^{\frac{1}{\alpha}}} \\ \Theta(\sigma) &= \frac{\beta(\alpha+1)\sigma_1^2}{\alpha\Theta^{\frac{1}{\alpha}}}, \quad \Theta(\sigma) = \left[\frac{\alpha}{\beta}(\alpha+1)\sigma_1^2 \right]^{\frac{\alpha}{\alpha+1}}. \end{aligned} \quad (42)$$

$$u = -\frac{1}{2} \frac{\beta(\alpha+1)}{\alpha\Theta^{\frac{1}{\alpha}}} \sigma_1. \quad (43)$$

Now using (34), we get

$$s = \sigma_1 - \sigma_1(t_0) - \int_{t_0}^t u dt \quad (44)$$

Taking the derivative of first Eqn. of (44) and substituting the value of σ_1 from (41), one can write

$$\dot{s} = \hat{\varphi} + \Delta\varphi + (\hat{\gamma} + \Delta\gamma)\nu - u, \quad (45)$$

Choosing gain K , as par Eqn.(33) η -reachability condition is satisfied. Hence $s = 0$ in finite time and the equivalent control during sliding is obtained by putting $\dot{s} = 0$. Therefore one can write

$$\dot{\sigma}_1 = u = -\frac{1}{2} \frac{\beta(\alpha+1)}{\alpha\Theta^{\frac{1}{\alpha}}} \sigma_1. \quad (46)$$

Putting Θ from (42) in the above equation, we get

$$\dot{\sigma}_1 = -\frac{1}{2} \frac{\beta}{\alpha \left[\frac{\alpha}{\beta} (\alpha + 1) \right]^{\frac{1}{\alpha+1}}} \sigma_1^{\frac{\alpha-1}{\alpha+1}}. \quad (47)$$

Obviously (47) is finite time stable for $\alpha \geq 1$ and $\beta > 0$. Equation of the controller from Theorem 2 is given by

$$\begin{aligned} \nu &= \hat{\gamma}^{-1} [-\hat{\varphi} + u - K \text{sign}(s)], \\ s &= \sigma_1 - \sigma_1(0) - \int_{t_0}^t u dt \\ &= \sigma_1 - \sigma_1(0) - \int_{t_0}^t \frac{1}{2} \frac{\beta}{\alpha \left[\frac{\alpha}{\beta} (\alpha + 1) \right]^{\frac{1}{\alpha+1}}} \sigma_1^{\frac{\alpha-1}{\alpha+1}} dt \end{aligned} \quad (48)$$

The second order sliding mode algorithm is,

$$\begin{aligned} \dot{\sigma}_1 &= \sigma_2 \\ \dot{\sigma}_2 &= \hat{\varphi} + \Delta\varphi + (\hat{\gamma} + \Delta\gamma)\nu, \end{aligned} \quad (49)$$

where $A_{11} = 0$, $A_{12} = 1$, σ_1 and σ_2 be the output(sliding variable) and its derivative respectively. Using Eqns.(6) and (7), we get

$$N(\Theta) = \begin{pmatrix} \frac{2\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)^3}{(\alpha+1)(\alpha+2)(\alpha+3)} & -\frac{\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)^2}{(\alpha+1)(\alpha+2)} \\ -\frac{\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)^2}{(\alpha+1)(\alpha+2)} & -\frac{\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)}{(\alpha+1)} \end{pmatrix}, \quad (50)$$

$$N^{-1}(\Theta) = \begin{pmatrix} \frac{(\alpha+2)^2(\alpha+3)}{\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)^3} & \frac{(\alpha+2)(\alpha+3)}{\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)^2} \\ \frac{(\alpha+2)(\alpha+3)}{\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)^2} & \frac{2(\alpha+2)}{\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)} \end{pmatrix}. \quad (51)$$

$$\begin{aligned} \Theta(\sigma) &= \frac{(\alpha+2)^2(\alpha+3)\sigma_1^2}{\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)^3} + \frac{2(\alpha+2)(\alpha+3)\sigma_1\sigma_2}{\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)^2} \\ &+ \frac{(\alpha+2)\sigma_2^2}{\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)}. \end{aligned} \quad (52)$$

$u(\sigma)$ from (11) is given as

$$u(\sigma) = -\frac{(\alpha+2)(\alpha+3)\sigma_1}{2\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)^2} - \frac{(\alpha+2)\sigma_2}{\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)}. \quad (53)$$

Now using (34), we get

$$\begin{aligned} s &= \sigma_2 - \sigma_2(t_0) - \int_{t_0}^t u dt \\ \Rightarrow \sigma_2 &= s + \sigma_2(t_0) + \int_{t_0}^t u dt \end{aligned} \quad (54)$$

Substituting σ_2 in the first equation of (49) the same becomes $\dot{\sigma}_1 = s + \sigma_2(t_0) + \int_{t_0}^t u dt$. Also differentiating s and substituting the value of $\dot{\sigma}_2$ from (49), one can write

$$\dot{s} = \hat{\varphi} + \Delta\varphi + (\hat{\gamma} + \Delta\gamma)\nu - u, \quad (55)$$

Choosing gain K , as par Eqn.(33) η -reachability condition is satisfied. Hence $s = 0$ in finite time and during sliding equivalent control can be obtained by putting $\dot{s} = 0$, therefore

$$\begin{aligned} \dot{\sigma}_1 &= \sigma_2(t_0) + \int_{t_0}^t u dt, \\ 0 &= \sigma_2 - \sigma_2(t_0) - \int_{t_0}^t u dt, \\ \dot{\sigma}_2 &= u \\ \Rightarrow \dot{\sigma}_1 &= \sigma_2, \quad \dot{\sigma}_2 = u. \end{aligned} \quad (56)$$

Obviously (56) is finite time stable using controllability function method based u , as discussed in Section III, for $\alpha \geq 1$ and $\beta > 0$. Equation of the controller from Theorem 2 is given by

$$\begin{aligned} \nu &= \hat{\gamma}^{-1} [-\hat{\varphi} + u - K \text{sign}(s)], \\ s &= \sigma_2 - \sigma_2(t_0) - \int_{t_0}^t u dt \\ u &= -\frac{(\alpha+2)(\alpha+3)\sigma_1}{2\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)^2} - \frac{(\alpha+2)\sigma_2}{\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)}. \end{aligned} \quad (57)$$

The third order sliding mode algorithm is,

$$\begin{aligned} \dot{\sigma}_1 &= \sigma_2 \\ \dot{\sigma}_2 &= \sigma_3 \\ \dot{\sigma}_3 &= \hat{\varphi} + \Delta\varphi + (\hat{\gamma} + \Delta\gamma)\nu, \end{aligned} \quad (58)$$

where $A_{11} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A_{12}^T = [0 \ 1]$, σ_1 , σ_2 and σ_3 are the output(sliding variable), first derivative of output and second derivative of output respectively. Using Eqns.(6) and (7), we get

$$N(\Theta) = \begin{pmatrix} \frac{4\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)^5}{(\alpha+1)\cdots(\alpha+5)} & \frac{3\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)^4}{(\alpha+1)\cdots(\alpha+4)} & \frac{2\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)^3}{(\alpha+1)\cdots(\alpha+3)} \\ -\frac{3\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)^4}{(\alpha+1)\cdots(\alpha+4)} & \frac{2\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)^3}{(\alpha+1)\cdots(\alpha+3)} & -\frac{\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)^2}{(\alpha+1)(\alpha+2)} \\ \frac{2\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)^3}{(\alpha+1)\cdots(\alpha+3)} & -\frac{\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)^2}{(\alpha+1)(\alpha+2)} & \frac{\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)}{(\alpha+1)} \end{pmatrix} \quad (59)$$

$$\begin{aligned} \Theta^{1+\frac{5}{\alpha}} &= \frac{3\beta}{\alpha}(\alpha+3)\sigma_3^2\Theta^{\frac{4}{\alpha}} \\ &+ \frac{6\beta^2}{\alpha^2}(\alpha+3)(\alpha+4)\sigma_2\sigma_3\Theta^{\frac{3}{\alpha}} \\ &+ \frac{\beta^3}{\alpha^3}(\alpha+3)(\alpha+4)[(5\alpha+16)\sigma_2^2+2(\alpha+5)\sigma_1\sigma_3]\Theta^{\frac{2}{\alpha}} \\ &+ \frac{4\beta}{\alpha}(\alpha+3)^2(\alpha+4)(\alpha+5)\sigma_1\sigma_2\Theta^{\frac{1}{\alpha}} \\ &+ \frac{\beta^5}{\alpha^5}(\alpha+3)^2(\alpha+4)^2(\alpha+5)\sigma_1^2 \end{aligned} \quad (60)$$

$u(\sigma)$ from (11) is given as

$$\begin{aligned} u &= -\frac{(\alpha+3)(\alpha+4)(\alpha+5)\sigma_1}{2\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)^3} - \frac{3(\alpha+3)(\alpha+4)\sigma_2}{2\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)^2} \\ &- \frac{3(\alpha+3)\sigma_3}{2\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)}. \end{aligned} \quad (61)$$

Equation of the controller from Theorem 2 is given by

$$\begin{aligned} \nu &= \hat{\gamma}^{-1}[-\hat{\varphi} + u - K \text{sign}(s)] \\ s &= \sigma_3 - \sigma_3(t_0) - \int_{t_0}^t u dt \\ u &= -\frac{(\alpha+3)(\alpha+4)(\alpha+5)\sigma_1}{2\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)^3} - \frac{3(\alpha+3)(\alpha+4)\sigma_2}{2\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)^2} \\ &- \frac{3(\alpha+3)\sigma_3}{2\left(\frac{\alpha}{\beta}\Theta^{\frac{1}{\alpha}}\right)}. \end{aligned} \quad (62)$$

6 Simulation Results

Stabilization of Variable Length Pendulum using the Proposed Second Order Controller

Consider an example of a variable-length pendulum [22] with motions restricted to some vertical plane. System dynamics is given as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2\frac{\dot{R}(t)}{R(t)}x_2 - \frac{g}{R(t)}\sin(x_1) + \frac{1}{mR^2(t)}u. \end{aligned} \quad (63)$$

with (x_1, x_2) the angular positions and velocity of the rod, $m = 1\text{kg}$ the rod mass, $g = 9.81\text{ms}^{-2}$ the gravitational constant, $R(t)$ the distance from the fix point and the mass, and u the control torque. $R(t)$ is a non-measured disturbance and given by $R(t) = 0.8 + 0.1\sin(8t) + 0.3\cos(4t)$, $m = 1\text{kg}$, $g = 9.81\text{ms}^{-2}$. Function $R(t)$ and its time derivative $\dot{R}(t)$ are such that $0.4515 \leq R(t) \leq 1.1485$ and $-2.5226 \leq \frac{\dot{R}(t)}{R(t)} \leq 1.4989$. After applying the proposed controller (57)

with $K = 100$, $\alpha = 2$, $\beta = 1$, $x_1(0) = 2$, $x_2(0) = -2$, system state trajectories converges to origin in finite time as shown in Figure(1).

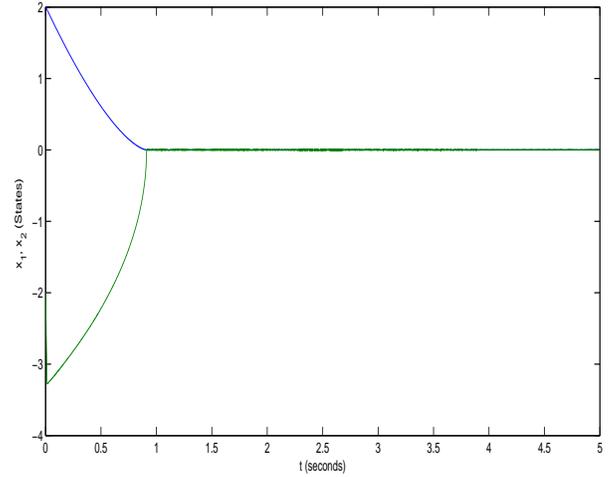


Figure 1: Evolution of states w.r.t. time

Simulation Results of the Proposed Third Order Sliding Mode Controller

For the simulation of the proposed third order sliding mode controller, initial values of sliding surface and its derivatives are chosen as $\sigma_1 = -0.5$, $\sigma_2 = 1$, $\sigma_3 = -0.1$, the controller gain $K = 50$ and controllability function parameters as $\alpha = 2$ and $\beta = 1$. The Figure (2) shows that the sliding surface σ_1 , it derivatives σ_2 and σ_3 converge towards origin in finite time.

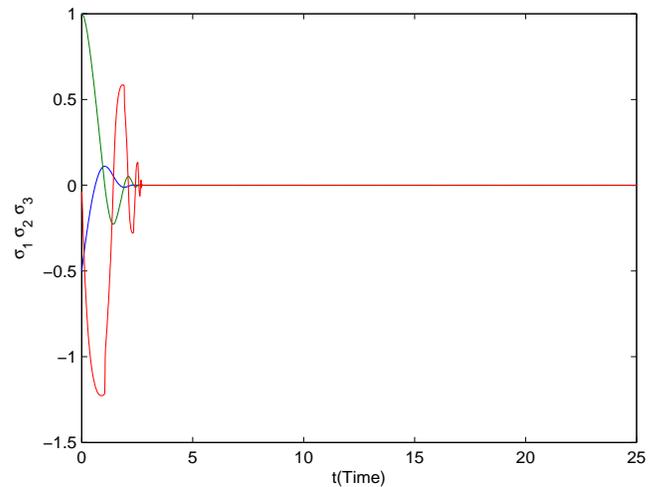


Figure 2: Trajectories of sliding variable and its derivative w.r.t time

7 Conclusion

In this paper a generalized higher order sliding mode controller has been presented. Finite time convergence of the output and its higher order derivatives was proved by means of the controllability function based Lyapunov function $\Theta(x) = x^T N^{-1}(\Theta)x$ and $V = \frac{1}{2}s^2$ using appropriate switching variable. Indeed, this is the main contribution proposed in this paper. Also, the application of this idea has been tested on a second order variable length pendulum of relative degree 2. Simulation results verifies the proposed controller scheme.

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